Sweeps, Arrangements and Signotopes

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Abstract. Sweeping is an important algorithmic tool in geometry. In the first part of this paper we define sweeps of arrangements and use the 'Sweeping Lemma' to show that Euclidean arrangements of pseudolines can be represented by wiring diagrams and zonotopal tilings.

In the second part we introduce a further representation for Euclidean arrangements of pseudolines. This representation records an 'orientation' for each triple of lines. It turns out that a 'triple orientation' corresponds to an arrangement exactly if it obeys a generalized transitivity law. Moreover, the 'triple orientations' carry a natural order relation which induces an order relation on arrangements. A closer look on the combinatorics behind this leads to a series of signotope orders closely related to higher Bruhat orders. We investigate the structure of higher Bruhat orders and give new purely combinatorial proofs for the main structural properties. Finally, we reconnect the combinatorics of the second part to geometry. In particular we show that maximum chains in the higher Bruhat orders correspond to sweeps.

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1 Introduction

Sweeping is an important algorithmic tool in geometry. In the first part of this paper (Sections 2 and 3) we define sweeps of arrangements and use the 'Sweeping Lemma' to prove representations of Euclidean arrangements by allowable sequences, wiring diagrams (c.f. [11]) and zonotopal tilings (c.f. [26]). We also use the Sweeping Lemma to give a new proof of Levi's Extension Lemma.

In the second part (Section 4) we introduce a further representation for Euclidean arrangements of pseudolines. This representation records an 'orientation' for each triple of lines. It turns out that a 'triple orientation' corresponds to an arrangement exactly if it obeys a generalized transitivity law. Moreover, the 'triple orientations' carry a natural order relation which induces an order relation on arrangements. A closer look on the combinatorics behind this leads to a series of orders $S_r(n)$ whose elements will be called *signotopes.* These orders exist for all pairs (r, n) with $1 \leq r \leq n$. $S_r(n)$ is closely related to the *higher Bruhat order* $B(n, r-1)$ defined by Manin and Schechtman [17] and further studied by Ziegler [26]. We investigate the structure of these orders and give a purely combinatorial proof for the main structural result on higher Bruhat orders: There is a surjective mapping, $C \to \alpha_C$, from maximum chains in $S_{r-1}(n)$ to elements of $S_r(n)$

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In Section 5 the combinatorics of the second part is reconnected to geometry: A signotope $\alpha \in S_r(n)$ represents an arrangement $\mathcal{A}(\alpha)$ of n pseudohyperplanes in $\mathbb{R}^r$ and a maximum chain C in $S_{r-1}(n)$ represents a sweep of the arrangement $\mathcal{A}(\alpha_C)$ in $\mathbb{R}^r$.

Section 6 concludes with a brief collection of open problems.

1.1 Arrangements of Pseudolines

A *pseudoline* is a curve in the Euclidean plane whose removal from the plane leaves two unbounded connected components. In other words: a pseudoline is a simple curve which goes to infinity on both sides. An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection, where the two pseudolines cross. Since in this paper we are not concerned about realizability questions we abbreviate and say arrangement when we really mean arrangement of pseudolines, we also say line instead of pseudoline.

An arrangement is *simple* if no three pseudolines have a common point of intersection. The *order* of an arrangement is the number of its pseudolines. An arrangement partitions the plane into cells of dimensions 0, 1 or 2, the *vertices, edges* and *faces* of the arrangement. Two arrangements are *isomorphic* if there is an isomorphism of the induced cell decompositions respecting the labeling of the lines. Edges and faces of the arrangement may either be bounded or unbounded. Let F be an unbounded cell of arrangement $\mathcal{A}$ and let $\overline{F}$ be the *complementary face of* F, i.e., the face separated from F by all pseudolines. We may orient all pseudolines such that F is in the left halfspace and $\overline{F}$ in the right halfspace of every line. This orientation of pseudolines induces an orientation of the edges of the arrangement. The pair $(\mathcal{A}, F)$ is a *marked arrangement* or an *arrangement with northface F and southface $\overline{F}$*. If there is no explicit reference to the northface of a marked arrangement $\mathcal{A}$ embedded in a coordinatized plane we assume that the northface is the face containing the ray to $(0, \infty)$. Two marked arrangements are *isomorphic* if there is an isomorphism of the induced cell decompositions respecting the labeling and the orientation of the edges. See Figure 1 for an illustration.

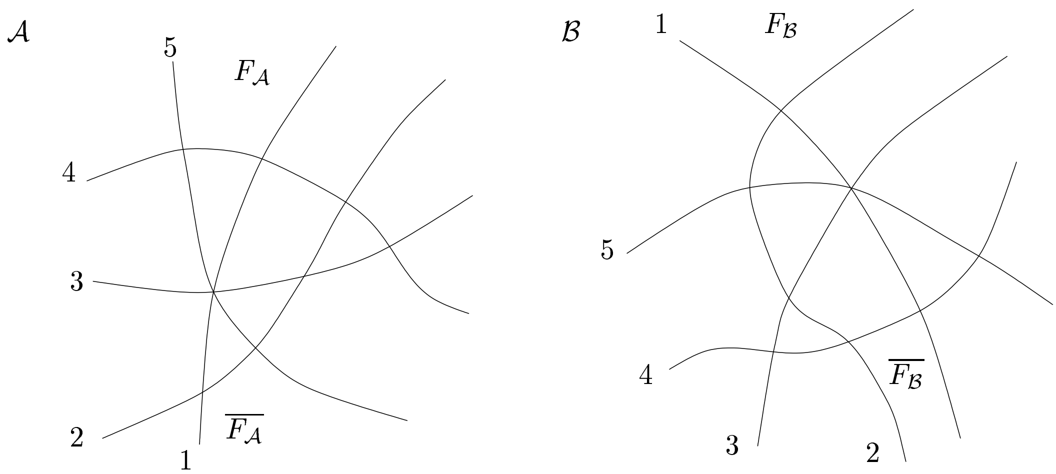


Figure 1: Arrangements $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as arrangements but non-isomorphic as marked arrangements.

2 Sweeping the Plane

Our main tool in proving a number of combinatorial encodings for Euclidean arrangements in Section 3 will be the ability of sweeping the arrangement. In this section we set up this tool, the main result is the *Sweeping Lemma* (Lemma 1) showing that Euclidean arrangements can be swept. This result is not new, we are aware of at least two sources. Snoeyink and Hershberger [22] have a theorem implying the Sweeping Lemma for simple arrangements. In the book on oriented matroids [1] a result equivalent to the Sweeping Lemma is derived as a consequence of Levi's extension lemma. Here we revert the direction and prove Levi's extension lemma (Lemma 6) using sweep techniques.

To begin with we formalize the notion of a sweep. Let $(\mathcal{A}, F)$ be a marked arrangement. A *sweep of $\mathcal{A}$ with northpole in F* is a sequence $c_0, c_1, \ldots c_r$, of curves such that each curve c_i has fixed points $\overline{x} \in \overline{F}$ and $x \in F$ as endpoints. Further requirements are:

(1) None of the curves c_i contains a vertex of arrangement $\mathcal{A}$.

(2) Each curve c_i has exactly one point of intersection with each line l_j.

(3) Besides at their endpoints any two curves c_i and c_j are disjoint.

(4) For any two consecutive curves c_i, c_{i+1} of the sequence there is exactly one vertex of arrangement $\mathcal{A}$ between them, i.e., in the interior of the closed curve $c_i \cup c_{i+1}$.

(5) Every vertex of the arrangement is between a unique pair of consecutive curves, hence, the interior of the closed curve $c_0 \cup c_r$ contains all vertices of $\mathcal{A}$.

See Figure 2 for an example of a sweep for the arrangement $\mathcal{A}$ of Figure 1.

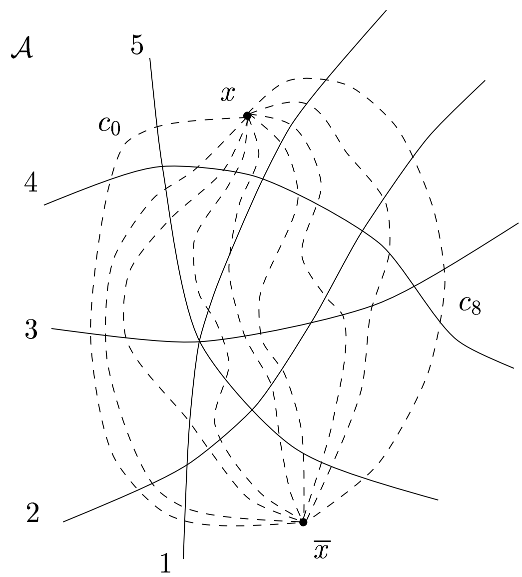


Figure 2: A sweep for arrangement $\mathcal{A}$

Note that if $c_0, \ldots, c_r$ is a sweep for $\mathcal{A}$ then the reversed sequence is also a sweep for $\mathcal{A}$. One of these sweeps is from left to right and the other from right to left. As usual we will always think of a sweep as a left to right sweep. A discrete sweep as defined here can be transformed into a continuous sweep by appropriate interpolation between any pair c_i, c_{i+1} of curves. The dependency on the chosen points x and $\overline{x}$ can also be eliminated.

Lemma 1 (Sweeping Lemma) *Let $(\mathcal{A}, F)$ be a marked Euclidean arrangement of pseudolines. Then there is a sweep sequence of curves for $\mathcal{A}$, i.e., $\mathcal{A}$ can be swept.*

Proof. Let $G = (V, E)$ be the graph such that the vertices V of G are the vertices of $\mathcal{A}$ and the edges of G are the finite edges of the arrangement $\mathcal{A}$. Let $\overrightarrow{E}$ be the orientation of the edges of G induced by the orientation of pseudolines (the northface is in the left halfplane of each pseudoline).

Claim A. The orientation $\overrightarrow{E}$ is an acyclic orientation of G.

Walking 'at infinity' and clockwise from $\overline{F}$ to F the pseudolines of $\mathcal{A}$ are met in some order. Let permutation π be the corresponding order of the labels, w.l.o.g. we assume that π is the identity.

We prove the above claim by contradiction: Assuming that $\overrightarrow{E}$ is not acyclic we choose a cycle C such that the area enclosed by the corresponding curve in $\mathcal{A}$ is minimal. It is easy to conclude that C corresponds to the boundary of a face of $\mathcal{A}$. With respect to this face the cycle C may be oriented clockwise or counterclockwise. We consider the first case (clockwise) the other is symmetric.

Let $e_1, e_2, \ldots, e_k$ be edges of C and let i_j be the supporting pseudoline of e_j. Since e_j and e_{j+1} are consecutive on C the lines i_j and i_{j+1} cross at a vertex of C. From the definition of π and the clockwise orientation of C it follows that $i_j < i_{j+1}$ (see Figure 3). Hence $i_1 < i_2 < \ldots < i_k < i_1$ a contradiction. △

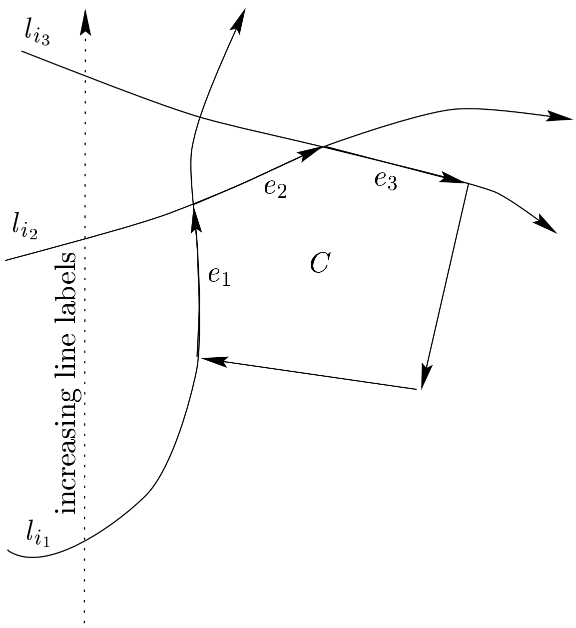


Figure 3: Assuming an oriented cycle.

Since $\overrightarrow{G} = (V, \overrightarrow{E})$ is acyclic there exists a topological sorting $v_1, v_2, \ldots, v_r$ of $\overrightarrow{G}$. Fix points $x \in F$ and $\overline{x} \in \overline{F}$.

Claim B. There exists a sweep of curves $c_0, c_1, \ldots, c_r$ such that vertices $v_1, \ldots, v_i$ are to the left of c_i and vertices $v_{i+1}, \ldots, v_r$ are to the right of c_i for all $i = 1, \ldots, r$.

Proof. Let R be the union of the closed bounded cells of $\mathcal{A}$. Define c_0 as the union of three curves. The first and the second connect x to R within F and $\overline{x}$ to R within $\overline{F}$, the third is the left boundary of an ϵ-tube of the left boundary of R and connected to the two other curves. For an appropriate ϵ this gives a curve as desired.

Now suppose that c_{i-1}, $i \leq r$, has been defined. Let $i_1, \ldots, i_t$ be the lines of $\mathcal{A}$ containing vertex v_i and assume $i_1 < \ldots < i_t$. Let T be the triangle defined by curve c_{i-1} and the two lines i_1 and i_t. Since v_i is a source (minimal) in the restriction of $\overrightarrow{G}$ to $v_i, \ldots, v_r$ and $v_1, \ldots, v_{i-1}$ are left of c_{i-1} vertex v_i is the unique vertex of $\mathcal{A}$ in the triangular region T. Define c_i as the right boundary of an ϵ-tube around c_{i-1} and T. For an appropriate ϵ this gives a curve as desired, see Figure 4. $\triangle$

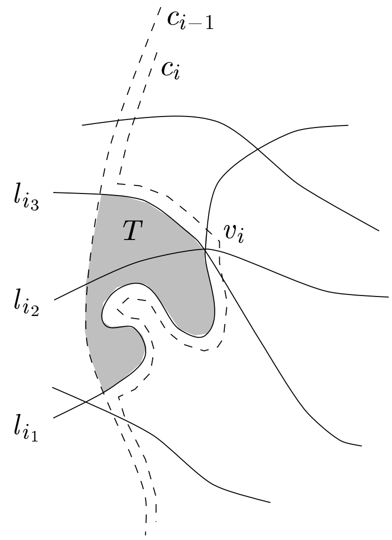


Figure 4: Defining c_i based on c_{i-1} and the shaded triangular region T.

This concludes the proof of the lemma. $\square$

3 Applications of Sweeping

In combinatorial geometry it is often useful to encode a geometric object by a combinatorial structure and further work with this structure. There are several combinatorial encodings for arrangements. In the first part of this section we review *allowable sequences* and *wiring diagrams*. These representations have been introduced by Goodman and Pollack [11]. The same authors [12] give an overview on work related to allowable sequences and mention some applications. There are two reasons to including a complete treatment of this subject here. The relation between allowable sequences and arrangements of pseudolines is a special case of a more general phenomenon in the theory of signotopes (higher Bruhat orders) which will be the topic of Subsection 4.4, Theorem 13. Furthermore we believe that sweeps are the natural approach to these representations.

In Subsection 3.2 we prove the equivalence between arrangements and *zonotopal tilings*. This is a special (rank 3) case of the celebrated Bohne-Dress Theorem which states a bijection between zonotopal tilings and oriented matroid liftings. No elementary proof of the special case was known. Recently we learned that Elnitsky [4] found another simple proof for the correspondence. We will make use of zonotopal tilings in our figures since they provide us with canonical pictural representations of arrangements, see e.g. Figure 8.

Further sources for encodings of arrangements are Goodman and Pollack [12], Edelsbrunner [3], Felsner [5] and Knuth [15].

In the last application we use the sweep technique to prove Levi's extension lemma.

3.1 Allowable Sequences and Wiring Diagrams

Let $c_0, c_1, \ldots, c_r$ be a sweep sequence of curves for the marked arrangement $(\mathcal{A}, F)$ of order n. Traversing curve c_i from $\overline{x}$ to x we meet the lines of $\mathcal{A}$ in some order. Since each line is met by c_i exactly once the order of the crossings corresponds to a permutation π_i of $[n]$.

Consider the labels of lines crossing at vertex v_i. Since the region T defined in the proof of Claim B is empty of vertices of $\mathcal{A}$ and by property 2 of the sweep curve c_i the lines $i_1, \ldots, i_t$ containing vertex v_i are a consecutive substring of π_{i-1}. Moreover, in permutation π_{i-1} these lines are in the reversed order and this is the only difference between π_{i-1} and π_i. Relabeling the lines of $\mathcal{A}$ appropriately we may assume that π_0 is the identity permutation.

Example A. The sequence of permutations obtained from the sweep of Figure 2 is $(1,2,3,4,5) \overset{4,5}{\to} (1,2,3,5,4) \overset{1,2}{\to} (2,1,3,5,4) \overset{1,3,5}{\to} (2,5,3,1,4) \overset{2,5}{\to} (5,2,3,1,4) \overset{1,4}{\to}$ $(5,2,3,4,1) \overset{2,3}{\to} (5,3,2,4,1) \overset{2,4}{\to} (5,3,4,2,1) \overset{3,4}{\to} (5,4,3,2,1)$.

The sequence $\pi_0, \ldots, \pi_r$ has the following properties:

(1) π_0 is the identity permutation and π_r is the reverse permutation on $[n]$.

(2) Each permutation π_i, $1 \le i \le r$ is obtained by the reversal of a consecutive substring M_i from the preceding permutation π_{i-1}.

(3) Any two elements $x, y \in [n]$ are joint members of exactly one move M_i, i.e., reverse their order exactly once.

A sequence $\Sigma = \pi_0, \ldots, \pi_r$ of permutations with properties (1), (2) and (3) is called an *allowable sequence of permutations*. If each move from π_{i-1} to π_i consists in the reversal of just one pair of elements, i.e., an adjacent transposition, we have $r = \binom{n}{2}$ and the sequence Σ is called a *simple allowable sequence*. We have thus seen how to obtain an allowable sequence of permutations from every marked arrangement $(\mathcal{A}, F)$. However, more can be said:

Every topological sorting of the graph $\overrightarrow{G}$ of $(\mathcal{A}, F)$ induces an allowable sequence. Consider the allowable sequences Σ and Σ' corresponding to topological sortings σ and σ' of $\overrightarrow{G}$ with the property that $\sigma = v_1, \ldots, v_i, v_{i+1}, \ldots, v_r$ and $\sigma' = v_1, \ldots, v_{i+1}, v_i, \ldots, v_r$, i.e., σ and σ' differ in an adjacent transposition. It follows that v_i and v_{i+1} are both minimal elements in the restriction of $\overrightarrow{G}$ to $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_r\}$. Hence, there is no line in $\mathcal{A}$ that contains vertices v_i and v_{i+1} and the labels of lines involved in the moves $M_i : \pi_{i-1} \to \pi_i$ and $M_{i+1} : \pi_i \to \pi_{i+1}$ in Σ are disjoint. In fact for $j \ne i, i+1$ the permutations π_j and π'_j in Σ and Σ' coincide and $M'_i = M_{i+1}$ and $M'_{i+1} = M_i$. Call two allowable sequences Σ and Σ' *elementary equivalent* if Σ can be transformed into Σ' by interchanging two disjoint adjacent moves. Two allowable sequences Σ and Σ' are called *equivalent* if there exists a sequence $\Sigma = \Sigma_1, \Sigma_2, \ldots, \Sigma_m = \Sigma'$ such that Σ_i and Σ_{i+1} are elementary equivalent for $1 \leq i < m$. It is well known that it is possible to transform any topological sorting of a directed acyclic graph $\overrightarrow{G}$ into any other by a sequence of adjacent transpositions, i.e., reversals of adjacent pairs of unrelated vertices. Therefore, any two allowable sequences corresponding to the same marked arrangement $(\mathcal{A}, F)$ are equivalent.

Theorem 2 *There is a bijection between equivalence classes of allowable sequences and marked arrangements of pseudolines. Moreover, this bijection maps simple allowable sequences to simple arrangements.*

Proof. We have already seen how to define the equivalence class of allowable sequences corresponding to a marked arrangement.

Let Σ be an allowable sequence. Start drawing n horizontal lines called *wires* and vertical lines $p_0, \ldots, p_r$. Label the crossing of the ith wire from below with p_j with the label $p_j(i)$. Draw pseudoline i such that it interpolates the crossings with its label as in Figure 5.

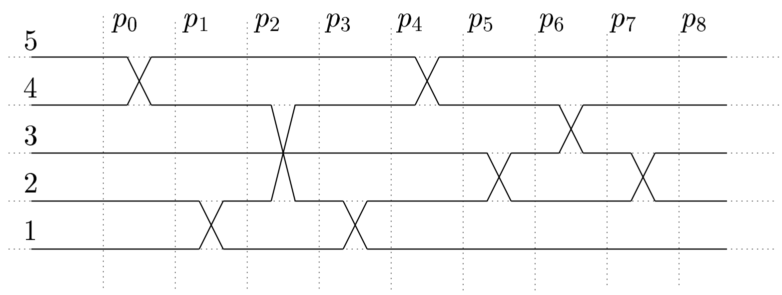


Figure 5: A wiring diagram for the arrangement of Figure 2

Following Goodman [9] we call the arrangement thus obtained a *wiring diagram* for Σ. Since the vertical lines $p_0, \ldots, p_r$ essentially are a sweep sequence of curves for the wiring diagram we see that the mapping from arrangements to allowable sequences is surjective. Let $(\mathcal{A}, F)$ be any marked arrangements $(\mathcal{A}, F)$ such that Σ corresponds to a sweep of $c_0, \ldots, c_r$ of $\mathcal{A}$. It is obvious that the part of $\mathcal{A}$ between c_{i-1} and c_i is isomorphic to the part of the wiring diagram between p_{i-1} and p_i. These isomorphisms for $i = 1, .., r$ can be glued together to an isomorphism of the arrangements. This proves injectivity and hence the first part of the theorem.

The second part of the theorem is obvious. $\square$

It is interesting to ask for the change in the representation when the northface is changed. Let $(\mathcal{A}, F)$ be a marked arrangement and redefine the northface to be the unbounded 2-cell F' to the left of F. Cells F and F' are separated by line n. The directed graph $\overrightarrow{G}'$ is obtained from $\overrightarrow{G}$ by reverting the orientations of all edges with supporting line n. Now choose a topological sorting σ for $\overrightarrow{G}$ such that all vertices of $\mathcal{A}$ which are right or (below) line n precede the vertices on n and all vertices left or (above) line n

come later. Let $v_1, \ldots, v_{i-1}$, be the left block of σ, $v_i, \ldots, v_{j-1}$ be the middle block, i.e., the ordered sequence of vertices on line n, and $v_j, \ldots, v_r$ be the right block. It follows that $v_1, \ldots, v_{i-1}, v_{j-1}, \ldots, v_i, v_j, \ldots, v_r$ is a topological sorting of $\overrightarrow{G}'$. Note that the order in which the lines enter v_k for $i \leq k \leq j$ has also changed, in $\overrightarrow{G}$ line n was the highest line entering v_k and in $\overrightarrow{G}'$ line n is the lowest line entering v_k. Hence, from the allowable sequence Σ of $(\mathcal{A}, F)$ with moves $M_1, \ldots, M_r$ corresponding to $v_1, \ldots, v_r$ we obtain a sequence Σ_0' with moves $M_1, \ldots, M_{i-1}, M_{j-1}^*, \ldots, M_i^*, M_j, \ldots, M_r$, where M_k^* is obtained from M_k by moving element n from the top to the bottom. An allowable sequence Σ' for $(\mathcal{A}, F')$ is obtained from Σ_0' by relabeling $n \to 1 \to 2 \to \ldots \to n-1 \to n$.

We briefly mention another representation for marked arrangements where the change from the representation of $(\mathcal{A}, F)$ to the representation $(\mathcal{A}, F')$ is more transparent. Let α_i be the permutation of $\{1, .., n\} \setminus i$ reporting the order from left to right in which the other pseudolines cross line i, for $i = 1, .., n$. Goodman and Pollack [11] call this the *local sequences of unordered switches* of the arrangement. Felsner [5] used sweeps to show that local sequences are a representation for marked arrangements. In case of non-simple arrangements local sequences are slightly more general structures than permutations since several lines can cross line i in the same point. For the arrangement of Figure 2 the local sequences are $\alpha_1 = [2, \{3,5\}, 4]$, $\alpha_2 = [1,5,3,4]$, $\alpha_3 = [\{1,5\}, 2, 4]$, $\alpha_4 = [5,1,2,3]$ and $\alpha_5 = [4, \{1,3\}, 2]$. To change from the local sequences of $(\mathcal{A}, F)$ to those of $(\mathcal{A}, F')$ we revert sequence α_n and relabel $n \to 1 \to 2 \to \ldots \to n-1 \to n$ as before. In Section 4 Theorem 8 we characterize those $(\alpha_i)_{i=1..n}$ corresponding to simple marked arrangements.

3.2 Zonotopal Tilings

A particularly nice representation of arrangements of pseudolines is the representation by 'zonotopal tilings'. Basically this is a standardized drawing of the 'dual graph' of the arrangement. Figure 6 should make the connection clear. Below, in Theorem 3 we prove a bijection between zonotopal tilings and arrangements.

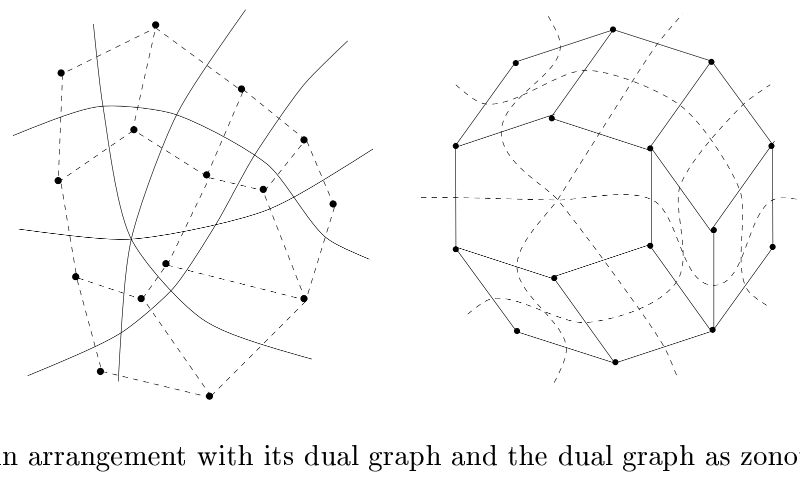


Figure 6: An arrangement with its dual graph and the dual graph as zonotopal tiling.

A 2-dimensional *zonotope* is a centrally symmetric $2n$-gon, or equivalently the Minkowski sum of a set of line segments in $\mathbb{R}^2$. With a vector v_i we associate the line segment $[-v_i, +v_i]$. The Minkowski sum of the line segments corresponding to $V = \{v_1, \ldots, v_n\}$ is the set

$$Z(V) = \left\{ \sum_{i=1}^{n} c_i v_i : -1 \leq c_i \leq 1 \text{ for all } 1 \leq i \leq n \right\}.$$

A *zonotopal tiling* $\mathcal{T}$ is a tiling of $Z(V)$ by translates of zonotopes $Z(V_i)$ with $V_i \subset V$. A zonotopal tiling is a *simple zonotopal tiling* if all tiles are rhombi, i.e., $|V_i| = 2$ for all i. A zonotopal tiling together with a distinguished vertex x of the boundary of $Z(V)$ is a *marked zonotopal tiling.* The next theorem is a precise statement for the correspondence suggested by Figure 6. The proof of the theorem is based on a Sweeping Lemma for zonotopal tilings, Lemma 4.

Theorem 3 *Let V be a set of n pairwise non-collinear vectors in $\mathbb{R}^2$.*

(1) *There is a bijection between marked zonotopal tilings of $Z(V)$ and marked arrangements of order n.*

(2) *Via this bijection simple tilings correspond to simple arrangements.*

Before going into the proof let us comment on the broader context Theorem 3. the theorem is equivalent to the rank 3 version of the Bohne-Dress Theorem which gives a bijection between zonotopal tilings of d-dimensional zonotopes and oriented matroids of rank $d+1$ with a realizable one-element contraction. The correspondence between oriented matroids and arrangements is given by the representation theorem for oriented matroids. This theorem states that oriented matroids of rank $d+1$ are in bijection with arrangements of pseudohyperplanes in d-dimensiononal projective space. An accessible treatment of these connections is given by Zigler [27]. A more geometric proof of the Bohne-Dress Theorem was given by Richter-Gebert and Ziegler [20].

In the first part of the proof we give the mapping from zonotopal tilings to equivalence classes of allowable sequences. Let $Z(V)$ be a marked zonotope with V a set of n pairwise non-collinear vectors. The zonotope $Z = Z(V)$ is a centrally symmetric $2n$-gon. Rotate Z such that the distinguished vertex x is the unique highest vertex of Z, in particular the boundary of Z has no horizontal edge. Assume that the vectors in V are labeled such that along the left boundary of Z, i.e., on the left path from the lowest vertex $\overline{x}$ to x, the segments correspond to $v_1, v_2, \ldots, v_n$ in this order.

Given a zonotopal tiling $\mathcal{T}$ consider the set of y-monotone path along segments of $\mathcal{T}$ from $\overline{x}$ to x. We define a *sweep of $\mathcal{T}$ with northpole x* as a sequence $p_0, p_1, \ldots, p_r$ of y-monotone path from $\overline{x}$ to x in $\mathcal{T}$ with the following properties.

(1) Any two consecutive paths p_i, p_{i+1} of the sequence have exactly one tile T_i of tiling $\mathcal{T}$ between them, i.e., in the interior of the closed curve $p_i \cup p_{i+1}$.

(2) Every tile is between a unique pair of consecutive paths, therefore, $p_0 \cup p_r$ is the boundary of $Z(V)$.

As we did for sweeps of arrangements we further assume that the sweep of $\mathcal{T}$ is from left to right, i.e., p_0 is the left boundary of $Z(V)$.

Remark. There is some interest in the maximum number $m(n)$ of y-monotone $\overline{x}$ to x path a marked zonotopal tiling can have. Knuth [15, page 39] conjectures that $m(n) \leq n2^{n-2}$. Via an inductive argument this would imply that the number of marked arrangements of

n pseudolines is bounded by $\Pi_{k=1}^{n} m(k)$. Therefore, the conjectured bound would show that this number is at most $2^{n^2/2+o(n^2)}$ which would be an improvement over the currently best upper bound $2^{0.69n^2}$, Felsner [5].

A sweep of tiling $\mathcal{T}$ induces a total order $T_1, T_2, \ldots, T_r$ on the tiles of $\mathcal{T}$ with the property that after removing the tiles of any initial segment $T_1, \ldots, T_{i-1}$ tile T_i can be separated from the remaining tiles $T_{i+1}, \ldots, T_r$ by a translation to the left parallel to the x-axis, we call this the *separation property.* Conversely, an order $T_1, T_2, \ldots, T_r$ of the tiles with the separation property corresponds to a sweep: Define path p_i as the right boundary of the union of $T_1, \ldots, T_i$. To proof that a zonotopal tiling $\mathcal{T}$ can be swept it is therefore sufficient to show that there is a total order of the tiles with the separation property.

Guibas and Yao [14] observed that given any set $C_1, C_2, \ldots, C_r$ of disjoint convex objects in the plane there is at least one object C_i that can be translated to the left parallel to the x-axis without ever colliding with another object from the set. Hence, by induction every set of disjoint convex objects admits a total ordering $C_1, C_2, \ldots, C_r$ with the separation property, i.e., for $i = 1, .., r$ given the sets $C_i, \ldots, C_r$ we can separate C_i from the remaining sets by a translation to the left parallel to the x-axis. As a special case we obtain:

Lemma 4 *Every marked zonotopal tiling $\mathcal{T}$ can be swept.*

Define a graph $G = (V, E)$ such that the vertices V of G are the tiles of $\mathcal{T}$ and the edges of G are pairs of tiles sharing a common segment. Let $\overrightarrow{E}$ be an orientation of the edges of G such that an edge $\{T, T'\}$ of G points from the tile on the left side of the segment $T \cap T'$ to the tile on the right side. Since the boundary of Z consists entirely of non horizontal edges this orientation is well defined. The orientation of the edges of G represents the 'immediate blocking relation' with respect to translations parallel to the x-axis. From Lemma 4 we obtain:

Fact A. The orientation $\overrightarrow{E}$ is an acyclic orientation of G.

From the correspondence between marked zonotopal tilings and marked arrangements indicated in Figure 6 we see that we met graph G and its orientation already in the proof of Lemma 1. A formal proof of this 'obvious fact' will be implicit in the next lemma. For later use we note:

Fact B. Every topological sorting of $\overrightarrow{G}$ has the separation property.

The next lemma is the 'zonotopal equivalent' of Theorem 2.

Lemma 5 *There is a bijection between marked zonotopal tilings and equivalence classes of allowable sequences. Moreover, this bijection maps simple zonotopal tilings to classes of simple allowable sequences.*

Proof. First we show how to associate an allowable sequence to every sweep of a zonotopal tiling. Recall that sweeps of $\mathcal{T}$ correspond to topological sortings of $\overrightarrow{G}$. Given a sweep sequence $p_0, \ldots, p_r$ of paths we associate to each path p_i a sequence π_i recording the labels of the vectors which define the segments along the path in the order of the path from $\overline{x}$ to x. The sequence π_0 is a permutation, the identity. Any two consecutive sequences π_i and π_{i+1} only differ in a substring where path p_i takes the left boundary and path p_{i+1}

takes the right boundary of tile T_i. Since T_i is a zonotope the same labels appear on both boundaries but in reversed order. Hence, all π_i are permutations, moreover, $\pi_i \to \pi_{i+1}$ is a move as in part (2) of the definition of allowable sequences. We also note that π_r is the reverse permutation.

It remains to prove property (3) of allowable sequences, namely, that any two elements $a, b \in [n]$ are reversed in exactly one move. This is shown by an argument involving volumes. Due to a formula of McMullen (see Shephard [21, Prop. 2.2.12]) the volume of a 2-dimensional zonotope $Z(v_1, \ldots, v_n)$ is given as follows

$$\mathrm{vol}(Z(v_1, \ldots, v_n)) = \sum_{i<j} \mathrm{vol}(Z(v_i, v_j) = \sum_{i<j} 4|\det(v_i, v_j)|.$$

A move reverting $i_1 < i_2 < .. < i_s$ corresponds to a tile $T = Z(v_{i_1}, .., v_{i_s})$ of volume $\sum_{i_j < i_k} 4|\det(v_{i_j}, v_{i_k})|$. Each pair has to be reversed at least once and this exhausts the volume of the zonotope $Z(V)$. Hence there can be no additional reversals and property (3) is established.

Now we have to show that the set of sweeps of $\mathcal{T}$ maps to an equivalence class of allowable sequences. From Fact B we already know that the sweeps of $\mathcal{T}$ are in one-to-one correspondence with topological sortings of $\overrightarrow{G}$.

Consider topological sortings σ and σ' of $\overrightarrow{G}$ which only differ in an adjacent transposition and let Σ and Σ' be the two corresponding allowable sequences. From $\sigma = T_1, \ldots, T_i, T_{i+1}, \ldots, T_r$ and $\sigma' = T_1, \ldots, T_{i+1}, T_i, \ldots, T_r$ it follows that the tiles T_i and T_{i+1} are both minimal elements in the restriction of $\overrightarrow{G}$ to $\{T_i, T_{i+1}, T_{i+2}, \ldots, T_r\}$. Hence there is no horizontal line intersecting both of them. From the y-monotonicity of p_{i-1} and the fact that π_{i-1} is a permutation we conclude that $V_i \cap V_{i+1} = \emptyset$ when $T_i = Z(V_i)$ and $T_{i+1} = Z(V_{i+1})$. This shows that the moves $M_i : \pi_{i-1} \to \pi_i$ and $M_{i+1} : \pi_i \to \pi_{i+1}$ in Σ are disjoint and hence Σ and Σ' are elementary equivalent. The argument can be read backwards to show that if Σ and Σ' are elementary equivalent allowable sequences and Σ corresponds to a topological sorting of $\overrightarrow{G}$ then so does Σ'.

For the inverse mapping we have to associate a marked zonotopal tiling to an equivalence class of allowable sequences. Build the tiling from left to right starting with the left boundary of $Z(V)$. After placing i tiles, three properties remain invariant:

(1) The union of the already placed tiles together with the left boundary of Z is a simply connected region.

(2) The right boundary of this region is a y-monotone path p_i.

(3) The segments along path p_i are in the order given by π_i.

From this it is obvious that we can place the tile T_{i+1} corresponding to move M_{i+1} such that the invariant remains valid. Since the last permutation π_r is the reverse of the identity path p_r is the right boundary of $Z(V)$. The volume formula implies that the tiles have been placed without overlap. Therefore, the placement of tiles $T_1, \ldots, T_r$ is a tiling $\mathcal{T}$ of $Z(V)$.

It is easily seen that equivalent allowable sequences lead to the same tiling while non-equivalent allowable sequences produce different tilings. □

Theorem 3 is now easily obtained:

proof (Theorem 3). Statement (1) is a direct consequence of Theorem 2 and Lemma 5. Combining the two bijections it is seen that the graph of edges of the marked zonotopal tiling corresponding is the dual of the graph of the corresponding marked arrangement with the marked face F of the arrangement and the marked vertex x of the tiling dually corresponding to each other. For statement (2) we additionally note that an arrangement is simple exactly if all bounded regions of the dual graph are quadrangles. $\square$

Remark. Richter-Gebert and Ziegler [20] use a similar volume argument in their proof of the Bohne-Dress Theorem. A proof of Theorem 3 avoiding the volume argument was recently given by Elnitsky [4] in the context of reduced decompositions.

3.3 Levi's Extension Lemma

Lemma 6 *Let $\mathcal{A}$ be an arrangement of order n and let p, q be two points in the plane which do not both lie on any of the lines of $\mathcal{A}$. Then there is a pseudoline c containing p and q such that $\mathcal{A} \cup c$ is an arrangement of order $n+1$.*

The original source for the lemma stated for projective arrangements is Levi [16], an English transcription is found in Grünbaum [13]. A proof using a variant of sweeps, namely cyclic sweeps, was given by Snoeyink and Hershberger [22]. Here we use the projective space as auxiliary tool.

Proof. We detail the proof for the case where p and q are not incident to a line of $\mathcal{A}$ and leave the obvious modifications to include special cases to the interested reader.

Let p be contained in face F_p of $\mathcal{A}$. Let $l_1, \ldots, l_n$ be the pseudolines of $\mathcal{A}$ and without loss of generality let l_1 contain an edge e of the boundary of F_p. Add the line at infinity l_∞ to the arrangement and map it back to Euclidean space such that l_1 is the line at infinity thus obtaining an arrangement $\mathcal{A}'$ with lines $l_\infty, l_2, \ldots, l_n$. Mark $\mathcal{A}'$ such that $p \in F_p$ is the northpole. Apply the Sweeping Lemma to find a curve c crossing the face F_q containing q. Line c can be bent in F_q to make q a point on c. Extending c from p to infinity we see that $\mathcal{A}' \cup c$ is an arrangement of order $n+1$. Adding the line at infinity, i.e., l_1 we obtain a projective arrangement of order $n+2$ which is mapped back to the Euclidean plane using l_∞ as line at infinity. This gives an arrangement of lines $l_1, \ldots, l_n, c$ with both points p and q on line c. $\square$

It is notable that higher dimensional analogs of the Extension Lemma fail. Examples can be given of arrangements of pseudoplanes in three-space such that for some triples of points p, q, r no pseudoplane can be added to extend the arrangement and contain the three points (see Goodman and Pollack [10]). Richter-Gebert [19] has constructed examples showing that the above non-existence result is already true for two points instead of three.

4 Flips, Arrangements and Signotopes

In the first part of the paper we have studied arrangements of pseudolines as individual objects. In this part we will change the focus and consider the set of all arrangements. More precisely we consider a graph $\mathcal{G}_n$ whose vertices are all combinatorially different simple marked arrangements of n pseudolines in the Euclidean plane and edges corresponding to elementary *flips* (see Figure 7), i.e., arrangements $\mathcal{A}$ and $\mathcal{B}$ are adjacent if they only differ in the orientation of a single triangle. Figure 8 shows the graph $\mathcal{G}_n$ for $n = 5$ with the arrangements represented by zonotopal tilings.

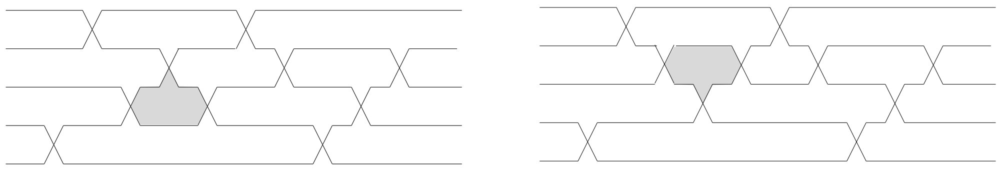


Figure 7: Elementary flip at the shaded triangle.

In Subsection 4.1 we introduce an encoding of arrangements by triangle orientations. This encoding imposes a natural orientation on $\mathcal{G}_n$. In Subsection 4.2 we generalize the patterns and define an order $S_r(n)$, for all $1 \leq r \leq n$, such that $S_1(n)$ is the Boolean lattice, $S_2(n)$ is the weak Bruhat order of the symmetric group and $S_3(n)$ is the abovementioned orientation of $\mathcal{G}_n$. The elements of $S_r(n)$ will be called signotopes. Subsection 4.3 gives some constructions for new signotopes from old ones. The main structural result about signotopes is the surjective mapping from maximum chains in $S_{r-1}(n)$ to the elements of $S_r(n)$, this result is derived in Subsection 4.4. Note that we have already seen a special case of this mapping in Theorem 2: Maximum chains in the weak Bruhat order $S_2(n)$ are simple allowable sequences and elements of $S_3(n)$ are marked simple arrangements of pseudolines.

4.1 Encoding arrangements by triangle orientations

Flips are nicely described in the different encodings of arrangements. In the encoding by zonotopal tilings the projection of a cube is replaced by the view of the cube from the other side. In the encoding by local sequences an adjacent transposition of elements i and j is applied to the local sequence α_k of line k and similarly to local sequences α_i and α_j when the flip-triangle is confined by lines i, j and k.

In the representation by allowable sequences the transformation is not that obvious. The change is easy to describe if we recall that the allowable sequences of a marked arrangement $(\mathcal{A}, F)$ correspond to topological sortings of a directed graph $\overrightarrow{G}$. The change on $\overrightarrow{G}$ is again a local one.

We now introduce a further representation for simple marked arrangements of pseudolines. Let $(\mathcal{A}, F)$ be such an arrangement of n pseudolines. Consider the arrangement induced by a triple of $\{i, j, k\}$ of lines of $\mathcal{A}$, where we assume $i < j < k$. Note that these three lines can induce two combinatorial different arrangements. Either the crossing of lines i and k is above line j, denote this by the symbol $-$ or the crossing is below line j, denoted by $+$. The shaded triangles of Figure 7 are a $-$ triangle on the left side and a $+$ triangle on the right side. With this convention a marked simple arrangement induces a *triangle-sign function* $f : \binom{[n]}{3} \to \{-,+\}$.

Note that for $i < j$ and all $k \neq i, j$ we have $f(\{i, j, k\}) = -$ iff on line k, the crossing with line i precedes the crossing with line j, i.e., on the local sequence α_k the pair (i, j) is a non-inversion. Since local sequences encode marked arrangements, i.e., arrangements with the same local sequences are isomorphic, it follows that the above defined sign patterns $f : \binom{[n]}{3} \to \{-,+\}$ also encode marked simple arrangements of pseudolines.

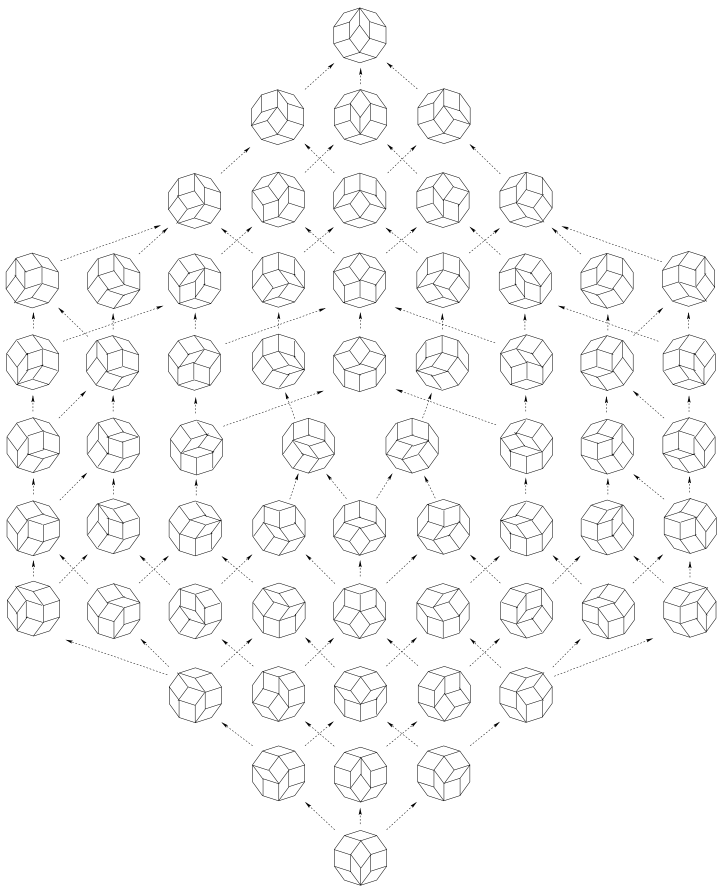


Figure 8: The graph $\mathcal{G}_5$ as diagram of the signotope order $S_3(n)$.

Clearly not every possible sign pattern $f : \binom{[n]}{3} \to \{-,+\}$ will correspond to an arrangement, there are simply too many such functions. Below we derive an obvious necessary condition on the sign patterns of arrangements. Later it will be shown that this necessary condition is already sufficient.

Consider a quadruple of pseudolines h, i, j, k of $\mathcal{A}$. These lines induce a marked arrangement of four pseudolines. Since there is only one (unmarked) arrangement of four lines with eight unbounded faces we easily enumerate the eight possible patterns of triangle-sign functions for $n = 4$. The following list shows them, the signs are given in lexicographical order of the three-sets, i.e, as { sign(1,2,3), sign(1,2,4), sign(1,3,4), sign(2,3,4) }.

$$\{-,-,-,-\},\ \{+,-,-,-\},\ \{+,+,-,-\},\ \{+,+,+,-\},$$
$$\{-,-,-,+\},\ \{-,-,+,+\},\ \{-,+,+,+\},\ \{+,+,+,+\}$$

From this we obtain a necessary condition for the functions f induced by an arrangement. For $A \in \binom{[n]}{4}$ and $1 \le i \le 4$ we let $A^{\lfloor i \rfloor}$ denote the set A minus the ith largest element of A, e.g., $\{2,4,5,9\}^{\lfloor 3 \rfloor} = \{2,4,9\}$. If f corresponds to an arrangement $\mathcal{A}$ then the restriction of $\mathcal{A}$ to the four lines of A has a pattern $\{\, \operatorname{sign} A^{\lfloor 4 \rfloor}, \operatorname{sign} A^{\lfloor 3 \rfloor}, \operatorname{sign} A^{\lfloor 2 \rfloor}, \operatorname{sign} A^{\lfloor 1 \rfloor} \,\}$ from the above list. Order the set $\{-,+\}$ of signs by $- \prec +$. Inspecting the above enumeration we see that the legal sign patterns are characterized by the following property: For every

4 element subset P of $[n]$ and all $1 \leq i < j < k \leq 4$ either $f(P^{\lfloor i \rfloor}) \preceq f(P^{\lfloor j \rfloor}) \preceq f(P^{\lfloor k \rfloor})$ or $f(P^{\lfloor i \rfloor}) \succeq f(P^{\lfloor j \rfloor}) \succeq f(P^{\lfloor k \rfloor})$. This property is called *monotonicity.*

Theorem 7, whose proof will be given in the next section, shows that monotonicity already characterizes the sign patterns $f : \binom{[n]}{3} \to \{-,+\}$ encoding arrangements.

Theorem 7 *A function $f : \binom{[n]}{3} \to \{-,+\}$ is the triangle-sign function of a marked simple arrangements $\mathcal{A}_f$ of order n if and only if f is monotone on all 4-element subsets of $[n]$.*

It is a useful exercise to verify that monotonicity of the triangle-sign function induced by an arrangement is equivalent to the transitivity of non-inversions and of inversions of the local sequences α_k, hence, equivalent to α_k being a permutation. Combining these remarks with Theorem 7 we obtain:

Theorem 8 *A set $(\alpha_i)_{i=1..n}$ with α_i a permutation of $[n] \setminus \{i\}$ is the set of local sequences of a simple marked arrangement of order n if and only if for all $i < j < k$ the pairs $(i,j),(i,k),(j,k)$ are inversions in $\alpha_k, \alpha_j, \alpha_i$ or they are all three non-inversions.*

An equivalent characterization theorem has been obtained by Streinu [24] in the context of generalized configurations of points.

4.2 Signotopes and their Orders

In this section we generalize the concept of triangle-sign functions. Recall some notations. The set $[n] = \{1,..,n\}$ is equipped with the natural linear order. The set of r element subsets of $[n]$ is $\binom{[n]}{r}$. For $A \in \binom{[n]}{r}$ with $r \geq i$ we let $A^{\lfloor i \rfloor}$ denote the set A minus the ith largest element of A. The set $\{-,+\}$ of signs is ordered by $- \prec +$.

Definition 1 *For integers $1 \leq r \leq n$ a r–*signotope *on $[n]$ is a function α from the r elements subsets of $[n]$ to $\{-,+\}$ such that for every $r+1$ element subset P of $[n]$ and all $1 \leq i < j < k \leq r+1$ either $\alpha(P^{\lfloor i \rfloor}) \preceq \alpha(P^{\lfloor j \rfloor}) \preceq \alpha(P^{\lfloor k \rfloor})$ or $\alpha(P^{\lfloor i \rfloor}) \succeq \alpha(P^{\lfloor j \rfloor}) \succeq \alpha(P^{\lfloor k \rfloor})$. We refer to this property as* monotonicity.

*Let $S_r(n)$ denote the set of all r-signotopes on $[n]$ equipped with the order relation $\alpha \leq \beta$ if $\alpha(A) \preceq \beta(A)$ for all $A \in \binom{[n]}{r}$. Call $S_r(n)$ the r–*signotope order.

Note that for $r = 3$ the definitions reflect our observations for the encodings of marked simple arrangements of pseudolines made in the previous section. In particular Theorem 7 implies that $S_3(n)$ is a partial order on the set of marked arrangements of n pseudolines. Indeed $S_3(n)$ is an orientation of the graph $\mathcal{G}_n$, see Figure 8.
The list below collects some other special cases and easy observations.

(1) For $r = 1$ monotonicity is vacuous and $S_1(n)$ is just the lattice of subsets of $[n]$.

(2) For all $n \geq r \geq 1$ there is a unique minimal and a unique maximal element in $S_r(n)$, namely the constant $-$ and the constant $+$ function.

(3) The diagram of $S_r(r+1)$ is a $(2r+2)$-gon for all $r \geq 1$.

(4) There is a natural correspondence between 2-signotopes on $[n]$ and permutations of n. Permutation π and 2-signotope α correspond to each other if a pair (i,j) is

an inversion of π iff $\alpha(i,j) = +$. For the proof that this is a bijection, note that monotonicity of α corresponds to transitivity of the inversion relation and transitivity of the non-inversion relation for π. In the weak Bruhat order of the symmetric group, the permutations are ordered by inclusion of their inversion sets. By the indicated correspondence between 2-signotopes and permutations, $S_2(n)$ is isomorphic to the weak Bruhat order of S_n.

Manin and Schechtman [17] introduced the *higher Bruhat order* $B(n, r-1)$ which is an order relation on the set of r-signotopes on $[n]$. The higher Bruhat order relation $\leq_{HB}$ is defined as follows: Let α and β be two r-signotopes with $\alpha(A) = \beta(A)$ for all r-subsets A of $[n]$ but just one A^* where $\alpha(A^*) = -$ and $\beta(A^*) = +$ in this case we call the pair (α, β) a *single-step*. The order relation $\leq_{HB}$ is the transitive closure of the single-step relation, i.e, $\alpha \leq_{HB} \beta$ iff there is a sequence $\alpha = \alpha_0, \alpha_1, \ldots \alpha_t = \beta$ such that for $i = 1, \ldots, t$ the pair (α_{i-1}, α_i) is a single-step. Higher Bruhat orders were further studied by Voevodskij and Kapranov [25] and Ziegler [26]. In particular, Ziegler showed that the higher Bruhat order $B(n, r-1)$ and the signotope order $S_r(n)$ are not equal in general. His example is $B(8,3) \neq S_4(8)$. For $r \leq 2$ obviously $B(n, r-1) = S_r(n)$. Ziegler also shows that $B(n, n-k-1) = S_{n-k}(n)$ for $k \leq 3$. The question whether $B(n,2) = S_3(n)$ was left open by Ziegler. This problem was resolved affirmatively by Felsner and Weil [7].

It should also be mentioned that Ziegler [26] gives a geometric interpretation of signotopes. We give a different interpretation in Theorem 7 (dimension 2) and Section 5 (general dimension). In terms of the closely related theory of oriented matroids our geometric objects are the adjoints of the duals of Ziegler's; see [8] for details.

4.3 New Signotopes from Old

Various operations can be performed on signotopes. As in matroids we can perform deletion and contraction but there exist other constructions of new signotopes from old. In this subsection we review these operations. Some of the constructions, e.g. deletion, contraction and weak boundary, will be useful later.

(1) For an r-signotope α the *complement* $\overline{\alpha}$ is obtained by exchanging all signs of α. $\overline{\alpha}$ is a r-signotope.

(2) For an r-signotope α on a linearly ordered set X and $Y \subseteq X$ with $|X \setminus Y| \geq r$ define the *deletion* $\alpha{\uparrow_Y}$ to be the induced function on $\binom{X \setminus Y}{r}$. Deletion of Y gives a r-signotope on $X \setminus Y$.

(3) For an r-signotope α on a set X and $Y \subseteq X$ with $|Y| < r$ define the *contraction* $\alpha{\downarrow_Y}$ to be the function on $\binom{X \setminus Y}{r-|Y|}$ with $\alpha{\downarrow_Y}(A) = \alpha(A \cup Y)$. Contraction of Y gives a $(r - |Y|)$-signotope on $X \setminus Y$.

Let α be an r-signotope on $[n-1]$. A *one-element expansion* of α is an r-signotope β in $S_r(n)$ such that $\alpha = \beta{\uparrow_n}$.

Lemma 9 *The one-element expansions of $\alpha \in S_r(n-1)$ form a lattice in $S_r(n)$.*

Proof. Let β and β' be expansions of α. Let $\gamma : \binom{n}{r} \to \{-,+\}$ be the function with $\gamma(A) = +$ if $\beta(A) = +$ or $\beta'(A) = +$. We claim that γ is a r-signotope and hence the least upper bound for β and β'. For the claim note first that every $r+1$ element set P has $\beta(P^{\lfloor r+1 \rfloor}) = \beta'(P^{\lfloor r+1 \rfloor}) = \alpha(P^{\lfloor r+1 \rfloor})$. It follows that restricted to P the signotopes β and β' are comparable, i.e., the restrictions are comparable in $S_r(P)$. On P the function γ equals the larger of the restrictions of β and β'. Hence for all $(r+1)$-sets P monotonicity of γ is inherited from either β or β'. $\square$

We give geometric interpretations for the above constructions in the two-dimensional case, i.e., for $r = 3$. Proofs for the correspondences can be derived from Theorem 7. Let $(\mathcal{A}, F)$ be the marked arrangement with lines labeled by X corresponding to α. The arrangement corresponding to $\overline{\alpha}$ is $(\mathcal{A}, \overline{F})$. Delete the lines of Y from $\mathcal{A}$ to obtain the arrangement corresponding to $\alpha{\uparrow_Y}$. Let x be an element of X; the contraction $\alpha{\downarrow_x}$ is the local sequence α_x of line x in $\mathcal{A}$. One-element expansions of $\mathcal{A}$ are obtained by adding a pseudoline n compatible with $\mathcal{A}$ that enters the plane in F and leaves in $\overline{F}$. The new northface is the right one of the two faces obtained from F, i.e., the face above n. Lemma 9 has the intuitive explanation that with two expansion lines n and ln' the right boundary of the region R obtained as union of the left halfplanes of n and n' is again an expansion line.

Ziegler [26] proposes two constructions of $(r+1)$-signotopes from a r-signotope.

(4) For a r-signotope α on $[n]$ let $\partial\alpha : \binom{[n]}{r+1} \to \{-,+\}$ be defined by $\partial\alpha(P) = +$ iff $\alpha(P^{\lfloor 1 \rfloor}) = -$ and $\alpha(P^{\lfloor r+1 \rfloor}) = +$. The *boundary* $\partial\alpha$ of α is an $(r+1)$-signotope (see [26]).

(5) For a r-signotope α on $[n]$ let $\hat{\alpha} : \binom{[n+1]}{r+1} \to \{-,+\}$ be the unique function with $\hat{\alpha}{\uparrow_{n+1}} = \partial\alpha$ and $\hat{\alpha}{\downarrow_{n+1}} = \alpha$. The *extension* $\hat{\alpha}$ is an $(r+1)$-signotope (see [26]).

Very much in the spirit of these constructions we define:

(6) For a r-signotope α on $[n]$ let $\partial^*\alpha : \binom{[n]}{r+1} \to \{-,+\}$ be defined by $\partial^*\alpha(P) = +$ iff $\alpha(P^{\lfloor r+1 \rfloor}) = +$.

Claim. The *weak boundary* $\partial^*\alpha$ of α is an $(r+1)$-signotope.

Proof. Let Q be a $r+2$ element set and let $P = Q^{\lfloor r+2 \rfloor}$. Note that $Q^{\lfloor i \rfloor \lfloor r+1 \rfloor} = P^{\lfloor i \rfloor}$ for all $i < r+2$. Hence, $\partial^*\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor i \rfloor \lfloor r+1 \rfloor}) = \alpha(P^{\lfloor i \rfloor})$. It follows from the monotonicity of α that for $1 \le i < j < k < r+2$ either $\partial^*\alpha(Q^{\lfloor i \rfloor}) \preceq \partial^*\alpha(Q^{\lfloor j \rfloor}) \preceq \partial^*\alpha(Q^{\lfloor k \rfloor})$ or $\partial^*\alpha(Q^{\lfloor i \rfloor}) \succeq \partial^*\alpha(Q^{\lfloor j \rfloor}) \succeq \partial^*\alpha(Q^{\lfloor k \rfloor})$.

If $k = r+2$ and $j < r+1$ we note that $Q^{\lfloor k \rfloor \lfloor r+1 \rfloor} = P^{\lfloor r+1 \rfloor}$ and the monotonicity condition of $\partial^*\alpha$ for indices i, j, k follows from the condition for $i, j, k-1$. Finally if $k = r+2$ and $j = r+1$ we find that $Q^{\lfloor j \rfloor \lfloor r+1 \rfloor} = Q^{\lfloor k \rfloor \lfloor r+1 \rfloor}$, hence, $\partial^*\alpha(Q^{\lfloor j \rfloor}) = \partial^*\alpha(Q^{\lfloor k \rfloor})$ and this implies the monotonicity condition of $\partial^*\alpha$ for i, j, k. $\triangle$

(7) For a r-signotope α on $[n]$ let $\tilde{\alpha} : \binom{[n+1]}{r+1} \to \{-,+\}$ be the unique function with $\tilde{\alpha}{\uparrow_{n+1}} = \partial^*\alpha$ and $\tilde{\alpha}{\downarrow_{n+1}} = \alpha$. The *weak extension* $\tilde{\alpha}$ is a $r+1$-signotope.

Remark. Weak extensions have been studied by Rambau [18], using the name expansion for these objects, he shows that $\alpha \to \tilde{\alpha}$ is an order preserving embedding from $B(n, r-1)$ to $B(n+1, r)$.

4.4 Maximum Chains of Signotopes

This subsection is devoted to the proof of the main structural theorem on signotopes, Theorem 13. This result is already part of publications on higher Bruhat orders [17, 26]. While Ziegler refers to some non-trivial results from the theory of oriented matroids in his proof the approach we take remains completely within elementary combinatorics.

With an r-signotope α on $[n]$ associate a directed graph with vertices the $r-1$ element subsets of $[n]$ and edges $\to_\alpha$ defined by: For $P \in \binom{[n]}{r}$ and $1 \leq i < j \leq r$, if $\alpha(P) = +$ let $P^{\lfloor i \rfloor} \to_\alpha P^{\lfloor j \rfloor}$ and if $\alpha(P) = -$ let $P^{\lfloor j \rfloor} \to_\alpha P^{\lfloor i \rfloor}$.

Lemma 10 *For an r-signotope α on $[n]$ the graph with vertices $\binom{[n]}{r-1}$ and edges $\to_\alpha$ is acyclic.*

Proof. For $r = 2$ and arbitrary n, relation $\to_\alpha$ is the transitive tournament corresponding to the permutation whose inversion set is the set of pairs (i, j) with $\alpha(i, j) = +$.

For $n = r$, relation $\to_\alpha$ is the lexicographic order on the $r-1$ subsets of $[r]$ if $\alpha([r]) = -$, otherwise, if $\alpha([r]) = +$ it is the reverse-lexicographic order.

Let $n > r > 2$ and let β be the signotope obtained from α by deletion of $\{n\}$. By induction $\to_\beta$ is acyclic on $\binom{[n-1]}{r-1}$. Let γ be the signotope obtained from α by contraction of $\{n\}$ and view $\to_\gamma$ as graph on the vertex set $Y = \{A \in \binom{[n]}{r-1} : n \in A\}$. By induction $\to_\gamma$ is acyclic.

Let $X^- = \{A \in \binom{[n-1]}{r-1} : \alpha(A \cup \{n\}) = -\}$ and $X^+ = \{A \in \binom{[n-1]}{r-1} : \alpha(A \cup \{n\}) = +\}$. The three sets X^-, X^+, Y partition the $r-1$ element subsets of $[n]$, moreover, the subgraph of $\to_\alpha$ induced by each of the three blocks of the partition is acyclic: It agrees with the subgraph induced by $\to_\beta$ in case of X^- and X^+ and with the subgraph induced by $\to_\gamma$ in the case of Y. Now consider the edges of $\to_\alpha$ between the blocks. By definition of X^- all edges with one end in X^- and the other end in Y are oriented from X^- to Y. Also all edges with one end in X^+ and the other end in Y are oriented from Y to X^+. Therefore, the acyclicity of $\to_\alpha$ is readily established if we show that all edges with one end in X^- and the other end in X^+ are oriented from X^- to X^+. This follows from the next claim:

Claim. $A \in X^-$ and $B \to_\beta A$ implies $B \in X^-$,i.e., X^- is an ideal in the partial order defined by the transitive closure of $\to_\beta$.

From $B \to_\beta A$ it follows that $P = A \cup B$ is a r subset $[n]$. Let i, j be such that $B = P^{\lfloor i \rfloor}$ and $A = P^{\lfloor j \rfloor}$. For $Q = P \cup \{n\}$ we then obtain $Q^{\lfloor i \rfloor} = B \cup \{n\}$, $Q^{\lfloor j \rfloor} = A \cup \{n\}$ and $Q^{\lfloor r+1 \rfloor} = A \cup B = P$. We use the monotonicity of α on Q and distinguish two cases:
(1) If $i < j$ then $B \to_\beta A$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = +$. From $A \in X^-$ it follows that $\alpha(Q^{\lfloor j \rfloor}) = \alpha(A \cup \{n\}) = -$. Monotonicity forces $\alpha(Q^{\lfloor i \rfloor}) = \alpha(B \cup \{n\}) = -$, i.e., $B \in X^-$.
(2) If $j < i$ then $B \to_\beta A$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = -$. From $A \in X^-$ it follows that $\alpha(Q^{\lfloor j \rfloor}) = \alpha(A \cup \{n\}) = -$. Monotonicity forces $\alpha(Q^{\lfloor i \rfloor}) = \alpha(B \cup \{n\}) = -$, i.e., $B \in X^-$.

□

Proposition 11 *For a r-signotope α on $[n]$ there exist a chain $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{r-1}}$ of $(r-1)$-signotopes in $S_{r-1}(n)$ such that for $t = 1, \ldots, \binom{n}{r-1}$ the signs of β_{t-1} and β_t differ at only one $(r-1)$-set A_t.*

Proof. Let $A_1, A_2, \ldots, A_{\binom{n}{r-1}}$ be a topological sorting of $\to_\alpha$ and define $\beta_t(A) = -$ if $A = A_i$ for some $i > t$ and $\beta_t(A) = +$ if $A = A_i$ for some $i \leq t$. To prove the lemma it remains to show that each β_t is a $(r-1)$-signotope.

For every r element set P and all i, j, k with $1 \leq i < j < k \leq r$ we either have $P^{\lfloor i \rfloor} \to_\alpha P^{\lfloor j \rfloor} \to_\alpha P^{\lfloor k \rfloor}$ or $P^{\lfloor k \rfloor} \to_\alpha P^{\lfloor j \rfloor} \to_\alpha P^{\lfloor i \rfloor}$. In the first case we have $\beta_t(P^{\lfloor i \rfloor}) \succeq \beta_t(P^{\lfloor j \rfloor}) \succeq \beta_t(P^{\lfloor k \rfloor})$ for all t and in the second case $\beta_t(P^{\lfloor i \rfloor}) \preceq \beta_t(P^{\lfloor j \rfloor}) \preceq \beta_t(P^{\lfloor k \rfloor})$ for all t. This proves monotonicity for β_t. $\square$

Based on this lemma will next give the proof of Theorem 7. The main motivation for including this here is to illustrate the interpretations of the abstract combinatorial objects we are playing with.

Proof. [Theorem 7] Let α be a 3-signotope, i.e., a function $\alpha : \binom{[n]}{3} \to \{-, +\}$ obeying monotonicity on 4-subsets of $[n]$. From Proposition 11 we obtain a chain $\beta_0, \ldots, \beta_{\binom{n}{2}}$ in $S_2(n)$ corresponding to α. Each β_i encodes a permutation of $[n]$. β_0 is the identity and $\beta_{\binom{n}{2}}$ the reverse permutation. Moreover, two permutations β_t and β_{t+1} differ in a single sign where β_t is $-$ and β_{t+1} is $+$. Hence, there is a single pair (i, j) being a non-inversion of β_t but an inversion in β_{t+1}. This pair is an adjacent pair of both permutations. This shows that $\beta_0, \ldots, \beta_{\binom{n}{2}}$ is a simple allowable sequence. From Theorem 2 we obtain that via $\beta_0, \ldots, \beta_{\binom{n}{2}}$ signotope α encodes an arrangement $\mathcal{A}$. From the construction it is easily verified that the triangle induced by lines i, j, k in $\mathcal{A}$ is a $+$ triangle exactly when $\alpha(ijk) = +$. This proves the bijection. $\square$

The next proposition can be seen as a generalization of Theorem 2; it shows that saturated chains of $(r-1)$-signotopes can be used to encode r-signotopes.

Proposition 12 *Let $1 < r \leq n$ and $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{r-1}}$ be a maximum chain in $S_{r-1}(n)$. For $t = 1, \ldots, \binom{n}{r-1}$ let A_t be the unique $(r-1)$-set with $\beta_{t-1}(A_t) = -$ and $\beta_t(A_t) = +$. There exists a r-signotope α on $[n]$ so that $A_1, \ldots, A_{\binom{n}{r-1}}$ is a topological sorting of $\to_\alpha$.*

Proof. For a set $A \in \binom{[n]}{r-1}$ let $\rho(A)$ be the index of A in the list $A_1, \ldots, A_{\binom{n}{r-1}}$. Note that monotonicity of the β_t's implies that for all $D \in \binom{[n]}{r}$ either $\rho(D^{\lfloor 1 \rfloor}) < \rho(D^{\lfloor 2 \rfloor}) < \ldots < \rho(D^{\lfloor r \rfloor})$ or $\rho(D^{\lfloor 1 \rfloor}) > \rho(D^{\lfloor 2 \rfloor}) > \ldots > \rho(D^{\lfloor r \rfloor})$. In the first case let $\alpha(D) = +$ in the second case $\alpha(D) = -$. We have to show that α is a r-signotope, i.e., that α is monotone at $r+1$ sets. Let $Q \in \binom{[n]}{r+1}$ and consider indices $1 \leq i < j < k \leq r+1$. Suppose $\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor k \rfloor}) = +$. Let $Q^{\lfloor i,j \rfloor}$ denote the set Q minus the ith largest and the jth largest element of Q, e.g., $\{1, 2, 5, 7, 8\}^{\lfloor 2,3 \rfloor} = \{1, 7, 8\}$. From $\alpha(Q^{\lfloor i \rfloor}) = +$ we obtain $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor i,k \rfloor})$. From $\alpha(Q^{\lfloor k \rfloor}) = +$ we obtain that $\rho(Q^{\lfloor i,k \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$. Hence $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$ which implies $\alpha(Q^{\lfloor j \rfloor}) = +$ as required. The argument for $\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor k \rfloor}) = -$ is symmetric. It is obvious that $A_1, \ldots, A_{\binom{[n]}{r-1}}$ is a topological sorting for the relation $\to_\alpha$. $\square$

Propositions 11 and 12 together prove the main structure theorem for signotopes.

Theorem 13 *There is a surjective mapping from maximum chains in $S_{r-1}(n)$ to $S_r(n)$.*

Note that whenever $S_r(n) = B(n, r-1)$ then for any two signotopes $\alpha < \beta$ in $S_r(n)$ there is a chain of maximum length containing both. In general we can show that at least every single element of $S_r(n)$ is contained in a chain of maximum length.

Proposition 14 *Every element of $S_r(n)$ is contained in a chain of length $\binom{n}{r} + 1$.*

Proof. Let $\alpha \in S_r(n)$ and consider the weak boundary $\partial^*\alpha$ of α. This defines the directed graph $\rightarrow_{\partial^*\alpha}$ on $\binom{[n]}{r}$. Note that $A \rightarrow_{\partial^*\alpha} B$ implies $\alpha(A) \preceq \alpha(B)$, i.e., the sets A with $\alpha(A) = -$ form an ideal in the order corresponding to $\rightarrow_{\partial^*\alpha}$. Let $A_1, A_2 \ldots, A_{\binom{n}{r}}$ be a linear extension of this order such that there is a t with $\alpha(A_i) = -$ for all $i \leq t$ and $\alpha(A_i) = +$ for all $i > t$. Define the sequence β_j of r-signotopes as in the proof of Proposition 11. The sequence of complements $\overline{\beta_j}$ is a chain of r signotopes with $\overline{\beta_t} = \alpha$. □

Proposition 11 implies that the mapping Π from maximum chains in $S_{r-1}(n)$ to elements of $S_r(n)$ described in the proof of Proposition 12 is surjective. The two propositions also imply that the preimage of α under Π is a set of maximum chains in $S_{r-1}(n)$ of the same size as the set of topological sortings of $\rightarrow_\alpha$, i.e., linear extensions of the transitive closure of $\rightarrow_\alpha$. We can even say more about this preimage.

Call two maximum chains in $S_{r-1}(n)$ *swap-equivalent* if one of them corresponds to the list $A_1, \ldots, A_{\binom{n}{r-1}}$ of $(r-1)$-sets and the list of the other chain differs only by an adjacent transposition, i.e., is of the form $A_1, .., A_{t-1}, A_{t+1}, A_t, A_{t+2}, .., A_{\binom{n}{r-1}}$ for some t.

Proposition 15 *For $r \geq 3$ the set of maximum chains in $S_{r-1}(n)$ mapped by Π to $\alpha \in S_r(n)$ is a complete swap-equivalence class.*

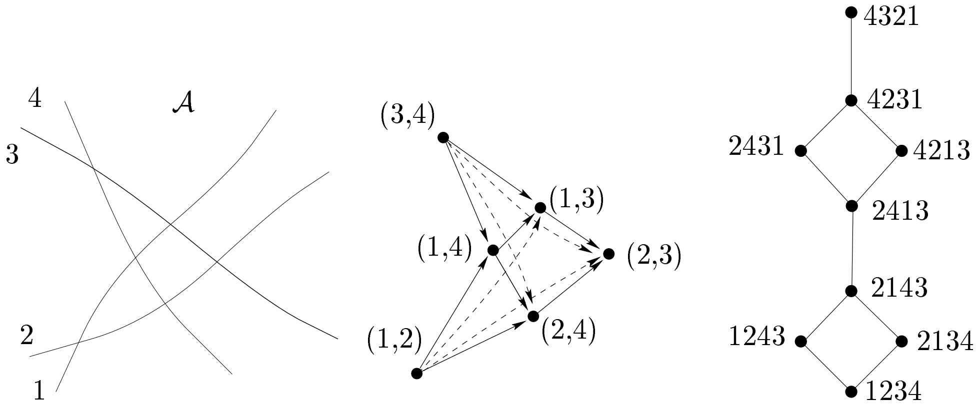
Proof. The proof follows from two facts.

First, it is possible to transform any topological sorting of a directed acyclic graph into any other by a sequence of adjacent transpositions, i.e., reversals of adjacent pairs of unrelated vertices. Therefore, the preimage of α is contained in one swap-equivalence class of chains in $S_{r-1}(n)$.

Now assume $r \geq 3$ that $A_1, \ldots, A_{\binom{n}{r-1}}$ is a topological sorting of $\rightarrow_\alpha$ and let list $A_1, .., A_{t-1}, A_{t+1}, A_t, A_{t+2}, .., A_{\binom{n}{r-1}}$ correspond to a maximum chain of $S_{r-1}(n)$. We claim that A_t and A_{t+1} are unrelated in $\rightarrow_\alpha$. Otherwise $P = A_t \cup A_{t+1}$ is a r-set and monotonicity only allows the signs of A_t and A_{t+1} to be changed in a row if there is an index i so that one of the two sets is $P^{\lfloor i \rfloor}$ and the other is $P^{\lfloor i+1 \rfloor}$. Consider sign and location in the list of a set of $P^{\lfloor j \rfloor}$, $j \neq i, i+1$, to obtain a contradiction to monotonicity. Hence, A_t and A_{t+1} are unrelated in $\rightarrow_\alpha$ and the second list also corresponds to a topological sorting of $\rightarrow_\alpha$.

□

These considerations about swap-equivalence of the Π preimages can be rephrased as follows: Given a r-signotope α the set of $(r-1)$-signotopes on maximum chains of $S_{r-1}(n)$ mapped to α by Π together with the edges (single-steps) used by these chains forms a lattice isomorphic to the lattice of antichains of the transitive closure of $\rightarrow_\alpha$ (An example of this is given in Example B below). In particular this shows that the orders $S_r(n)$ have a local lattice structure. What about global lattice structure? It is known that $S_r(n)$ is a lattice for $r \leq 2$. Ziegler [26] has shown that $S_r(n)$ is a lattice for $r \geq n-2$ and that $S_3(6)$ is not a lattice.



(a) The arrangement. (b) The graph $\rightarrow_\alpha$. (c) The lattice.

Figure 9: Illustrations for Example B.

Example B. Let $\mathcal{A}$ (as shown in Figure 9(a)) be the arrangement corresponding to a 3-signotope α. The directed graph $\rightarrow_\alpha$ is shown in Figure 9(b). Note that we met the transitive reduction of this graph (non-dashed edges) several times as $\overrightarrow{G}$ (see Lemma 1, Subsection 3.1 and Lemma 4). The maximum chains of 2-signotopes mapped by Π to α are the allowable sequences of $\mathcal{A}$. In Subsection 3.1 we have seen that they correspond bijectively to topological sortings of $\overrightarrow{G}$. It follows that the suborder of the weak Bruhat order induced by permutations π appearing in allowable sequences of $\mathcal{A}$ is a distributive lattice (see Figure 9(c)).

5 Geometric Interpretations for Signotopes

Ziegler [26] shows that there is a natural bijection between the uniform extension poset on the set of single element extensions of a cyclic hyperplane arrangement $\mathbf{X}_c^{n,d}$ in $\mathbb{R}^d$ and the higher Bruhat order $B(n, n-d-1)$. Felsner and Ziegler [8] note that from oriented matroid duality, $B(n, n-d-1)$ has another geometric representation as the set of 1–element liftings of $\mathbf{X}_c^{n,n-d}$. These liftings correspond to certain affine arrangements of pseudohyperplanes in $\mathbb{R}^{n-d-1}$. In this section we make the connection with the second class of geometric objects explicit; that is, we characterize a class of arrangements of pseudohyperplanes in $\mathbb{R}^d$ corresponding to signotopes $\alpha \in S_{d+1}(n)$.

A *pseudohyperplane* H in $\mathbb{R}^d$ is a homeomorph of a hyperplane such that the two connected components of $\mathbb{R}^d \setminus H$ are homeomorphic to the d-ball. A set $\{H_1, \ldots, H_n\}$ of pseudohyperplanes in $\mathbb{R}^d$ is an *arrangement of pseudohyperplanes* if every two pseudohyperplanes H_i and H_j intersect in an $(n-1)$ dimensional pseudohyperplane and they cross at their intersection. Moreover, for all j the set $\{H_i \cap H_j : i = 1, ..., j-1, j+1, ..., n\}$ is an arrangement of $n-1$ pseudohyperplanes in $H_j \cong \mathbb{R}^{d-1}$. We say d-arrangement to abbreviate for 'arrangement of pseudohyperplanes in $\mathbb{R}^d$'. A d-arrangement is *simple* if any set of $d+1$ pseudohyperplanes has empty intersection.

So far we have discussed arrangements of pseudolines which had been normalized by a marking face F and a specific labeling of the lines (increasing on a clockwise walk from $\overline{F}$ to F at infinity). For all arrangements of this section we assume that they are simple and that they are embedded in $\mathbb{R}^d$ in a normalized way as described in the next paragraph.

For $i = 1, ..., d-1$ let I_i be the $d-i$ dimensional space at infinity obtained by setting the last i coordinates equal to $-\infty$, i.e., with $x_d = -\infty, x_{d-1} = -\infty, \ldots, x_{d-i+1} = -\infty$ (if the reader feels uncomfortable with these 'spaces at infinity' he may assume that the arrangement is embedded in a d-dimensional unit hypercube $[0,1]^d$ and consider I_i as the side of this cube obtained by setting the last i coordinates equal to 0). We demand that the d-arrangement induces a $(d-i)$-arrangement with the same number of pseudohyperplanes on I_i. Moreover, the pseudohyperplanes are labeled by increasing x_1 coordinate at their intersection with I_{d-1}. We call an arrangement with these properties *normal*.

The intersection of every set of $d-1$ pseudohyperplanes of an arrangement $\mathcal{A}$ determines a line of the arrangement. If the arrangement is normal we consider these lines and the edges they support as oriented away from I_1. A normal d-arrangement induces a sign function $f : \binom{[n]}{d+1} \to \{-,+\}$ by the following rule: Given $i_1 < i_2 < \ldots < i_{d+1}$ let $f(i_1, ..., i_{d+1}) = -$ iff on the intersection line of the pseudohyperplanes $h_{i_3}, ..., h_{i_{d+1}}$ the intersection with h_{i_1} comes before the intersection with h_{i_2}.

Hurrying ahead we define: A normal d-arrangement $\mathcal{A}$ is called a C_d-arrangement if the normal $(d-1)$-arrangement induced by $\mathcal{A}$ on I_1 corresponds to the minimal signotope $\alpha_0 \in S_d(n)$; the minimal signotope α_0 is the signotope with all signs $-$. It should be remarked that the arrangement corresponding to $\alpha_0 \in S_d(n)$ is the cyclic arrangement $\mathbf{X}_c^{n,d}$.

Theorem 16 *There is a bijection between C_d-arrangements with n pseudohyperplanes and signotopes in $S_{d+1}(n)$. The signotope corresponding to a C_d-arrangement $\mathcal{A}$ is the sign function of $\mathcal{A}$ as defined above.*

Proof. We use induction on d. Theorem 7 covers the case $d = 2$ and may serve as basis for the induction. For the induction step we also use that if (α, α') is a single step in $S_d(n)$ then the associated C_{d-1}-arrangements $\mathcal{A}$ and $\mathcal{A}'$ are related by a flip at a simplicial cell bounded by the hyperplanes corresponding to the unique d element set A with $\alpha(A) = -$ and $\alpha'(A) = +$.

For d dimensions we first consider normal arrangements of $d+1$ pseudohyperplanes labeled by the elements of $A = [d+1]$. Such an arrangement $\mathcal{A}$ has just one bounded cell which is a (pseudo)simplex. The set of bounded edges of $\mathcal{A}$ forms the skeleton graph of the simplex, i.e., a complete graph K_{d+1}. The vertex of this graph determined by the intersection of the pseudohyperplanes in $A^{\lfloor i \rfloor}$ will itself be denoted $A^{\lfloor i \rfloor}$.

Claim A. The orientation of lines induces an acyclic orientation on the graph of bounded edges of $\mathcal{A}$.

Let $A^{\lfloor i \rfloor}$, $A^{\lfloor j \rfloor}$ and $A^{\lfloor k \rfloor}$ be any three vertices of the graph. The three lines $A^{\lfloor i,j \rfloor}$, $A^{\lfloor i,k \rfloor}$, $A^{\lfloor j,k \rfloor}$ are supported by the plane $A^{\lfloor i,j,k \rfloor}$ which is a homeomorph of a disk D. The intersection of $A^{\lfloor i,j,k \rfloor}$ with I_1 corresponds to an interval on the boundary of D in which all three lines begin. Since lines and edges are oriented away from I_1 the orientation of the triangle with vertices $A^{\lfloor i \rfloor}$, $A^{\lfloor j \rfloor}$ and $A^{\lfloor k \rfloor}$ is acyclic. An orientation of the complete graph K_{d+1} with all triangles acyclic is acyclic. $\triangle$

Claim B. For C_d-arrangements the orientation of K_{d+1} is either the transitive closure of $A^{\lfloor 1\rfloor} \to A^{\lfloor 2\rfloor} \to \ldots \to A^{\lfloor d+1\rfloor}$ in which case the sign of the arrangement is $+$ or of $A^{\lfloor d+1\rfloor} \to A^{\lfloor d\rfloor} \to \ldots \to A^{\lfloor 1\rfloor}$ in which case the sign is $-$.

Since the graph is acyclic we can sweep arrangement $\mathcal{A}$ starting with I_1. Meaning, we find a sequence $s_0, s_1, \ldots, s_{d+1}$ of pseudohyperplanes such that they all share the pseudosphere at infinity with $I_1 = s_0$ and between any two consecutive pseudohyperplanes s_i, s_{i+1} there is exactly one vertex of the arrangement. Since the arrangement is a C_d arrangement we know that the first vertex to be swept corresponds to a simplicial cell in the arrangement of the minimal element of $S_d(d+1)$. This arrangement has only two simplicial cells one bounded by the pseudohyperplanes in $A^{\lfloor 1\rfloor}$ and the other by those in $A^{\lfloor d+1\rfloor}$. The arrangement induced on s_1 is thus obtained by flipping one of these cells. After this first flip one of the two branches of $S_d(d+1)$ which as we recall has the structure of $(2d+2)$-gon is determined. Playing with the bijection between the arrangements induced on the sweep-planes s_i and the corresponding signotopes we see that the sweep has to follow the chosen branch of $S_d(d+1)$. This results in one of the above orderings of the vertices of K_{d+1}. The statement about the sign of the arrangement follows from considering the orientation of the edge between $A^{\lfloor 1\rfloor}$ and $A^{\lfloor 2\rfloor}$. $\triangle$

From the previous claim we obtain generalized criteria for determining the sign of a $d+1$ element set A in a C_d-arrangement. Consider any two vertices $A^{\lfloor i\rfloor}$ and $A^{\lfloor j\rfloor}$ with $i < j$ of the arrangement induced by A. The sign of A is $+$ iff $A^{\lfloor i\rfloor}$ precedes $A^{\lfloor j\rfloor}$ on the line $A^{\lfloor i,j\rfloor}$.

With this at hand we can show monotonicity for the sign functions of a C_d-arrangement $\mathcal{A}$ with more then $d+1$ pseudohyperplanes: Let α be the sign function corresponding to $\mathcal{A}$ and let P be a $d+2$ element set of pseudohyperplanes. For $1 \leq i < j < k \leq d+2$ we have to show that $\alpha(P^{\lfloor i\rfloor}) = +$ together with $\alpha(P^{\lfloor j\rfloor}) = -$ implies $\alpha(P^{\lfloor k\rfloor}) = -$ and $\alpha(P^{\lfloor i\rfloor}) = -$ together with $\alpha(P^{\lfloor j\rfloor}) = +$ implies $\alpha(P^{\lfloor k\rfloor}) = +$. We only prove the first implication, the other being similar. From $\alpha(P^{\lfloor i\rfloor}) = +$ we obtain that vertex $P^{\lfloor i,j\rfloor}$ precedes vertex $P^{\lfloor i,k\rfloor}$ on the line $P^{\lfloor i,j,k\rfloor}$. From $\alpha(P^{\lfloor j\rfloor}) = -$ we obtain that vertex $P^{\lfloor j,k\rfloor}$ precedes vertex $P^{\lfloor i,j\rfloor}$ on the line $P^{\lfloor i,j,k\rfloor}$. From transitivity $P^{\lfloor j,k\rfloor}$ precedes $P^{\lfloor i,k\rfloor}$ and hence $\alpha(P^{\lfloor k\rfloor}) = -$.

So far we have seen that the sign function of a C_d-arrangement of n pseudohyperplanes is a signotope in $S_{d+1}(n)$. Given a C_d-arrangement with signotope α the next thing to prove is the correspondence between simplicial cells in $\mathcal{A}$ and single steps involving α. For the first half note that a simplicial cell of $\mathcal{A}$ can be flipped leading to $\mathcal{A}'$. Since $\mathcal{A}'$ is a C_d-arrangement it has a corresponding signotope α'. Now compare the ordering of vertices on lines of $\mathcal{A}$ and $\mathcal{A}'$ to see that α and α' differ in just one sign. On the other hand, if α and α' only differ in the sign A then it is possible to show that for all i, j in $\mathcal{A}$ the two vertices $A^{\lfloor i\rfloor}$ and $A^{\lfloor j\rfloor}$ are adjacent along the line $A^{\lfloor i,j\rfloor}$. Therefore, the simplicial cell corresponding to A is not penetrated by any further pseudohyperplane.

Given any C_d-arrangement $\mathcal{A}$, we may move to any other C_d-arrangement (of same dimension with same number of pseudohyperplanes) using flips. This is due to the connectedness of $S_{d+1}(n)$ (Lemma 14). Therefore, the missing link for a complete proof is the existence of a single C_d-arrangement with n pseudohyperplanes. This can be provided by checking that the cyclic arrangements have the required properties. Here we indicate a construction which is similar in spirit to the construction of wiring diagrams as representatives of pseudolinearrangements:

Given $\alpha \in S_{d+1}(n)$ choose a chain $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{d}}$ in $S_d(n)$ mapped by Π to α. By induction β_0 corresponds to a C_{d-1}-arrangement $\mathcal{B}_0$ of n pseudohyperplanes. Let A be the unique d-set with different sign in β_0 and β_1. We know that the pseudohyperplanes from A bound a simplicial cell in $\mathcal{B}_0$. Construct $\mathcal{B}_1$ by applying a simplicial-flip to this cell in $\mathcal{B}_0$. Repeat this to obtain a sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{\binom{n}{d}}$ of arrangements in $\mathbb{R}^{d-1}$ corresponding to $\beta_0, \beta_1, \ldots, \beta_{\binom{n}{d}}$. Introduce a new dimension x_d and place arrangement $\mathcal{B}_i$ in the affine $(d-1)$-dimensional space at $x_d = i$. The pseudohyperplane h_i of the arrangement $\mathcal{A}$ corresponding to α is obtained by properly interpolating between the ith pseudohyperplane in $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ for $j = 0, \ldots, \binom{n}{d} - 1$ and extending the ith pseudohyperplane of $\mathcal{B}_0$ and $\mathcal{B}_{\binom{n}{d}}$ to $x_d = -\infty$ and $x_d = \infty$ respectively. □

Note that, as a consequence of Theorem 16 C_d-arrangements can be swept. This means that starting with the sweep-pseudohyperplane I_1 the sweep never gets stuck. While this property is clearly shared by realizable arrangements there are reasons to believe that "most" higher dimensional arrangements can not be swept (e.g. the examples constructed by Richter-Gebert [19]). In fact it is not even known whether every d-arrangement of $n > d$ pseudohyperplanes contains a simplicial cell.

6 Conclusion and Open Problems

Summarized in three phrases the contributions of this paper are: Sweeps are an effective tool in dealing with planar arrangements. In the simple case the mapping from allowable sequences to marked arrangements is a special case of the general existence of surjective mappings from maximal chains in $S_{r-1}(n)$ to elements of $S_r(n)$. And that elements of $S_r(n)$ correspond to a special class of arrangements (C_{r-1}-arrangements) of pseudohyperplanes in $\mathbb{R}^{r-1}$ which admits sweeps.

Hence, restricted to C_{r-1}-arrangements maximal chains in $S_{r-1}(n)$ can be seen as an $r-1$ dimensional generalization of allowable sequences. Goodman and Pollack [11] had already asked for higher dimensional analogs of allowable sequences. Can these ideas be carried further to give such analogs for a larger class of arrangements of pseudohyperplanes? Are there other sets of conditions which guarantee the sweepability of an arrangement?

Already Manin and Schechtman [17] mention that maximal chains in the weak Bruhat order have a nice encoding in terms of Young tableaux [23, 2]. They ask for a generalization to higher dimension, i.e., for encodings of chains in $S_r(n)$, for $r > 2$. It seems that so far there is no progress concerning this question.

It would be very interesting to understand more of the structure of the graph $\mathcal{G}_r(n)$ whose elements are r-signotopes and edges correspond to single-steps, i.e., two r-signotopes are connected by an edge if they differ only in the sign of a single r-set. Only little is known: Ziegler [26] shows that the higher Bruhat order is homotopy equivalent to a sphere. Felsner and Ziegler [8] have shown that these graphs contain large subgraphs which form the skeleton of zonotopes and characterize those pairs (r, n) where $\mathcal{G}_r(n)$ actually is the skeleton graph of a zonotope in $\mathbb{R}^{n-r+1}$. Questions like minimum and maximum degree of $G_r(n)$ or connectedness are wide open in general. Even for $r = 3$ the question concerning the minimum degree has only recently been solved by Felsner and Kriegel [6]. They showed that every simple Euclidean arrangement of n pseudolines contains $n-2$ triangles. We venture the following conjecture: Minimum degree and connectedness of $G_r(n)$ are both $n - r + 1$.

We conclude with a problem that was brought to our attention by a referee. Does the equivalent of Levi's extension lemma hold for C_d-arrangements?

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