Note: Semi-Order Dimension Two is a Comparability Invariant

Stefan Felsner[1]

Freie Universität Berlin, Fachbereich Mathematik, Institut für Informatik, Takustr. 9, 14195 Berlin, Germany
E-mail: felsner@inf.fu-berlin.de

Rolf H. Möhring

Technische Universität Berlin, Fachbereich Mathematik, MA 6-1, Straße des 17. Juni 135, 10623 Berlin, Germany
E-mail: mohring@math.tu-berlin.de

Abstract. A partial order $P = (X, <_P)$ is a semi-order if it is an interval order admitting an interval representation such that all the intervals are of unit length. The semi-order dimension of P is the smallest k for which there exist k semi-order extensions of P which realize P. In [HKM] the question whether semi-order dimension is a comparability invariant was posed. We prove that for $k = 2$ this is the case.

AMS subject classification (1991). 06A07

Key words. Partially ordered sets, dimension, comparability invariance

1. Introduction

A partial order $P = (X, <_P)$ is a *semi-order* if it is an interval order admitting an interval representation such that all the intervals are of length 1, i.e., $I_x = (s_x, s_x + 1)$. The *semi-order dimension* of P, denoted $dim_{\mathcal{S}}(P)$, is the smallest k for which there exist k semi-order extensions of P which realize P. Since linear orders are semi-orders and semi-orders are interval orders we trivially obtain that order dimension is an upper bound and interval dimension is a lower bound for semi-order dimension. Rabinovich [Ra] has shown that order dimension and semi-order dimension differ only by constant factors between 1 and 3. This might be one of the reasons why semi-order dimension has for quite a while received little attention.

In [HKM] it was observed that all techniques developed to proof the comparability invariant of dimension, interval dimension and some related notions of dimension fail to work in the case of semi-order dimension.

The *comparability graph* of an order $P = (X, <)$ is the undirected graph $G_P = (X, E)$ with $\{x, y\} \in E$ iff either $x <_P y$ or $y <_P x$. In general there can be nonisomorphic orders P and Q with $G_P = G_Q$. A property ψ of orders is a *comparability invariant* if it depends on the comparability graph only, i.e., if $P \in \psi$ and $G_P = G_Q$ together imply $Q \in \psi$.

A subset of elements A of X is called *autonomous* in P, if the relation of elements in A to an element outside A is independent of the element of A. More formally, if for any $x \notin A$, whenever $x < a$, $x > a$ or $x||a$ holds for some $a \in A$, then the same holds for all $a \in A$. We say that an order $P' = (X, <')$ is obtained by a *reversal* from $P = (X, <)$ if there is an autonomous set $A \subset X$ so that:

(1) If not both of x and y are in A, then $x <' y$ if and only if $x < y$.

(2) If $x, y \in A$, then $x <' y$ if and only if $y < x$.

The order obtained from P by reversing the autonomous subset A will be denoted by $P|A$.

The importance of the notion of autonomous sets is due to a theorem of Gallai [Ga]: Two orders P and Q on a set X have the same comparability graph, exactly if there exists a finite sequence $P = R_0, R_1, \ldots, R_k = Q$ of orders on X, such that R_i is obtained by a reversal from R_{i-1} for $i = 1, \ldots, k$.

[1] supported by the DFG under grant FE 340/2-1

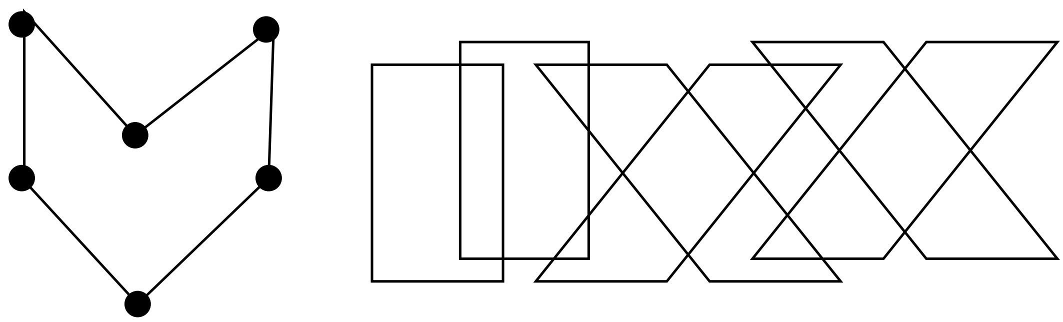

Figure 1. The Chevron with a parallelogram representation.

As consequence of this theorem we obtain a simple scheme for proving the comparability invariance of a property. We only have to show that if P has the property and P' is obtained by a reversal from P, then the property holds for P' too. In the next section we use this scheme to show that in the finite case semi-order dimension two is a comparability invariant. Recently we learned that David Kelly has independently obtained this result with a different proof technique.

2. Flipping in semi-order dimension two

Let P be an order of semi-order dimension two and let S_1, S_2 be a realizer of P. In the proof we will refer to a representation of P by parallelograms induced by representations of S_1 and S_2. We think of this representation as being embedded in the Euclidean plane, so let L_1 be the line $y = 0$ and let L_2 be the line $y = 1$. Now assume that the intervals $(s_x, s_x + 1)$ for $x \in X$ form a semi-order representation of S_1 on L_1 and that S_2 is represented by the intervals $(t_x, t_x + 1)$ for $x \in X$ on L_2. The parallelogram $p(x)$ corresponding to an element $x \in X$ is the stripe joining the intervals corresponding to x on L_1 and L_2 (see Figure 1). That is, $p(x)$ is the convex hull of the four points $(s_x, 0), (s_x + 1, 0), (t_x, 1), (t_x + 1, 1)$. Note that the parallelograms represent the order relation of P, i.e., $x <_P y$ exactly if $p(x)$ is completely to the left of $p(y)$ and $x||y$ iff $p(x) \cap p(y) \neq \emptyset$.

With an autonomous set A in P we associate the convex region $C(A)$ spanned by the parallelograms $p(x)$ with $x \in A$. The four corners of $C(A)$ in clockwise order are $(l_1, 0), (l_2, 1), (r_2, 1)$ and $(r_1, 0)$, if for $i = 1, 2$ we define l_i as the leftmost and r_i as the rightmost point corresponding to the interval of an element of A in S_i. That is $l_1 = \min_{x\in A} s_x$ and $r_1 = \max_{x\in A} s_x + 1$, similarly, $l_2 = \min_{x\in A} t_x$ and $r_2 = \max_{x\in A} t_x + 1$.

Note that the parallelogram representation of an order of semi-order dimension two is a representation by unit-parallelograms, i.e., by parallelograms of width one. In this article we use the term parallelogram as equivalent to unit-parallelogram.

Let P be an order with an autonomous subset A and a parallelogram representation $x \to p(x)$. The first naive idea for obtaining a parallelogram representation of $P|A$ is to *flip the representation of A relative to $C(A)$* and leave the rest of the representation unchanged. That is, for $x \notin A$ we let $p'(x) = p(x)$, while for $x \in A$ we take $p'(x)$ as the parallelogram spanned by $(l_1 + r_1 - s_x - 1, 0), (l_1 + r_1 - s_x, 0), (l_2 + r_2 - t_x - 1, 1), (l_2 + r_2 - t_x, 1)$ (see Figure 2).

In general this flipping will not lead to a representation of $P|A$. Already in the cases of dimension or interval dimension the analogous idea fails. However, in these two cases geometric proofs for comparability invariance are known. In these proofs the given representation is first modified so that the flip in the new representation gives a representation of $P|A$, see e.g. [HKM]. The

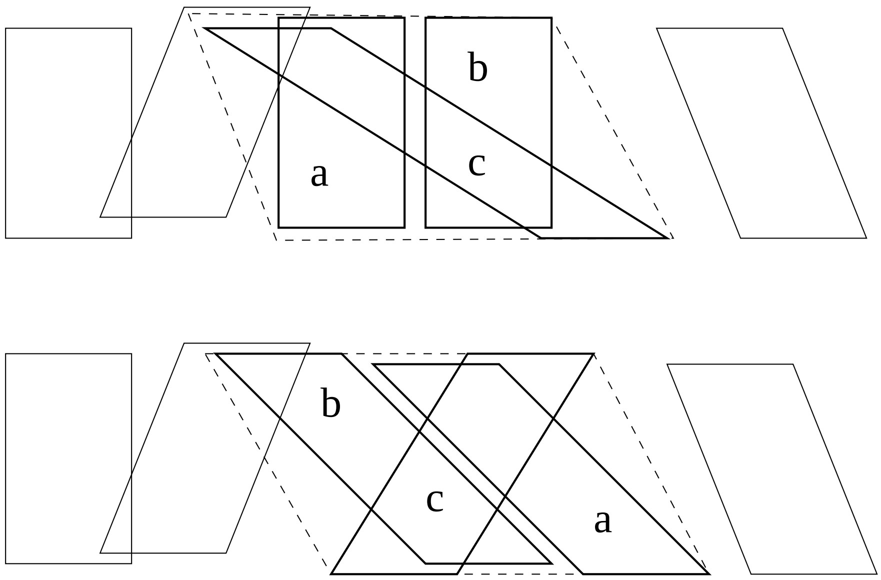

Figure 2. Flipping the set $A = \{a, b, c\}$.

problem with semi-order dimension is that semi-orders are not invariant under the modifications used.

To indicate the role taken by the flipping procedure we now give a brief outline of our proof for the comparability invariance of semi-order dimension two. Let A be an autonomous set in $P = (X, <)$ and let B be the set of all elements $x \in X$ with $p(x) \subset C(A)$, obviously $A \subseteq B$.

Claim 1. The set B is autonomous in P.

Claim 2. If we flip the representation of B relative to $C(B) = C(A)$ and leave the rest of the representation unchanged we obtain a representation of $P_1 = P|B$.

Claim 3. If $A_1 = B \setminus A$ then A_1 is autonomous and $P|A = P_1|A_1$.

We then repeat the process with P_1 and A_1 replacing P and A. The claim is that, after a finite number of steps, we end up with a P_i such that $A_i = \emptyset$, i.e., with a representation of $P|A$. This can be proven by showing that a function measuring the size of A_i is decreasing. Assuming that the representation P_i is such that no two parallelograms share a common corner it can be shown that Area($C(A_{i+1})$) <Area($C(A_i)$) and $|A_{i+2}| < |A_i|$.

The rigorous proof, as given in the next section, is based on the ideas indicated above. However, the proof will be indirect and will require some more technicalities.

3. The main theorem

Theorem 3.1. *Semi-order dimension two is a comparability invariant.*

Suppose the theorem is false. It follows, that there exist pairs (P, A), where $P = (X, <)$ is an order with $dim_{\mathcal{S}}(P) = 2$ and A is an autonomous set of P such that $dim_{\mathcal{S}}(P|A) \geq 3$. Among all such pairs (P, A) we choose one with $|A|$ minimal.

Let P be realized by semi-orders S_1 and S_2. We assume that S_1 and S_2 are represented by unit length intervals so that no two intervals share a common endpoint. For $x \in X$ let $(s_x, s_x + 1)$

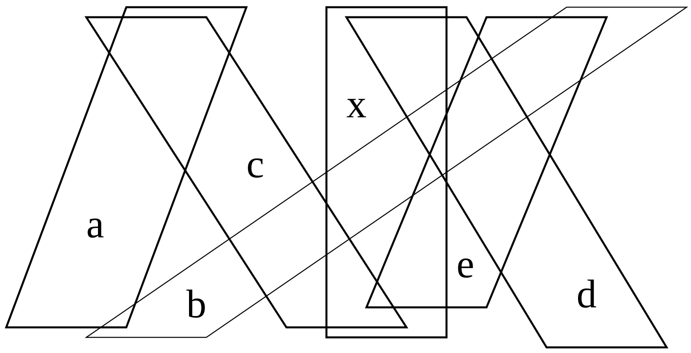

Figure 3. If $A = \{a, b, c, d, e\}$ then $A^* = \{a, c, d, e\}$ and $B = \{a, c, d, e, x\}$.

and $(t_x, t_x + 1)$ be the intervals of x in S_1 and S_2, also let $p(x)$ denote the parallelogram of $x \in X$ induced by the representations of S_1 on L_1 and S_2 on L_2.

An element $a \in A$ is called *isolated* if $a \| a'$ for all $a' \in A \setminus \{ a \}$. We call an autonomous set *proper* if it contains no isolated elements. With $A^* = A \setminus \{ a : a \text{ is isolated in } A \}$ we denote the largest proper subset of an autonomous set A. The next trivial observation is stated as a lemma for ease of referencing.

Lemma 3.2. *If A is autonomous in P, then A^* is autonomous and $P|A = P|A^*$.*

We are ready now to state our key lemma.

Lemma 3.3. *Let A be a autonomous set in P and let $x \in X$ such that $p(x) \cap C(A^*) \neq \emptyset$. Then*
(1) *$p(x) \subset C(A^*)$ or*
(2) *$p(x)$ intersects with every width one parallelogram q contained in $C(A^*)$.*

Proof. Assume $p(x) \not\subset C(A^*)$. Then there is a corner c of $p(x)$ with $c \notin C(A^*)$. By symmetry we may suppose that c is the upper right corner of $p(x)$ and that this corner is to the right of the upper right corner of $C(A^*)$, i.e., $t_x + 1 > r_2$.

We first show that $x \| a$ for some $a \in A^*$ and hence for all $a \in A$. If $t_x < r_2$, then x is incomparable to a_r^2, the element of A^* with $t_{a_r^2} + 1 = r_2$. Otherwise, if $t_x > r_2$, then $s_x < r_1$ and $p(x)$ intersects with $p(a_r^1)$ if $a_r^1 \in A^*$ is the element with $s_{a_r^1} + 1 = r_1$.

Let a_0 be an element of A^* with $s_{a_0} = l_1$. Since A^* is proper there is an element $a_1 \in A^*$ comparable to a_0, necessarily $a_0 < a_1$. Therefore, $t_{a_0} + 1 < t_{a_1} \leq r_2 - 1 < t_x$ and in S_2 the element a_0 is less than x. Hence in S_1 we have $a_0 \not< x$, i.e., $s_x < s_{a_0} + 1 = l_1 + 1$. This shows that the line segment connecting s_x and t_x intersects every width one parallelogram contained in $C(A^*)$. △

Let A be a autonomous set in P and $B = \{ x : p(x) \subset C(A^*) \}$. The definition is illustrated in Figure 3.

Lemma 3.4. *The set B is autonomous in P and a parallelogram representation p_1 of $P|B$ is obtained by flipping the representation of B relative to $C(B) = C(A^*)$.*

Proof. If $x \notin B$ and $x < b$ for some $b \in B$, then by Lemma 3.3. $p(x) \cap C(A^*) = \emptyset$. Also $x < b$ implies that $p(x)$ is to the left of $p(b)$. This proves that $p(x)$ is completely to the left of $C(A^*)$ and hence $x < b$ for all $b \in B$.

If $x \notin B$ and $x > b$ for some $b \in B$, then symmetric to the previous case we obtain $x > b$ for all $b \in B$.

If $x \notin B$ and $x\|b$, then $\emptyset \neq p(x) \cap p(b) \subseteq p(x) \cap C(A^*)$. Lemma 3.3. implies $p(x) \cap p(b) \neq \emptyset$ for all $b \in B$, i.e., $x\|b$ for all $b \in B$. This concludes the proof that B is autonomous.

The above argument can be summarized by saying that the relation (left, right, intersecting) of $p(x)$, $x \notin B$, to all possible width one parallelograms $q \subset C(A^*)$ is the same. If we flip B in $C(A^*)$ the new parallelogram $p_1(b)$ assigned to $b \in B$ is again contained in $C(A^*)$. Therefore, the flipping will not change the relation of $b \in B$ to any $x \notin B$. For $x, y \notin B$ the parallelograms remain unchanged, i.e., $p(x) = p_1(x)$ and $p(y) = p_1(y)$, hence their relation in the order P_1 defined by p_1 remains the same as in P. If $b, c \in B$, then $p_1(b)$ is to the left of $p_1(c)$ exactly if $p(c)$ is to the left of $p(b)$. Altogether this shows that $P_1 = P|B$. △

Lemma 3.5. *If $A_1 = B \setminus A^*$ then A_1 is autonomous in $P|B$ and $P|A = P|B|A_1$.*

Proof. If $x \notin B$ then, since B is autonomous, the relation of x with any $a \in A_1$ is shared by all $b \in B$ and hence by all $b \in A_1$. If $x \in B$ but $x \notin A_1$ then $x \in A^*$ and, since A^* is autonomous, $x\|a$ for all $a \in A_1$. This shows that A_1 is autonomous.

Since all elements of A^* are incomparable to all elements of A_1 the order relation on B is the parallel composition of the order relations on A^* and A_1. Therefore, $P|A^*|A_1 = P|B$ and $P|A^* = P|B|A_1$. Together with Lemma 3.2. this completes the proof. △

Remove isolated elements from A_1 to obtain A_1^* and note that $P|A = P|B|A_1^*$. Let $C(A_1^*)$ be the convex region spanned by the parallelograms $p_1(x)$ with $x \in A_1^*$ and define $B_1 = \{x : p_1(x) \subset C(A_1^*)\}$. By Lemma 3.4. a parallelogram representation p_2 of $P|B|B_1$ is obtained by flipping the representation of B_1 in $C(A_1^*) = C(B_1)$. Now let $A_2 = B_1 \setminus A_1^*$, Lemma 3.5. shows that A_2 is autonomous in $P|B|B_1$ and $P|A = P|B|B_1|A_2$.

Our assumptions about P together with the lemmas imply that $dim_{\mathcal{S}}(P|B|B_1) = 2$ and $dim_{\mathcal{S}}(P|B|B_1|A_2^*) = 3$. Therefore $(P|B|B_1, A_2^*)$ was a candidate pair at the beginning of the proof. The choice of (P, A) reveals $|A| \leq |A_2^*|$. With the next lemma we obtain a contradiction which completing the proof of the theorem.

Lemma 3.6. *If p is a parallelogram representation of P such that no two parallelograms have a corner in common then $|A_2^*| < |A^*|$.*

Proof. First, note that $C(A_2^*) \subseteq C(A_1^*) \subsetneq C(A^*)$, where the strict inequality is a consequence of the assumption about the corners of the parallelograms.

We now claim that $A_2^* \subset A^*$. Let $x \in A_2^*$. Note that $p_1(x) \subset C(A_1^*) \subset C(A^*)$ and hence also $p(x) \subset C(A^*)$, i.e., $x \in B = A^* \cup A_1$. By definition $x \notin A_1^*$. Therefore, either $x \in A^*$ or x is an isolated element of A_1. If x is an isolated element of A_1, then it is also an isolated element of A_2 and therefore $x \notin A_2^*$. This contradiction proves the claim.

Consider $a_l^1 \in A^*$, the element defining the lower left corner of $C(A^*)$, then $p(a_l^1) \not\subset C(A_1^*)$, hence, $a_l^1 \notin A_2^*$. △

The extension of parallelogram representations to higher semi-order dimensions is obvious, however, we see no analog of Lemma 3.3. in this case. Hence the general open problem remains: Is semi-order dimension a comparability invariance? A second interesting question is whether orders of semi-order dimension two can be recognized efficiently.

References

[Ga] T. Gallai, *Transitiv orientierbare Graphen,* Acta. Math. Acad. sci. Hungar., **18** (1967), 25–66.

[HKM] M. Habib, D. Kelly and R.H. Möhring, *Comparability invariance of geometric notions of order dimension,* Preprint TU-Berlin 320 (1992).

[Ra] I. Rabinovich, *The dimension of semiorders,* Jour. Comb. Th. (A), **25** (1978), 275–279.