Sweeps, Arrangements and Signotopes

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Abstract. Sweeping is an important algorithmic tool in geometry. In the first part of this paper we define sweeps of arrangements and use the 'Sweeping Lemma' to show that Euclidean arrangements of pseudolines can be represented by wiring diagrams and zonotopal tilings.

In the second part we introduce a new representation for Euclidean arrangements of pseudolines. This representation records an 'orientation' for each triple of lines. It turns out that a 'triple orientation' corresponds to an arrangement exactly if it obeys a generalized transitivity law. Moreover, the 'triple orientations' carry a natural order relation which induces an order relation on arrangements. A closer look on the combinatorics behind this leads to a series of signotope orders closely related to higher Bruhat orders. We investigate the structure of higher Bruhat orders and give new purely combinatorial proofs for the main structural properties. We answer a question of Ziegler and show that two orderings of the higher Bruhat order $B(n,2)$ coincide. Finally, we reconnect the combinatorics of the second part to geometry. In particular we show that maximum chains in the higher Bruhat orders correspond to sweeps.

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1 Introduction

Sweeping is an important algorithmic tool in geometry. In the first part of this paper (Sections 1–3) we define sweeps of arrangements and use the 'Sweeping Lemma' to prove representations of Euclidean arrangements by wiring diagrams (c.f. [8]) and zonotopal tilings (c.f. [22]). We also use the Sweeping Lemma to give a new proof of Levi's Extension Lemma.

In the second part (Sections 4–5) we introduce a new representation for Euclidean arrangements of pseudolines. This representation records an 'orientation' for each triple of lines. It turns out that a 'triple orientation' corresponds to an arrangement exactly if it obeys a generalized transitivity law. Moreover, the 'triple orientations' carry a natural order relation which induces an order relation on arrangements. A closer look on the combinatorics behind this leads to a series of orders closely related to the *higher Bruhat orders* defined by Manin and Schechtman [14] and further studied by Ziegler [22]. We investigate the structure of higher Bruhat orders and give new purely combinatorial proofs for the main results of [14] and [22].

In Sections 6 we answer a question of Ziegler and show that two orderings of the higher Bruhat order $B(n,2)$ coincide. Finally, we reconnect the combinatorics of the second

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part to geometry: Elements of the higher Bruhat order $B(n,k)$ represent arrangements of n pseudohyperplanes in $\mathbb{R}^k$ and maximum chains in $B(n,k)$ correspond to sweeps of arrangements in $\mathbb{R}^{k+1}$.

1.1 Arrangements of Pseudolines

Let a *pseudoline* be a curve in the Euclidean plane which is unbounded on both sides and has no self-intersections, in particular, removing a pseudoline form the plane leaves two connected components and both components are unbounded. An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection where the two pseudolines cross. Since in this paper we are not concerned about realizability questions we will briefly say arrangement when we really mean arrangement of pseudolines. In some cases we even write line when we mean pseudoline.

An arrangement is *simple* if no three pseudolines have a common point of intersection. The *order* of an arrangement is the number of its pseudolines. Given an arrangement $\mathcal{A}$ of order n we will always assume that the pseudolines are labeled with the elements of $[n] = \{1, .., n\}$.

An arrangement partitions the plane into cells of dimensions 0, 1 or 2, the *vertices, edges* and *faces* of the arrangement. Two arrangements are *isomorphic* if there is an isomorphism of the induced cell complexes respecting the labeling of the lines. Edges and faces of the arrangement may either be bounded or unbounded. Let F be an unbounded cell of arrangement $\mathcal{A}$ and let $\overline{F}$ be the *complementary face of* F, i.e., the face separated from F by all pseudolines. We may orient all pseudolines such that F is in the left halfspace and $\overline{F}$ in the right halfspace of every line. This orientation of pseudolines induces an orientation of the edges of the arrangement. The pair $(\mathcal{A}, F)$ is a *marked arrangement* or an *arrangement with northface F and southface $\overline{F}$*. If there is no explicit reference to the northface of a marked arrangement $\mathcal{A}$ embedded in a coordinized plane we assume that the northface is the face containing the ray to $(0, \infty)$. Two marked arrangements are *isomorphic* if there is an isomorphism of the induced cell complexes respecting the orientation of the edges. See Figure 1 for an illustration.

2 Sweeping the Plane

In this section we discuss sweeps for Euclidean arrangements. The main result is the *Sweeping Lemma* (Lemma 1) which states that every such arrangement can be swept. Snoeyink and Hershberger [20] have a theorem that contains the Sweeping Lemma for the special case of simple arrangements.

Let $(\mathcal{A}, F)$ be a marked arrangement. A *sweep of $\mathcal{A}$ with northpole in F* is a sequence $c_0, c_1, \ldots c_r$, of curves such that each curve c_i has fixed points $\overline{x} \in \overline{F}$ and $x \in F$ as endpoints. Further requirements are:

(1) Non of the curves c_i contains a vertex of arrangement $\mathcal{A}$.

(2) Each curve c_i has exactly one point of intersection with each line l_j.

(3) Besides at their endpoints any two curves c_i and c_j are disjoint.

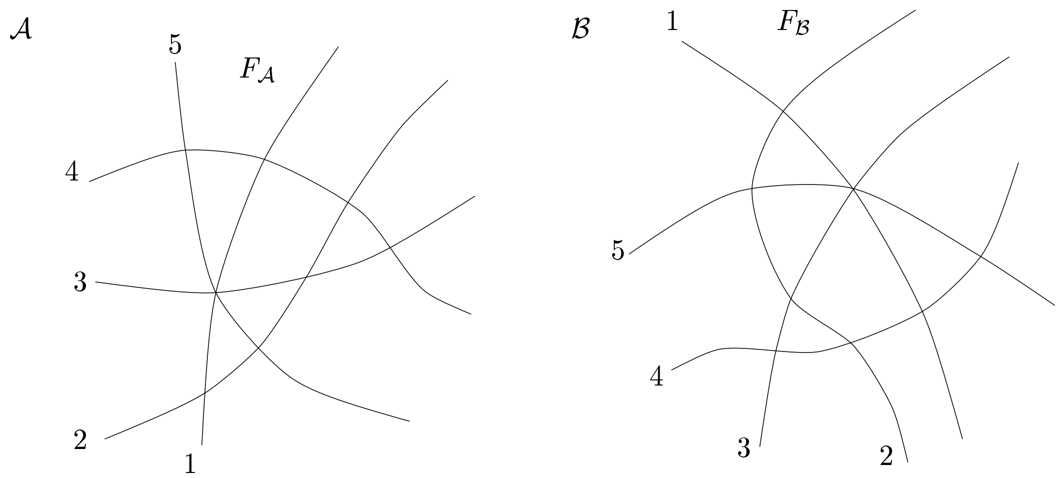


Figure 1: Arrangements $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as arrangements but non-isomorphic as marked arrangements.

(4) For any two consecutive curves c_i, c_{i+1} of the sequence there is exactly one vertex of arrangement $\mathcal{A}$ between them, i.e., in the interior of the closed curve $c_i \cup c_{i+1}$.

(5) Every vertex of the arrangement is between a unique pair of consecutive curves, hence, the interior of the closed curve $c_0 \cup c_r$ contains all vertices of $\mathcal{A}$.

See Figure 2 for an example of a sweep for the arrangement $\mathcal{A}$ of Figure 1.

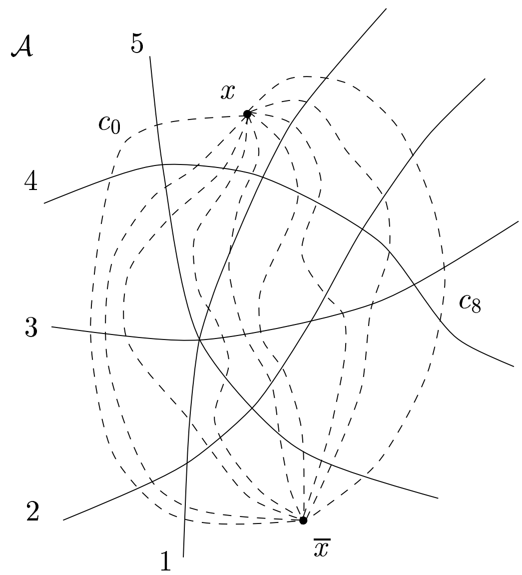


Figure 2: A sweep for arrangement $\mathcal{A}$

Note that if $c_0, \dots, c_r$ is a sweep for $\mathcal{A}$ then the reversed sequence is also a sweep for $\mathcal{A}$. One of these sweeps is from left to right and the other from right to left. As usual we will always think of a sweep as a left to right sweep. A discrete sweep as defined here can be transformed into a continuous sweep by appropriate interpolation between any pair c_i, c_{i+1} of curves. The dependency on the chosen points x and $\overline{x}$ can also be eliminated.

Lemma 1 (Sweeping Lemma) *Let $(\mathcal{A}, F)$ be a marked Euclidean arrangement of pseudolines. Then there is a sweep sequence of curves for $\mathcal{A}$, i.e., $\mathcal{A}$ can be swept.*

Proof. Let $G = (V, E)$ be the graph such that the vertices V of G are the vertices of $\mathcal{A}$ and the edges of G are the finite edges of the arrangement $\mathcal{A}$. Let $\overrightarrow{E}$ be the orientation of the edges of G induced by the orientation of pseudolines (the northface is in the left halfplane of each pseudoline).

Claim A. The orientation $\overrightarrow{E}$ is an acyclic orientation of G.

Walking 'at infinity' and clockwise from $\overline{F}$ to F the pseudolines of $\mathcal{A}$ are met in some order. Let permutation π be the corresponding order of the labels.

We prove the above claim by contradiction: Assuming that $\overrightarrow{E}$ is not acyclic we choose a cycle C such that the area enclosed by the corresponding curve in $\mathcal{A}$ is minimal. It is easy to conclude that C corresponds to the boundary of a face of $\mathcal{A}$. With respect to this face the cycle C may be oriented clockwise or counterclockwise. We consider the first case (clockwise) the other is symmetric.

Let $e_1, e_2, \ldots, e_k$ be edges of C and let l_{i_j} be the supporting pseudoline of e_j. Since e_j and e_{j+1} are consecutive on C the lines l_{i_j} and $l_{i_{j+1}}$ cross at a vertex of C. From the definition of π and the clockwise orientation of C it follows that i_j precedes i_{j+1} in π (see Figure 3). Hence $i_1 <_\pi i_2 <_\pi \ldots <_\pi i_k <_\pi i_1$ a contradiction. $\triangle$

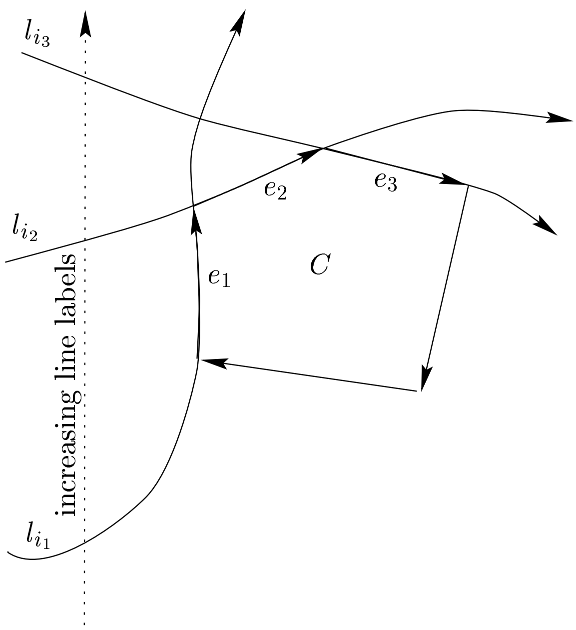


Figure 3: Permutation

Since $\overrightarrow{G} = (V, \overrightarrow{E})$ is acyclic there exists a topological sorting $v_1, v_2, \ldots, v_r$ of $\overrightarrow{G}$. Fix points $x \in F$ and $\overline{x} \in \overline{F}$.

Claim B. There exists a sweep of curves $c_0, c_1, \ldots, c_r$ such that vertices $v_1, \ldots, v_i$ are to the left of c_i and vertices $v_{i+1}, \ldots, v_r$ are to the right of c_i for all $i = 1, \ldots, r$.

Proof. Let R be the union of the closed bounded cells of $\mathcal{A}$. Define c_0 as the union of three curves. The first and the second connect x to R within F and $\overline{x}$ to R within $\overline{F}$, the third is the left boundary of an ϵ-tube of the left boundary of R and connected to the two other curves. For an appropriate ϵ this gives a curve as desired.

Now suppose that c_{i-1}, $i \leq r$, has been defined. Let $l_{i_1}, \ldots, l_{i_t}$ be the lines of $\mathcal{A}$ containing vertex v_i and assume $i_1 <_\pi \ldots <_\pi i_t$. Let T be the triangle defined by c_{i-1}, l_{i_1} and l_{i_t}. Since v_i is a source (minimal) in the restriction of $\overrightarrow{G}$ to $v_i, \ldots, v_r$ and $v_1, \ldots, v_{i-1}$ are left of c_{i-1} vertex v_i is the unique vertex of $\mathcal{A}$ in the triangular region T. Define c_i as the right boundary of an ϵ-tube around c_{i-1} and T. For an appropriate ϵ this gives a curve as desired, see Figure 4. $\triangle$

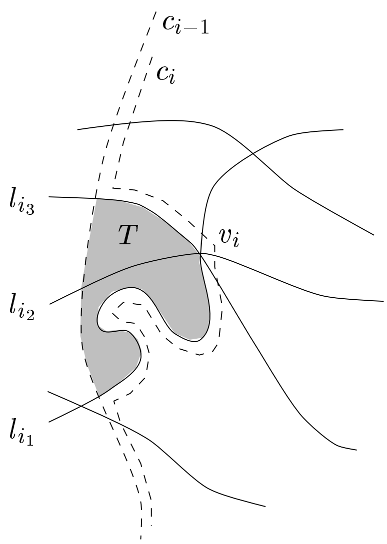


Figure 4: Defining c_i based on c_{i-1} and the shaded triangular region T.

This concludes the proof of the lemma. $\square$

3 Applications of Sweeping

3.1 Allowable Sequences and Wiring Diagrams

It is often convenient to work with purely combinatorial representations of arrangements. The representations discussed in this subsection have been introduced by Goodman and Pollack, see [8]. Further sources for representations of arrangements are Goodman and Pollack [9], Edelsbrunner [2], Felsner [3] and Knuth [12].

Let $c_0, c_1, \ldots, c_r$ be a sweep sequence of curves for the marked arrangement $(\mathcal{A}, F)$ of order n. Traversing curve c_i from $\overline{x}$ to x we meet the lines of $\mathcal{A}$ in some order. Since each line is met by c_i exactly once the order of the crossings corresponds to a permutation π_i of $[n]$.

Consider the labels of lines crossing at vertex v_i. Since the region T defined in the proof of Claim B is empty of vertices of $\mathcal{A}$ and by property 2 of the sweep curve c_i the lines $l_{i_1}, \dots, l_{i_t}$ containing vertex v_i are a consecutive substring of π_{i-1}. Moreover, in permutation π_{i-1} these lines are in the reversed order and this is the only difference between π_{i-1} and π_i. Relabeling the lines of $\mathcal{A}$ appropriately we may assume that π_0 is the identity permutation.

Example A. The sequence of permutations obtained from the sweep of Figure 2 is $(1,2,3,4,5) \overset{4,5}{\to} (1,2,3,5,4) \overset{1,2}{\to} (2,1,3,5,4) \overset{1,3,5}{\to} (2,5,3,1,4) \overset{2,5}{\to} (5,2,3,1,4) \overset{1,4}{\to}$ $(5,2,3,4,1) \overset{2,3}{\to} (5,3,2,4,1) \overset{2,4}{\to} (5,3,4,2,1) \overset{3,4}{\to} (5,4,3,2,1)$.

The sequence $\pi_0, \dots, \pi_r$ has the following properties:

(1) π_0 is the identity permutation and π_r is the reverse permutation on $[n]$.

(2) Each permutation π_i, $1 \le i \le r$ is obtained by the reversal of a consecutive substring M_i from the preceding permutation π_{i-1}.

(3) Any two elements $x, y \in [n]$ are joint members of exactly one move M_i, i.e., reverse their order exactly once.

A sequence $\Sigma = \pi_0, \dots, \pi_r$ of permutations with properties (1), (2) and (3) is called an *allowable sequence of permutations*. If each move from π_{i-1} to π_i consists in the reversal of just one pair of elements, i.e., a transposition, we have $r = \binom{n}{2}$ and the sequence Σ is called a *simple allowable sequence*. We have thus seen how to obtain an allowable sequence of permutations from every marked arrangement $(\mathcal{A}, F)$. However, we can say more:

Every topological sorting of the graph $\overrightarrow{G}$ of $(\mathcal{A}, F)$ induces an allowable sequence. Consider the allowable sequences Σ and Σ' corresponding to topological sortings σ and σ' of $\overrightarrow{G}$ with the property that $\sigma = v_1, \dots, v_i, v_{i+1}, \dots, v_r$ and $\sigma' = v_1, \dots, v_{i+1}, v_i, \dots, v_r$, i.e., σ and σ' differ in an adjacent transposition. It follows that v_i and v_{i+1} are both minimal elements in the restriction of $\overrightarrow{G}$ to $\{v_i, v_{i+1}, v_{i+2}, \dots, v_r\}$. Hence, there is no line in $\mathcal{A}$ that contains vertices v_i and v_{i+1} and the labels of lines involved in the moves $M_i : \pi_{i-1} \to \pi_i$ and $M_{i+1} : \pi_i \to \pi_{i+1}$ in Σ are disjoint. In fact for $j \ne i, i+1$ the permutations π_j and π'_j in Σ and Σ' coincide and $M'_i = M_{i+1}$ and $M'_{i+1} = M_i$. Call two allowable sequences Σ and Σ' *elementary equivalent* if Σ can be transformed into Σ' by interchanging two disjoint adjacent moves. Two allowable sequences Σ and Σ' are called *equivalent* if there exists a sequence $\Sigma = \Sigma_1, \Sigma_2, \dots, \Sigma_m = \Sigma'$ such that Σ_i and Σ_{i+1} are elementary equivalent for $1 \le i < m$. It is well known that it is possible to transform any topological sorting of a directed acyclic graph $\overrightarrow{G}$ into any other by a sequence of adjacent transpositions, i.e., reversals of adjacent pairs of unrelated vertices. Therefore, any two allowable sequences corresponding to the same marked arrangement $(\mathcal{A}, F)$ are equivalent.

Theorem 2 *There is a bijection between equivalence classes of allowable sequences and marked arrangements of pseudolines. Moreover, this bijection maps simple allowable sequences to simple arrangements.*

Proof. We have already seen how to define the equivalence class of allowable sequences corresponding to a marked arrangement.

Let Σ be an allowable sequence. Start drawing n horizontal lines called *wires* and vertical lines $p_0, \dots, p_r$. Label the crossing of the ith wire from below with p_j with the label $p_j(i)$. Draw pseudoline l_i such that it interpolates the crossings with its label as in Figure 5.

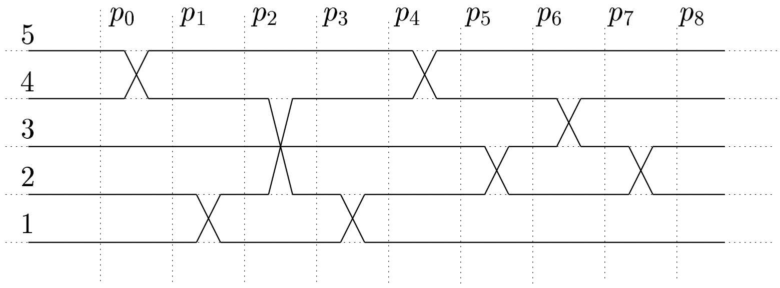


Figure 5: A wiring diagram for the arrangement of Figure 2

Following Goodman [6] we call the arrangement thus obtained a *wiring diagram* for Σ. Since the vertical lines $p_0, \ldots, p_r$ essentially are a sweep sequence of curves for the wiring diagram we see that the mapping from arrangements to allowable sequences is surjective. Let $(\mathcal{A}, F)$ be any marked arrangements $(\mathcal{A}, F)$ such that Σ corresponds to a sweep of $c_0, \ldots, c_r$ of $\mathcal{A}$. It is obvious that the part of $\mathcal{A}$ between c_{i-1} and c_i is isomorphic to the part of the wiring diagram between p_{i-1} and p_i. These isomorphisms for $i = 1, .., r$ can be glued together to an isomorphism of the arrangements. This proves injectivity and hence the first part of the theorem.

The second part of the theorem is obvious. $\square$

It is interesting to ask for the change in the representation when the northface is changed. Let $(\mathcal{A}, F)$ be a marked arrangement and redefine the northface to be the unbounded 2-cell F' to the left of F. Cells F and F' are separated by line l_n. The directed graph $\overrightarrow{G}'$ is obtained from $\overrightarrow{G}$ by reverting the orientations of all edges with supporting line l_n. Now choose a topological sorting σ for $\overrightarrow{G}$ such that all vertices of $\mathcal{A}$ which are right of (below) line l_n precede the vertices on l_n and all vertices left of (above) l_n come later. Let $v_1, \ldots, v_{i-1}$, be the left block of σ, $v_i, \ldots, v_{j-1}$ be the middle block, i.e., the ordered sequence of vertices on l_n, and $v_j, \ldots, v_r$ be the right block. It follows that $v_1, \ldots, v_{i-1}, v_{j-1}, \ldots, v_i, v_j, \ldots, v_r$ is a topological sorting of $\overrightarrow{G}'$. Note that the order in which the lines enter v_k for $i \leq k \leq j$ has also changed, in $\overrightarrow{G}$ line n was the highest line entering v_k and in $\overrightarrow{G}'$ line n is the lowest line entering v_k. Hence, from the allowable sequence Σ of $(\mathcal{A}, F)$ with moves $M_1, \ldots, M_r$ corresponding to $v_1, \ldots, v_r$ we obtain a sequence Σ'_0 with moves $M_1, \ldots, M_{i-1}, M^*_{j-1}, \ldots, M^*_i, M_j, \ldots, M_r$, where M^*_k is obtained from M_k by moving element n from the top to the bottom. An allowable sequence Σ' for $(\mathcal{A}, F')$ is obtained from Σ'_0 by relabeling $n \to 1 \to 2 \to \ldots \to n-1 \to n$.

We briefly mention another representation for marked arrangements where the change from the representation of $(\mathcal{A}, F)$ to the representation $(\mathcal{A}, F')$ is more transparent. Let α_i be the permutation of $\{1, .., n\} \setminus i$ reporting the order from left to right in which the other pseudolines cross line i, for $i = 1, .., n$. Goodman and Pollack [8] call this the *local sequences of unordered switches* of the arrangement. Felsner [3] used sweeps to show that local sequences are a representation for marked arrangements. In case of non-simple arrangements local sequences are slightly more general structures than permutations since several lines can cross line l_i in the same point. For the arrangement of Figure 2 the local sequences are $\alpha_1 = [2, \{3,5\}, 4]$, $\alpha_2 = [1,5,3,4]$, $\alpha_3 = [\{1,5\}, 2, 4]$, $\alpha_4 = [5,1,2,3]$ and $\alpha_5 = [4, \{1,3\}, 2]$. To change from the local sequences of $(\mathcal{A}, F)$ to those of $(\mathcal{A}, F')$ we revert sequence α_n and relabel $n \to 1 \to 2 \to \ldots \to n-1 \to n$ as before. In Section 4 Theorem 8 we characterize those $(\alpha_i)_{i=1..n}$ corresponding to simple marked arrangements.

3.2 Zonotopal Tilings

A particularly nice representation of arrangements of pseudolines is the representation by 'zonotopal tilings'. Basically this is a standardized drawing of the 'dual graph' of the arrangement. Figure 6 should make the connection clear. Below, in Theorem 3 we prove a bijection between zonotopal tilings and arrangements.

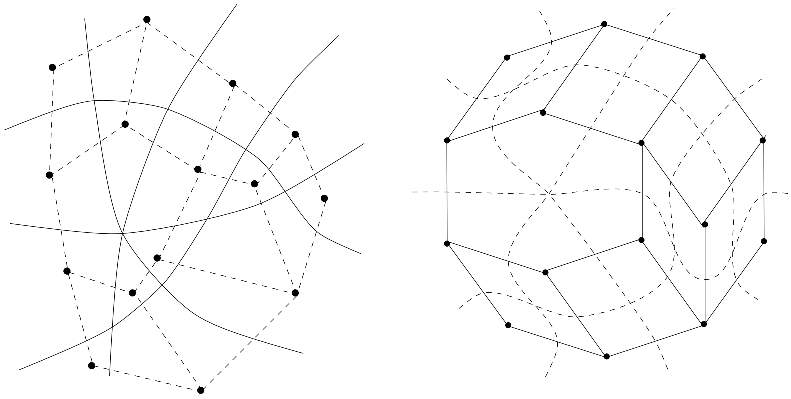


Figure 6: An arrangement with ist dual graph and the dual graph as zonotopal tiling.

A 2-dimensional *zonotope* is the Minkowski sum of a set of line segments in $\mathbb{R}^2$. With a vector v_i we associate the line segment $[-v_i, +v_i]$. The Minkowski sum of the line segments corresponding to $V = \{v_1, \ldots, v_n\}$ is the set

$$Z(V) = \Big\{ \sum_{i=1}^{n} c_i\, v_i : -1 \le c_i \le 1 \text{ for all } 1 \le i \le n \Big\}.$$

A *zonotopal tiling* $\mathcal{T}$ is a tiling of $Z(V)$ by translates of zonotopes $Z(V_i)$ with $V_i \subset V$. A zonotopal tiling is a *simple zonotopal tiling* if all tiles are rhombi, i.e., $|V_i| = 2$ for all i. A zonotopal tiling together with a distinguished vertex x of the boundary of $Z(V)$ is a *marked zonotopal tiling*. The next theorem is a precise statement for the correspondence suggested by Figure 6. The proof of the theorem is based on a Sweeping Lemma for zonotopal tilings, Lemma 4.

Theorem 3 *Let V be a set of n pairwise non-collinear vectors in $\mathbb{R}^2$.*

(1) *There is a bijection between marked zonotopal tilings of $Z(V)$ and marked arrangements of order n.*

(2) *Via this bijection simple tilings correspond to simple arrangements.*

Remark. Theorem 3 is equivalent to the rank 3 version of the Bohne-Dress Theorem which gives a bijection between zonotopal tilings of d-dimensional zonotopes and oriented matroids of rank $d+1$ with a realizable one-element contraction. The correspondence between oriented matroids and arrangements is given by the representation theorem for oriented matroids. This theorem states that oriented matroids of rank $d+1$ are in bijection with arrangements of pseudohyperplanes in d-dimensiononal projective space. An accessible treatment of these connections can be found in [23]. A more geometric proof of the Bohne-Dress Theorem was given by Richter-Gebert and Ziegler [16].

Let $Z(V)$ be a marked zonotope with V a set of n pairwise non-collinear vectors. The zonotope $Z = Z(V)$ is a centrally symmetric $2n$-gon. Rotate Z such that the distinguished vertex x is the unique highest vertex of Z, in particular the boundary of Z has no horizontal edge. Assume that the vectors in V are labeled such that along the left boundary of Z, i.e., on the left path from the lowest vertex $\overline{x}$ to x, the segments correspond to $v_1, v_2, \ldots, v_n$ in this order.

Given a zonotopal tiling $\mathcal{T}$ consider the set of y-monotone path along segments of $\mathcal{T}$ from $\overline{x}$ to x. We define a *sweep of* $\mathcal{T}$ *with northpole* x as a sequence $p_0, p_1, \ldots, p_r$ of y-monotone path from $\overline{x}$ to x in $\mathcal{T}$ with the following properties.

(1) Any two consecutive paths p_i, p_{i+1} of the sequence have exactly one tile T_i of tiling $\mathcal{T}$ between them, i.e., in the interior of the closed curve $p_i \cup p_{i+1}$.

(2) Every tile is between a unique pair of consecutive paths, therefore, $p_0 \cup p_r$ is the boundary of $Z(V)$.

As we did for sweeps of arrangements we further assume that the sweep of $\mathcal{T}$ is from left to right, i.e., p_0 is the left boundary of $Z(V)$.

Remark. There is some interest in the maximum number $m(n)$ of y-monotone $\overline{x}$ to x path a marked zonotopal tiling can have. Knuth [12, page 39] conjectures that $m(n) \leq n2^{n-2}$. Via an inductive argument this would imply that the number of marked arrangements of n pseudolines is bounded by $\Pi_{k=1}^{n} m(k)$. Therefore, the conjectured bound would show that this number is at most $2^{n^2/2+o(n^2)}$ which improves over the best known estimates, Felsner [3].

A sweep of tiling $\mathcal{T}$ induces a total order $T_1, T_2, \ldots, T_r$ on the tiles of $\mathcal{T}$ with the property that after removing the tiles of any initial segment $T_1, \ldots, T_{i-1}$ tile T_i can be separated from the remaining tiles $T_{i+1}, \ldots, T_r$ by a translation to the left parallel to the x-axis, we call this the *separation property*. Conversely, an order $T_1, T_2, \ldots, T_r$ of the tiles with the separation property corresponds to a sweep: Define path p_i as the right boundary of the union of $T_1, \ldots, T_i$. To proof that a zonotopal tiling $\mathcal{T}$ can be swept it is therefore sufficient to show that there is a total order of the tiles with the separation property.

Guibas and Yao [11] observed that given any set $C_1, C_2, \ldots, C_n$ of disjoint convex objects in the plane there is at least one object C_i that can be translated to the left parallel to the x-axis without ever colliding with another object from the set. Hence, by induction every set of disjoint convex objects admits a total ordering $C_1, C_2, \ldots, C_r$ with the separation property, i.e., for $i = 1..r$ given the sets $C_i, \ldots, C_r$ we can separate C_i from the remaining sets by a translation to the left parallel to the x-axis. As a special case we obtain:

Lemma 4 *Every marked zonotopal tiling* $\mathcal{T}$ *can be swept.*

Define a graph $G = (V, E)$ such that the vertices V of G are the tiles of $\mathcal{T}$ and the edges of G are pairs of tiles sharing a common segment. Let $\overrightarrow{E}$ be an orientation of the edges of G such that an edge $\{T, T'\}$ of G points from the tile on the left side of the segment $T \cap T'$ to the tile on the right side. Since the boundary of Z consists entirely of non horizontal edges this orientation is well defined. The orientation of the edges of G represents the 'immediate blocking relation' with respect to translations parallel to the x-axis. From Lemma 4 we obtain:

Fact A. The orientation $\overrightarrow{E}$ is an acyclic orientation of G.

From the correspondence between marked zonotopal tilings and marked arrangements indicated in Figure 6 we see that we met graph G and its orientation already in the proof of Lemma 1. For later use we note:

Fact B. Every topological sorting of $\overrightarrow{G}$ has the separation property.

The next lemma is the 'zonotopal equivalent' of Theorem 2.

Lemma 5 *There is a bijection between equivalence classes of allowable sequences and marked zonotopal tilings. Moreover, this bijection maps simple allowable sequences to simple arrangements.*

Proof. Recall that sweeps of $\mathcal{T}$ correspond to topological sortings of $\overrightarrow{G}$. Given a sweep sequence $p_0, \ldots, p_r$ of paths we associate to each path p_i a sequence π_i recording the labels of the vectors which define the segments along the path in the order of the path from $\overline{x}$ to x. The sequence π_0 is a permutation, the identity. Any two consecutive sequences π_i and π_{i+1} only differ in a substring where path p_i takes the left boundary and path p_{i+1} takes the right boundary of tile T_i. Since T_i is a zonotope the same labels appear on both boundaries but in reversed order. Hence, all π_i are permutations, moreover, $\pi_i \to \pi_{i+1}$ is a move as in part (2) of the definition of allowable sequences. We also note that π_r is the reverse permutation.

It remains to prove property (3) of allowable sequences, namely, that any two elements $a, b \in [n]$ are reversed in exactly one move. This is shown by an argument involving volumes. Due to a formula of McMullen (see Shephard [19, Prop. 2.2.12]) the volume of a 2-dimensional zonotope $Z(v_1, \ldots, v_n)$ is given as follows

$$\mathrm{vol}(Z(v_1, \ldots, v_n)) = \sum_{i<j} \mathrm{vol}(Z(v_i, v_j) = \sum_{i<j} 4|\det(v_i, v_j)|.$$

A move reverting $i_1 < i_2 < .. < i_s$ corresponds to a tile $T = Z(v_{i_1}, .., v_{i_s})$ of volume $\sum_{i_j < i_k} 4|\det(v_{i_j}, v_{i_k})|$. Each pair has to be reversed at least once and this exhausts the volume of the zonotope $Z(V)$. Hence there can be no additional reversals and property (3) is established.

Consider allowable sequences Σ and Σ' corresponding to topological sortings σ and σ' of $\overrightarrow{G}$ with the property that $\sigma = T_1, \ldots, T_i, T_{i+1}, \ldots, T_r$ and $\sigma' = T_1, \ldots, T_{i+1}, T_i, \ldots, T_r$, i.e., σ and σ' differ in an adjacent transposition. The tiles T_i and T_{i+1} are both minimal elements in the restriction of $\overrightarrow{G}$ to $\{T_i, T_{i+1}, T_{i+2}, \ldots, T_r\}$. Hence there is no horizontal line intersecting both of them. From the y-monotonicity of p_{i-1} and the fact that π_{i-1} is a permutation we conclude that $V_i \cap V_{i+1} = \emptyset$ when $T_i = Z(V_i)$ and $T_{i+1} = Z(V_{i+1})$. Therefore, the moves $M_i : \pi_{i-1} \to \pi_i$ and $M_{i+1} : \pi_i \to \pi_{i+1}$ in Σ are disjoint and Σ and Σ'

are equivalent. As in the proof of Theorem 2 we obtain that any two allowable sequences corresponding to the same marked zonotopal tiling are equivalent.

It remains to show how to associate a marked zonotopal tiling to an equivalence class of allowable sequences. Build the tiling from left to right starting with the left boundary of $Z(V)$. After placing i tiles three properties remain invariant:

(1) The union of the already placed tiles together with the left boundary of Z is a simply connected region.

(2) The right boundary of this region is a y-monotone path p_i.

(3) The segments along path p_i are in the order given by π_i.

From this it is obvious that we can place the tile T_{i+1} corresponding to move M_{i+1} such that the invariant remains valid. Since the last permutation π_r is the reverse of the identity path p_r is the right boundary of $Z(V)$. Hence, the placement of tiles $T_1, \ldots, T_r$ is a tiling $\mathcal{T}$ of $Z(V)$.

It is easily seen that equivalent allowable sequences lead to the same tiling while non-equivalent allowable sequences produce different tilings. □

Theorem 3 is now easily obtained.

proof (Theorem 3). Statement (1) is a direct consequence of Theorem 2 and Lemma 5. Combining the two bijections it is seen that the graph of edges of the marked zonotopal tiling corresponding is the dual of the graph of the corresponding marked arrangement with the marked face F of the arrangement and the marked vertex x of the tiling dually corresponding to each other. For statement (2) we additionally note that an arrangement is simple exactly if all bounded regions of the dual graph are quadrangles. □

3.3 Levi's Extension Lemma

Lemma 6 *Let $\mathcal{A}$ be an arrangement of order n and let p, q two points in the plane which do not both lie on any of the lines of $\mathcal{A}$. Then there is a pseudoline c containing p and q such that $\mathcal{A} \cup c$ is an arrangement of order $n+1$.*

The original source for the lemma stated for projective arrangements is Levi [13], an English transcription is found in Grünbaum [10]. A proof using a variant of sweeps, namely cyclic sweeps, was given by Snoeyink and Hershberger [20]. Here we use the projective space as auxiliary tool.

Proof. We detail the proof for the case where p and q are not incident to a line of $\mathcal{A}$. Let p be contained in face F_p of $\mathcal{A}$. Let $l_1, \ldots, l_n$ be the pseudolines of $\mathcal{A}$ and without loss of generality let l_1 contain an edge e of the boundary of F_p. Add the line at infinity l_∞ to the arrangement and map it back to Euclidean space such that l_1 is the line at infinity thus obtaining an arrangement $\mathcal{A}'$ with lines $l_\infty, l_2, \ldots, l_n$. Mark $\mathcal{A}'$ such that $p \in F_p$ is the northpole. Apply the Sweeping Lemma to find a curve c crossing the face F_q containing q. Line c can be bent in F_q to make q a point on c. Extending c from p to infinity we see that $\mathcal{A}' \cup c$ is an arrangement of order $n+1$. Adding the line at infinity, i.e., l_1 we obtain a projective arrangement of order $n+2$ which is mapped back to the Euclidean plane using l_∞ as line at infinity. This gives an arrangement of lines $l_1, \ldots, l_n, c$ with both points p and q on line c. □

It is notable that higher dimensional analogs of the Extension Lemma fail. Examples can be given of arrangements of pseudoplanes in three-space such that for some triples of points p, q, r no pseudoplane can be added to extend the arrangement and contain the three points (see Goodman and Pollack [7]).

4 Flips and Triangles in Arrangements of Pseudolines

Consider a graph $\mathcal{G}_n$ whose vertices are all combinatorially different simple marked arrangements of n pseudolines in the Euclidean plane and edges corresponding to elementary *flips* (see Figure 7), i.e., arrangements $\mathcal{A}$ and $\mathcal{B}$ are adjacent if they only differ in the orientation of a single triangle. Figure 8 shows the graph $\mathcal{G}_n$ for $n = 5$ with the arrangements represented by their corresponding zonotopal tilings.

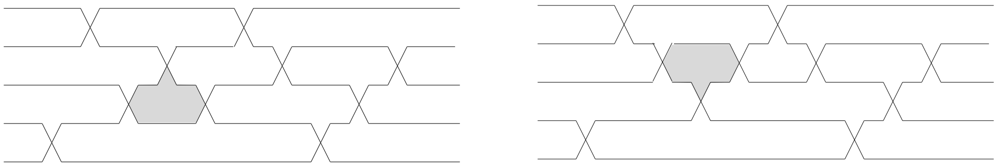


Figure 7: Elementary flip at the shaded triangle.

An arrangement $\mathcal{A}$ of n pseudolines has as many adjacent arrangements in $\mathcal{G}_n$ as it contains triangles. Felsner and Kriegel [4] have shown that a simple arrangement of order n contains at least $n-2$ triangles, hence, the minimum degree in $\mathcal{G}_n$ is $n-2$. From work of Roudneff [18] it follows that the maximum degree of $\mathcal{G}_n$, i.e., the maximal number of triangles in an arrangement of n pseudolines is $n(n-2)/3$.

Flips are nicely described in the different encodings of arrangements. In the encoding by zonotopal tilings the projection of a cube is replaced by the view of the cube from the other side. In the encoding by local sequences (page 7) an adjacent transposition of elements i and j is applied to the local sequence α_k of line l_k and similarly to local sequences α_i and α_j when the flip-triangle is confined by lines l_i, l_j and l_k.

In the representation by allowable sequences the transformation is not that obvious. The change is easy to describe if we recall that the allowable sequences of a marked arrangement $(\mathcal{A}, F)$ correspond to topological sortings of a directed graph $\overrightarrow{G}$. The change on $\overrightarrow{G}$ is again a local one.

We now introduce a further representation for simple marked arrangements of pseudolines. Let $(\mathcal{A}, F)$ be such an arrangement of n pseudolines. Consider the arrangement induced by a triple of $\{l_i, l_j, l_k\}$ of lines of $\mathcal{A}$, we assume $i < j < k$. Note that these three lines can induce two combinatorial different arrangements. Either the crossing of l_i and l_k is above l_j denote this by the symbol $-$ or the crossing is below l_j denoted by $+$. The shaded triangles of Figure 7 are a $-$ triangle on the left side and a $+$ triangle on the right side. With this convention a marked arrangement induces a *triangle-sign function* $f : \binom{[n]}{3} \to \{-, +\}$.

Consider a quadruple of pseudolines l_h, l_i, l_j, l_k of $\mathcal{A}$. These lines induce a marked arrangement of four pseudolines. Since there is only one arrangement of four lines with eight unbounded faces we easily enumerate the eight possible patterns of triangle-sign

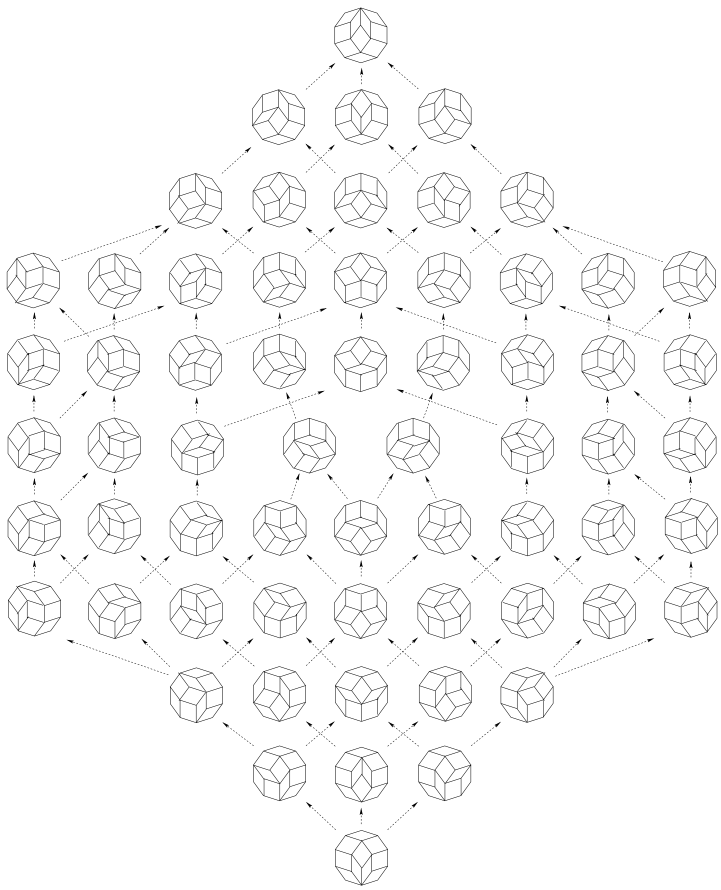


Figure 8: The graph $\mathcal{G}_5$ as diagram of the signotope order $S_3(n)$.

functions for $n = 4$. The following list shows them, the signs are given in lexicographical order of the three-sets, i.e, as { sign(1,2,3), sign(1,2,4), sign(1,3,4), sign(2,3,4) }.

$$\{-,-,-,-\},\ \{+,-,-,-\},\ \{+,+,-,-\},\ \{+,+,+,-\},$$
$$\{-,-,-,+\},\ \{-,-,+,+\},\ \{-,+,+,+\},\ \{+,+,+,+\}$$

From this we obtain a necessary condition for the functions f induced by an arrangement. For $A \in \binom{[n]}{4}$ and $1 \leq i \leq 4$ we let $A^{\lfloor i \rfloor}$ denote the set A minus the ith largest element of A, e.g., $\{2,4,5,9\}^{\lfloor 3 \rfloor} = \{2,4,9\}$. If f corresponds to an arrangement $\mathcal{A}$ then the restriction of $\mathcal{A}$ to the four lines of A has a pattern $\{\, \operatorname{sign} A^{\lfloor 4 \rfloor}, \operatorname{sign} A^{\lfloor 3 \rfloor}, \operatorname{sign} A^{\lfloor 2 \rfloor}, \operatorname{sign} A^{\lfloor 1 \rfloor} \,\}$ from the above list. Order the set $\{-,+\}$ of signs by $- \prec +$. Inspecting the above enumeration we see that the legal sign patterns are characterized by the following property: For every 4 element subset P of $[n]$ and all $1 \leq i < j < k \leq 4$ either $f(P^{\lfloor i \rfloor}) \preceq f(P^{\lfloor j \rfloor}) \preceq f(P^{\lfloor k \rfloor})$ or $f(P^{\lfloor i \rfloor}) \succeq f(P^{\lfloor j \rfloor}) \succeq f(P^{\lfloor k \rfloor})$. This property is called *monotonicity*.

Note that for $i < j$ and all $k \neq i, j$ we have $f(\{i,j,k\}) = -$ iff on line l_k the crossing with line l_i precedes the crossing with l_j, i.e., on the local sequence α_k the pair (i, j) is a non-inversion. Since local sequences encode marked arrangements, i.e., arrangements with the same local sequences are isomorphic, it follows, that the above defined sign patterns $f : \binom{[n]}{3} \to \{-,+\}$ also encode marked simple arrangements of pseudolines.

The next theorem whose proof will be given in the next section (page 17) shows that monotonicity already characterizes the sign pattern $f : \binom{[n]}{3} \to \{-,+\}$ which encode arrangements.

Theorem 7 *A function $f : \binom{[n]}{3} \to \{-,+\}$ is the triangle-sign function of a marked simple arrangements $\mathcal{A}_f$ of order n if and only if f is monotone on all 4-element subsets of $[n]$.*

It is a useful exercise to verify that monotonicity of the triangle-sign function induced by an arrangement is equivalent to the transitivity of non-inversions and of inversions of the local sequences α_k, hence, equivalent to α_k being a permutation. Combining these remarks with Theorem 7 we obtain.

Theorem 8 *A set $(\alpha_i)_{i=1..n}$ with α_i a permutation of $[n] \setminus \{i\}$ is the set of local sequences of a simple marked arrangement of order n if and only if for all $i < j < k$ the pairs $(i,j),(i,k),(j,k)$ are inversions in $\alpha_k, \alpha_j, \alpha_i$ or they are all three non-inversions.*

5 Signotopes and their Orders

In this section we generalize the concept of triangle-sign functions. Recall some notations. The set $[n] = \{1,..,n\}$ is equipped with the natural linear order. The set of r element subsets of $[n]$ is $\binom{[n]}{r}$. For $A \in \binom{[n]}{r}$ with $r \geq i$ we let $A^{\lfloor i \rfloor}$ denote the set A minus the ith largest element of A. The set $\{-,+\}$ of signs is ordered by $- \prec +$.

Definition 1 *For integers $1 \leq r \leq n$ a r–*signotope *on $[n]$ is a function α from the r elements subsets of $[n]$ to $\{-,+\}$ such that for every $r+1$ element subset P of $[n]$ and all $1 \leq i < j < k \leq r+1$ either $\alpha(P^{\lfloor i \rfloor}) \preceq \alpha(P^{\lfloor j \rfloor}) \preceq \alpha(P^{\lfloor k \rfloor})$ or $\alpha(P^{\lfloor i \rfloor}) \succeq \alpha(P^{\lfloor j \rfloor}) \succeq \alpha(P^{\lfloor k \rfloor})$. We refer to this property as* monotonicity.

*Let $S_r(n)$ denote the set of all r-signotopes on $[n]$ equipped with the order relation $\alpha \leq \beta$ if $\alpha(A) \preceq \beta(A)$ for all $A \in \binom{[n]}{r}$. Call $S_r(n)$ the r–*signotope order.

Easy observations:

(1) For $r = 1$ monotonicity is vacuous and $S_1(n)$ is just the lattice of subsets of $[n]$.

(2) For all $n \geq r \geq 1$ there is a unique minimal and a unique maximal element in $S_r(n)$, namely the constant $-$ and the constant $+$ function.

(3) The diagram of $S_r(r+1)$ is a $(2r+2)$-gon for all $r \geq 1$.

(4) There is a natural correspondence between 2-signotopes on $[n]$ and permutations of n. Permutation π and 2-signotope α correspond to each other if a pair (i,j) is an inversion of π iff $\alpha(i,j) = +$. For the proof that this is a bijection note that monotonicity of α corresponds to transitivity of the inversion relation and transitivity of the non-inversion relation for π. In the weak Bruhat order of the symmetric group the permutations of S_n are ordered by inclusion of their inversion sets. By the indicated correspondence between 2-signotopes and permutations $S_2(n)$ is isomorphic to the weak Bruhat order of S_n.

(5) For $r = 3$ the definitions reflect our observations for the encodings of marked simple arrangements of pseudolines made in the previous section. In view of Theorem 7 we see that $S_3(n)$ is nothing but an orientation of the graph $\mathcal{G}_n$, see Figure 8.

Manin and Schechtman [14] introduced signotopes, however, they defined a slightly different order relation on this set. The resulting structure corresponding to $S_r(n)$ is called the *higher Bruhat order* $B(n, r-1)$. The order relation $\leq_{HB}$ is defined as follows: Let α and β be two r-signotopes on groundset $[n]$ with $\alpha(A) = \beta(A)$ for all r-subsets A of $[n]$ but just one A^* where $\alpha(A^*) = -$ and $\beta(A^*) = +$ in this case we call the pair (α, β) a *single-step*. The order relation $\leq_{HB}$ is the transitive closure of the single-step relation, i.e, $\alpha \leq_{HB} \beta$ iff there is a sequence $\alpha = \alpha_0, \alpha_1, \dots \alpha_t = \beta$ such that for $i = 1, \dots, t$ the pair (α_{i-1}, α_i) is a single-step. Higher Bruhat orders were further studied by Voevodskij and Kapranov [21] and Ziegler [22]. In particular Ziegler shows that the higher Bruhat order $B(n, r-1)$ and the signotope order $S_r(n)$ are not equal in general. His example is $B(8, 3) \neq S_4(8)$. For $r \leq 2$ obviously $B(n, r-1) = S_r(n)$. Ziegler also shows that $B(n, n-k-1) = S_{n-k}(n)$ for $k \leq 3$. For $n \geq 7$ this leaves the question whether $B(n, 2) = S_3(n)$ open, in Section 6 we answer this in the affirmative.

It should also be mentioned that Ziegler [22] gives a geometric interpretation of signotopes. We give a different interpretation in Theorem 7 (dimension 2) and Section 7 (general dimension). In terms of the closely related theory of oriented matroids our geometric objects are the adjoints of the duals of Zieglers, see [5] for details.

5.1 New Signotopes from Old

In this section we give constructions of derived signotopes. Some of the constructions will be useful later.

(1) For a r-signotope α the *complement* $\overline{\alpha}$ is obtained by exchanging all signs of α. $\overline{\alpha}$ is a r-signotope.

(2) For a r-signotope α on a linearly ordered set X and $Y \subseteq X$ with $|X \setminus Y| \geq r$ define the *deletion* $\alpha{\uparrow_Y}$ to be the induced function on $\binom{X \setminus Y}{r}$. Deletion of Y gives a r-signotope on $X \setminus Y$.

(3) For a r-signotope α on a set X and $Y \subseteq X$ with $|Y| < r$ define the *contraction* $\alpha{\downarrow_Y}$ to be the function on $\binom{X \setminus Y}{r-|Y|}$ with $\alpha{\downarrow_Y}(A) = \alpha(A \cup Y)$. Contraction of Y gives a $(r - |Y|)$-signotope on $X \setminus Y$.

Let α be a r-signotope on $[n-1]$. A *one-element expansion* of α is a r-signotope β in $S_r(n)$ such that $\alpha = \beta{\uparrow_n}$.

Lemma 9 *The one-element expansions of $\alpha \in S_r(n-1)$ form a lattice in $S_r(n)$.*

Proof. Let β and β' be expansions of α. Let $\gamma : \binom{n}{r} \to \{-, +\}$ be the function with $\gamma(A) = +$ if $\beta(A) = +$ or $\beta'(A) = +$. We claim that γ is a r-signotope and hence the least upper bound for β and β'. For the claim note first that every $r+1$ element set P has $\beta(P^{\lfloor r+1 \rfloor}) = \beta'(P^{\lfloor r+1 \rfloor}) = \alpha(P^{\lfloor r+1 \rfloor})$. It follows that restricted to P the signotopes β and β' are comparable, i.e., the restrictions are comparable in $S_r(P)$. On P the function γ equals the larger of the restrictions of β and β'. Hence for all $(r+1)$-sets P monotonicity of γ is inherited from either β or β'. $\square$

We give geometric interpretations for the above constructions in the two-dimensional case, i.e., for $r = 3$. Proofs for the correspondences can be derived from Theorem 7. Let $(\mathcal{A}, F)$ be the marked arrangement with lines labeled by X corresponding to α. The arrangement corresponding to $\overline{\alpha}$ is $(\mathcal{A}, \overline{F})$. Delete the lines of Y from $\mathcal{A}$ to obtain the arrangement corresponding to $\alpha{\uparrow}_Y$. Let x be an element of X, the contraction $\alpha{\downarrow}_x$ is the local sequence α_x of line l_x in $\mathcal{A}$. One-element expansions of $\mathcal{A}$ are obtained by adding a pseudoline l_n compatible with $\mathcal{A}$ that enters the plane in F and leaves in $\overline{F}$. The new northface is the right one of the two faces obtained from F, i.e., the face above l_n. Lemma 9 has the intuitive explanation that with two expansion lines l_n and l'_n the right boundary of the region enclosed by $l_n \cup l'_n$ is again an expansion line.

Ziegler [22] proposes two constructions of $(r+1)$-signotopes from a r-signotope.

(4) For a r-signotope α on $[n]$ let $\partial\alpha : \binom{[n]}{r+1} \to \{-,+\}$ be defined by $\partial\alpha(P) = +$ iff $\alpha(P^{\lfloor 1 \rfloor}) = -$ and $\alpha(P^{\lfloor r+1 \rfloor}) = +$. The *boundary* $\partial\alpha$ of α is a $r+1$-signotope (see [22]).

(5) For a r-signotope α on $[n]$ let $\hat{\alpha} : \binom{[n+1]}{r+1} \to \{-,+\}$ be the unique function with $\hat{\alpha}{\uparrow}_{n+1} = \partial\alpha$ and $\hat{\alpha}{\downarrow}_{n+1} = \alpha$. The *extension* $\hat{\alpha}$ is a $r+1$-signotope (see [22]).

Very much in the spirit of these constructions we define:

(6) For a r-signotope α on $[n]$ let $\partial^*\alpha : \binom{[n]}{r+1} \to \{-,+\}$ be defined by $\partial^*\alpha(P) = +$ iff $\alpha(P^{\lfloor r+1 \rfloor}) = +$.

Claim. The *weak boundary* $\partial^*\alpha$ of α is a $r+1$-signotope.

Proof. Let Q be a $r+2$ element set and let $P = Q^{\lfloor r+2 \rfloor}$. Note that $Q^{\lfloor i \rfloor \lfloor r+1 \rfloor} = P^{\lfloor i \rfloor}$ for all $i < r+2$. Hence, $\partial^*\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor i \rfloor \lfloor r+1 \rfloor}) = \alpha(P^{\lfloor i \rfloor})$. It follows from the monotonicity of α that for $1 \leq i < j < k < r+2$ either $\partial^*\alpha(Q^{\lfloor i \rfloor}) \preceq \partial^*\alpha(Q^{\lfloor j \rfloor}) \preceq \partial^*\alpha(Q^{\lfloor k \rfloor})$ or $\partial^*\alpha(Q^{\lfloor i \rfloor}) \succeq \partial^*\alpha(Q^{\lfloor j \rfloor}) \succeq \partial^*\alpha(Q^{\lfloor k \rfloor})$.

If $k = r+2$ and $j < r+1$ we note that $Q^{\lfloor k \rfloor \lfloor r+1 \rfloor} = P^{\lfloor r+1 \rfloor}$ and the monotonicity condition of $\partial^*\alpha$ for indices i, j, k follows from the condition for $i, j, k-1$. Finally if $k = r+2$ and $j = r+1$ we find that $Q^{\lfloor j \rfloor \lfloor r+1 \rfloor} = Q^{\lfloor k \rfloor \lfloor r+1 \rfloor}$, hence, $\partial^*\alpha(Q^{\lfloor j \rfloor}) = \partial^*\alpha(Q^{\lfloor k \rfloor})$ and this implies the monotonicity condition of $\partial^*\alpha$ for i, j, k. $\triangle$

(7) For a r-signotope α on $[n]$ let $\tilde{\alpha} : \binom{[n+1]}{r+1} \to \{-,+\}$ be the unique function with $\tilde{\alpha}{\uparrow}_{n+1} = \partial^*\alpha$ and $\tilde{\alpha}{\downarrow}_{n+1} = \alpha$. The *weak extension* $\tilde{\alpha}$ is a $r+1$-signotope.

Remark. Weak extensions have been studied by Rambau [15], using the name expansion for these objects, he shows that $\alpha \to \tilde{\alpha}$ is an order preserving embedding from $B(n, r-1)$ to $B(n+1, r)$.

5.2 Maximum Chains of Signotopes

With a r-signotope α on $[n]$ associate a directed graph with vertices the $r-1$ element subsets of $[n]$ and edges $\to_\alpha$ defined by: For $P \in \binom{[n]}{r}$ and $1 \leq i < j \leq r$ if $\alpha(P) = +$ let $P^{\lfloor i \rfloor} \to_\alpha P^{\lfloor j \rfloor}$ and if $\alpha(P) = -$ let $P^{\lfloor j \rfloor} \to_\alpha P^{\lfloor i \rfloor}$.

Lemma 10 *For a r-signotope α on $[n]$ the graph with vertices $\binom{[n]}{r-1}$ and edges $\to_\alpha$ is acyclic.*

Proof. For $r = 2$ and arbitrary n relation $\rightarrow_\alpha$ is the transitive tournament corresponding to the permutation related to α.

For $n = r$ relation $\rightarrow_\alpha$ is a path traversing the $r-1$ subsets of $[r]$ in lexicographic order if $\alpha([r]) = -$ or in reverse-lexicographic order if $\alpha([r]) = +$.

Let $n > r > 2$ and let β be the signotope obtained from α by deletion of $\{n\}$. By induction $\rightarrow_\beta$ is acyclic on $\binom{[n-1]}{r-1}$. Let γ be the signotope obtained from α by contraction of $\{n\}$ and view $\rightarrow_\gamma$ as graph on the vertex set $Y = \{A \in \binom{[n]}{r-1} : n \in A\}$. By induction $\rightarrow_\gamma$ is acyclic.

Let $X^- = \{A \in \binom{[n-1]}{r-1} : \alpha(A \cup \{n\}) = -\}$ and $X^+ = \{A \in \binom{[n-1]}{r-1} : \alpha(A \cup \{n\}) = +\}$. The three sets X^-, X^+, Y partition the $r-1$ element subsets of $[n]$, moreover, the subgraph of $\rightarrow_\alpha$ induced by each of the three blocks of the partition is acyclic: It agrees with the subgraph induced by $\rightarrow_\beta$ in case of X^- and X^+ and with the subgraph induced by $\rightarrow_\gamma$ in the case of Y. Now consider the edges of $\rightarrow_\alpha$ between the blocks. By definition of X^- all edges with one end in X^- and the other end in Y are oriented from X^- to Y. Also all edges with one end in X^+ and the other end in Y are oriented from Y to X^+. Therefore, the acyclicity of $\rightarrow_\alpha$ is readily established if we show that all edges with one end in X^- and the other end in X^+ are oriented from X^- to X^+. This follows from the next claim:

Claim. $A \in X^-$ and $B \rightarrow_\beta A$ implies $B \in X^-$,i.e., X^- is an ideal in the partial order defined by the transitive closure of $\rightarrow_\beta$.

From $B \rightarrow_\beta A$ it follows that $P = A \cup B$ is a r subset $[n]$. Let i, j be such that $B = P^{\lfloor i \rfloor}$ and $A = P^{\lfloor j \rfloor}$. For $Q = P \cup \{n\}$ we then obtain $Q^{\lfloor i \rfloor} = B \cup \{n\}$, $Q^{\lfloor j \rfloor} = A \cup \{n\}$ and $Q^{\lfloor r+1 \rfloor} = A \cup B = P$. We use the monotonicity of α on Q and distinguish two cases:
(1) If $i < j$ then $B \rightarrow_\beta A$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = +$. From $A \in X^-$ it follows that $\alpha(Q^{\lfloor j \rfloor}) = \alpha(A \cup \{n\}) = -$. Monotonicity forces $\alpha(Q^{\lfloor i \rfloor}) = \alpha(B \cup \{n\}) = -$, i.e., $B \in X^-$.
(2) If $j < i$ then $B \rightarrow_\beta A$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = -$. From $A \in X^-$ it follows that $\alpha(Q^{\lfloor j \rfloor}) = \alpha(A \cup \{n\}) = -$. Monotonicity forces $\alpha(Q^{\lfloor i \rfloor}) = \alpha(B \cup \{n\}) = -$, i.e., $B \in X^-$.

□

Proposition 11 *For a r-signotope α on $[n]$ there exist a chain $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{r-1}}$ of $(r-1)$-signotopes in $S_{r-1}(n)$ such that for $t = 1, \ldots, \binom{n}{r-1}$ the signs of β_{t-1} and β_t differ at only one $(r-1)$-set A_t.*

Proof. Let $A_1, A_2, \ldots, A_{\binom{n}{r-1}}$ be a topological sorting of $\rightarrow_\alpha$ and define $\beta_t(A) = -$ if $A = A_i$ for some $i > t$ and $\beta_t(A) = +$ if $A = A_i$ for some $i \leq t$. To prove the lemma it remains to show that each β_t is a $(r-1)$-signotope.

For every r element set P and all i, j, k with $1 \leq i < j < k \leq r$ we either have $P^{\lfloor i \rfloor} \rightarrow_\alpha P^{\lfloor j \rfloor} \rightarrow_\alpha P^{\lfloor k \rfloor}$ or $P^{\lfloor k \rfloor} \rightarrow_\alpha P^{\lfloor j \rfloor} \rightarrow_\alpha P^{\lfloor i \rfloor}$. In the first case we have $\beta_t(P^{\lfloor i \rfloor}) \succeq \beta_t(P^{\lfloor j \rfloor}) \succeq \beta_t(P^{\lfloor k \rfloor})$ for all t and in the second case $\beta_t(P^{\lfloor i \rfloor}) \preceq \beta_t(P^{\lfloor j \rfloor}) \preceq \beta_t(P^{\lfloor k \rfloor})$ for all t. This proves monotonicity for β_t. □

Based on this lemma we now give the proof of Theorem 7.

Proof. [Theorem 7] Let α be a 3-signotope, i.e., a function $\alpha : \binom{[n]}{3} \rightarrow \{-, +\}$ obeying monotonicity on 4-subsets of $[n]$. From Proposition 11 we obtain a chain $\beta_0, \ldots, \beta_{\binom{n}{2}}$ in $S_2(n)$ corresponding to α. Each β_i encodes a permutation of $[n]$. β_0 is the identity and $\beta_{\binom{n}{2}}$ the reverse permutation. Moreover, two permutations β_t and β_{t+1} differ in a single sign where β_t is $-$ and β_{t+1} is $+$. Hence, there is a single pair (i,j) being a non-inversion of β_t but an inversion in β_{t+1}. This pair is an adjacent pair of both permutations. This shows that $\beta_0, \ldots, \beta_{\binom{n}{2}}$ is a simple allowable sequence. From Theorem 2 we obtain that via $\beta_0, \ldots, \beta_{\binom{n}{2}}$ signotope α encodes an arrangement $\mathcal{A}$. From the construction it is easily verified that the triangle induced by lines l_i, l_j, l_k in $\mathcal{A}$ is a $+$ triangle exactly when $\alpha(ijk) = +$. This proves the bijection. $\square$

The next lemma can be seen as a generalization of Theorem 2, it shows that saturated chains of $r-1$-signotopes can be used to encode r-signotopes.

Proposition 12 *Let $1 < r \leq n$ and $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{r-1}}$ be a maximum chain in $S_{r-1}(n)$. For $t = 1, \ldots, \binom{n}{r-1}$ let A_t be the unique $(r-1)$-set with $\beta_{t-1}(A_t) = -$ and $\beta_t(A_t) = +$. There exists a r-signotope α on $[n]$ so that $A_1, \ldots, A_{\binom{n}{r-1}}$ is a topological sorting of $\to_\alpha$.*

Proof. For a set $A \in \binom{[n]}{r-1}$ let $\rho(A)$ be the index of A in the list $A_1, \ldots, A_{\binom{n}{r-1}}$. Note that monotonicity of the β_t's implies that for all $D \in \binom{[n]}{r}$ either $\rho(D^{\lfloor 1 \rfloor}) < \rho(D^{\lfloor 2 \rfloor}) < \ldots < \rho(D^{\lfloor r \rfloor})$ or $\rho(D^{\lfloor 1 \rfloor}) > \rho(D^{\lfloor 2 \rfloor}) > \ldots > \rho(D^{\lfloor r \rfloor})$. In the first case let $\alpha(D) = +$ in the second case $\alpha(D) = -$. We have to show that α is a r-signotope, i.e., that α is monotone at $r+1$ sets. Let $Q \in \binom{[n]}{r+1}$ and consider indices $1 \leq i < j < k \leq r+1$. Suppose $\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor k \rfloor}) = +$. Let $Q^{\lfloor i,j \rfloor}$ denote the set Q minus the ith largest and the jth largest element of Q, e.g., $\{1,2,5,7,8\}^{\lfloor 2,3 \rfloor} = \{1,7,8\}$. From $\alpha(Q^{\lfloor i \rfloor}) = +$ we obtain $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor i,k \rfloor})$. From $\alpha(Q^{\lfloor k \rfloor}) = +$ we obtain that $\rho(Q^{\lfloor i,k \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$. Hence $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$ which implies $\alpha(Q^{\lfloor j \rfloor}) = +$ as required. The argument for $\alpha(Q^{\lfloor i \rfloor}) = \alpha(Q^{\lfloor k \rfloor}) = -$ is symmetric. It is obvious that $A_1, \ldots, A_{\binom{[n]}{r-1}}$ is a topological sorting for the relation $\to_\alpha$. $\square$

It is tempting to think that all maximal chains in $S_r(n)$ are chains of length $\binom{n}{r} + 1$. This, however, means that single-step inclusion and inclusion for signotopes are equal, i.e., that $B(n, r-1) = S_r(n)$. As already mentioned Ziegler [22] has shown that $B(8,3) \neq S_4(8)$.

The next lemma shows that at least every element of $S_r(n)$ is contained in a chain of maximum length, i.e., a chain in which each pair of consecutive elements form a single-step.

Lemma 13 *Every element of $S_r(n)$ is contained in a chain of length $\binom{n}{r} + 1$.*

Proof. Let $\alpha \in S_r(n)$ and consider the weak boundary $\partial^*\alpha$ of α. This defines the directed graph $\to_{\partial^*\alpha}$ on $\binom{[n]}{r}$. Note that $A \to_{\partial^*\alpha} B$ implies $\alpha(A) \preceq \alpha(B)$, i.e., the sets A with $\alpha(A) = -$ form an ideal in the order corresponding to $\to_{\partial^*\alpha}$. Let $A_1, A_2 \ldots, A_{\binom{n}{r}}$ be a linear extension of this order such that there is a t with $\alpha(A_i) = -$ for all $i \leq t$ and $\alpha(A_i) = +$ for all $i > t$. Define the sequence β_j of r-signotopes as in the proof of Proposition 11. The sequence of complements $\overline{\beta_j}$ is a chain of r signotopes with $\overline{\beta_t} = \alpha$. $\square$

Proposition 11 implies that the mapping Π from maximum chains in $S_{r-1}(n)$ to elements of $S_r(n)$ described in the proof of Proposition 12 is surjective. The two lemmas also imply that the preimage of α under Π is a set of maximum chains in $S_{r-1}(n)$ of the same size as the set of topological sortings of $\rightarrow_\alpha$, i.e., linear extensions of the transitive closure of $\rightarrow_\alpha$. We can even say more about this preimage.

Call two maximum chains in $S_{r-1}(n)$ *swap-equivalent* if one of them corresponds to the list $A_1, \ldots, A_{\binom{n}{r-1}}$ of $(r-1)$-sets and the list of the other chain differs only by an adjacent transposition, i.e., is of the form $A_1, .., A_{t-1}, A_{t+1}, A_t, A_{t+2}, .., A_{\binom{n}{r-1}}$ for some t.

Lemma 14 *For $r \geq 3$ the set of maximum chains in $S_{r-1}(n)$ mapped by Π to $\alpha \in S_r(n)$ is a complete swap-equivalence class.*

Proof. The proof follows from two facts.

First, it is possible to transform any topological sorting of a directed acyclic graph into any other by a sequence of adjacent transpositions, i.e., reversals of adjacent pairs of unrelated vertices. Therefore, the preimage of α is contained in one swap-equivalence class of chains in $S_{r-1}(n)$.

Now assume $r \geq 3$ that $A_1, \ldots, A_{\binom{n}{r-1}}$ is a topological sorting of $\rightarrow_\alpha$ and let list $A_1, .., A_{t-1}, A_{t+1}, A_t, A_{t+2}, .., A_{\binom{n}{r-1}}$ correspond to a maximum chain of $S_{r-1}(n)$. We claim that A_t and A_{t+1} are unrelated in $\rightarrow_\alpha$. Otherwise $P = A_t \cup A_{t+1}$ is a r-set and monotonicity only allows the signs of A_t and A_{t+1} to be changed in a row if there is an index i so that one of the two sets is $P^{\lfloor i \rfloor}$ and the other is $P^{\lfloor i+1 \rfloor}$. Consider sign and location in the list of a set of $P^{\lfloor j \rfloor}$, $j \neq i, i+1$, to obtain a contradiction to monotonicity. Hence, A_t and A_{t+1} are unrelated in $\rightarrow_\alpha$ and the second list also corresponds to a topological sorting of $\rightarrow_\alpha$.

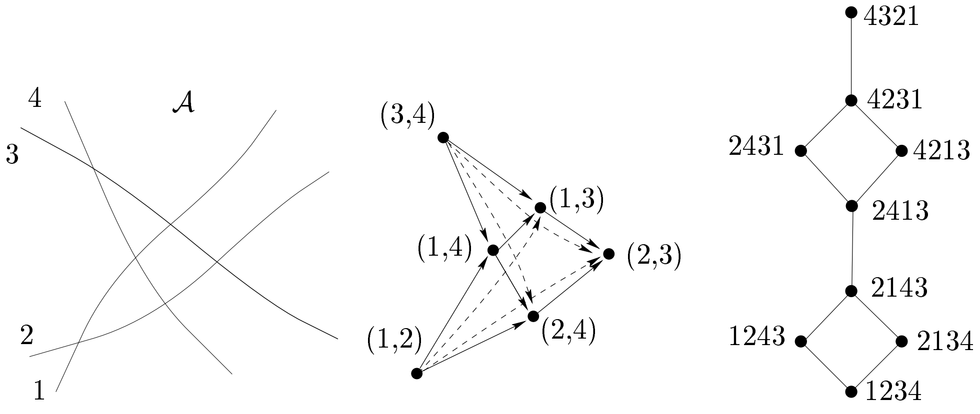
□

These considerations about swap-equivalence of the Π preimages can be rephrased as follows: Given a r-signotope α the set of $(r-1)$-signotopes on maximum chains of $S_{r-1}(n)$ mapped to α by Π together with the edges (single-steps) used by these chains forms a lattice isomorphic to the lattice of antichains of the transitive closure of $\rightarrow_\alpha$ (An example of this is given in Example B below). In particular this shows that the orders $S_r(n)$ have a local lattice structure. What about global lattice structure? It is known that $S_r(n)$ is a lattice for $r \leq 2$. Ziegler [22] has shown that $S_r(n)$ is a lattice for $r \geq n-2$ and that $S_3(6)$ is not a lattice.

Example B. Let $\mathcal{A}$ (as shown in Figure 9(a)) be the arrangement corresponding to a 3-signotope α. The directed graph $\rightarrow_\alpha$ is shown to in Figure 9(b). Note that we met the transitive reduction of this graph (non-dashed edges) several times as $\overrightarrow{G}$ (see Lemma 1, Subsection 3.1 and Lemma 4). The maximum chains of 2-signotopes mapped by Π to α are the allowable sequences of $\mathcal{A}$. In Subsection 3.1 we have seen that they correspond bijectively to topological sortings of $\overrightarrow{G}$. It follows that the suborder of the weak Bruhat order induced by permutations π appearing in allowable sequences of $\mathcal{A}$ is a distributive lattice (see Figure 9(c)).

6 $S_3(n) = B(n, 2)$

In this section we show that the single-step order and the inclusion order on 3-signotopes is the same. To prove the result we show that for any two signotopes $\alpha < \beta$ there is a signotope α' such that (α, α') is a single step and $\alpha' \leq \beta$. Iterating this argument we find a single-step chain $\alpha = \alpha_0, \alpha_1 \ldots \alpha_t = \beta$ connecting α and β.



(a) The arrangement. (b) The graph $\rightarrow_\alpha$. (c) The lattice.

Figure 9: Illustrations for Example B.

Given $\alpha < \beta$ we call a triple A with $\alpha(A) \neq \beta(A)$ a *difference triple.* From $\alpha < \beta$ it follows that $\alpha(A) = -$ and $\beta(A) = +$ for every difference triple A. On all other triples the signs of α and β are equal. Let $\mathcal{A}$ be a marked arrangement of pseudolines with signotope α. We will show that in $\mathcal{A}$ there is a triangular face F such that the three lines bounding F correspond to a difference triple, call such a triple *elementary.* Given such a triangle F we can apply an elementary flip to obtain an arrangement $\mathcal{A}'$ such that the signotope α' of $\mathcal{A}'$ has the desired properties, i.e., (α, α') is a single step and $\alpha' \leq \beta$.

For $i < j < k$ the *basis* of the triple is the piece of line l_j between the intersections with lines l_i and l_k. Clearly an elementary triple has a basis which is an edge of the graph G of $\mathcal{A}$. Call the basis of a triple which is an edge in the graph of $\mathcal{A}$ an *elementary basis.*

Let α_i denote the local sequence of line l_i in $\mathcal{A}$, i.e., the permutation of $[n] \setminus \{i\}$ recording the order in which line l_i is crossed by other lines. For every triple $\{i_1, i_2, i_3\}$ with $i_1 < i_2$ recall the following equivalence.

$$\alpha(i_1, i_2, i_3) = - \iff (i_1, i_2) \text{ is a non-inversion of } \alpha_{i_3}. \qquad (*)$$

Lemma 15 *There is a difference triple A with an elementary basis.*

Proof. Among all difference triples $\{i, j, k\}$ with $i < j < k$ choose one of minimal *width* $k - i$. Let this triple be $A = \{i, j, k\}$. From $\alpha(A) = -$ and $(*)$ we see that on line l_j the intersection with line l_i comes before the intersection with line l_k.

Claim A. For every x between i and k in the local sequence α_j either $x < i$ or $x > k$.

Proof. Suppose x with $i < x < k$ is between i and k on α_j denoted $i \prec x \prec k$. Now consider the order of i, x, k on β_j. From $\beta(i, j, k) = +$ and $(*)$ we obtain $k \prec i$ on β_j.

If $x \prec i$ on β_j we obtain from $(*)$ that $\{i, j, x\}$ is a difference triple. If $i < x < j$ the width of this triangle is $j - i$, otherwise, if $i < j < x < k$ the width is $x - i$. In both cases this contradicts our choice of $\{i, j, k\}$ as a difference triangle of minimal width.

If $x \not\prec i$ then $k \prec x$ on β_j. In this case $\{x, j, k\}$ is a difference triangle of width either $k - x$ or $k - j$. Again this contradicts our choice of $\{i, j, k\}$ as a difference triangle of minimal width. $\triangle$

Claim B. There exists an elementary basis on the segment of l_j between the crossings with l_i and l_k.

Proof. If i and k are adjacent elements of α_j we are done. Otherwise, by Claim A we can partition the elements between i and k into elements x with $x < i$ and elements y with $y > k$. For an x we note that from $i \prec x$ on α_j we obtain $\alpha(x, i, j) = +$. Hence, $\beta(x, i, j) = +$, i.e., $i \prec x$ on β_j. Since $k \prec i$ on β_j the triple $\{x, j, k\}$ is a difference triple. For an element y we obtain by an analogous argument that $\{i, j, y\}$ is a difference triple.

If the element to the right of i on α_j is a y the difference triple $\{i, j, y\}$ has an elementary basis and we are done. If the element to the left of k on α_j is a x the difference triple $\{x, j, k\}$ has an elementary basis and we are again done. If both these conditions fail then we find an adjacent pair (x, y) with $x < i$ and $y > k$ on α_j. On α_j we have $i \prec x \prec y \prec k$ while by the above considerations $y \prec k \prec i \prec x$ on β_j. This shows that $\{x, j, y\}$ is a difference triple. And it obviously has an elementary basis. $\triangle$

This completes the proof of the lemma. $\square$

We now consider the wiring diagram of $\mathcal{A}$. For an edge e of $\mathcal{A}$ we say e is on wire w if the horizontal portion of e is on wire w. Let $\{i, j, k\}$ be a difference triple with elementary basis such that the basis of $\{i, j, k\}$ is on the highest wire that contains elementary bases in the diagram.

Lemma 16 *The triple $\{i, j, k\}$ defined in the preceding paragraph is an elementary triple.*

Proof. Since the basis of $\{i, j, k\}$ is elementary any line l_x crossing the triangle of the three lines l_i, l_j, l_k enters the triangle through line l_i and leaves the triangle through line l_k. It follows that $i < x < k$.

If $i < x < j$ then $\alpha(i, x, j) = + = \beta(i, x, j)$. On β_i we therefore have $j \prec x$. With $k \prec j$ on β_i this shows that $\{i, x, k\}$ is a difference triple. Similarly, if $j < x < k$ then $\alpha(j, x, k) = + = \beta(j, x, k)$. Considering β_k we again obtain that $\{i, x, k\}$ is a difference triple.

Let F be the face of $\mathcal{A}$ above the edge on l_j corresponding to the basis of $\{i, j, k\}$. The boundary of F consists of the basis b and edges $e_0, \ldots, e_t$ in clockwise order. Note that in the wiring diagram of $\mathcal{A}$ the edges $e_0, \ldots, e_t$ are all on the wire above the wire of b.

Claim C. If $t > 1$ one of the edges $e_1, \ldots, e_{t-1}$ is an elementary basis.

If $t > 1$ we obtain a contradiction to the choice of the triple $\{i, j, k\}$ from Claim C. Therefore $t = 1$ and the face F is the triangle corresponding to the triple $\{i, j, k\}$. This shows that $\{i, j, k\}$ is an elementary triple. To prove the lemma it thus suffices to prove the claim.

Proof. If $t = 2$ let l_x be the supporting line of e_1. From the above considerations we know that $\{i, x, k\}$ is a difference triple. The basis of the triple is edge e_1 hence elementary.

If $t > 2$ let l_{x_s} be the supporting line of edge e_s for $s = 1, \ldots, t-1$. For $s = 1, \ldots, t-2$ note that $i \prec x_{s+1} \prec k$ on α_{x_s} and $k \prec i$ on β_{x_s}. Therefore, at least one of $\{i, x_s, x_{s+1}\}$ and $\{x_s, x_{s+1}, k\}$ is a difference triple. Let p_s be the vertex of $e_s \cap e_{s+1}$. Color p_s red if $\{i, x_s, x_{s+1}\}$ is a difference triple and blue otherwise.

If p_1 is a red edge then e_1 is an elementary basis. If p_{t-2} is a blue edge then e_{t-1} is an elementary basis. Now assume that p_1 is blue and p_{t-2} red then there is some s such that p_s is blue and p_{s+1} is red. Note that $x_s < x_{s+1} < x_{s+2}$ and $x_s \prec x_{s+2}$ on $\alpha_{x_{s+1}}$. From the definitions of red and blue vertices we obtain $x_{s+2} \prec k \prec i \prec x_s$ on $\beta_{x_{s+1}}$. Hence, $\{x_s, x_{s+1}, x_{s+2}\}$ is a difference triple with elementary basis e_{s+1}. This proves the claim.

As noted before this completes the proof of the lemma. □

Lemma 15 and Lemma 16 prove our theorem.

Theorem 17 *$S_3(n) = B(n, 2)$ for all n, i.e., single step-order and inclusion order on 3-signotopes are equal.*

As consequence of the theorem we obtain a strengthening of Lemma 13 for 3-signotopes.

Corollary 18 *Let α and β be two elements of $S_3(n)$ with $\alpha < \beta$ then there is a chain of length $\binom{n}{3} + 1$ in $S_3(n)$ containing both.*

7 Geometric Interpretations for Signotopes

Ziegler [22] shows that there is a natural bijection between the uniform extension poset on the set of single element extensions of a cyclic hyperplane arrangement $\mathbf{X}_c^{n,d}$ in $\mathbb{R}^d$ and the higher Bruhat order $B(n, n-d-1)$. Felsner and Ziegler [5] note that from oriented matroid duality $B(n, n-d-1)$ has another geometric representation as the set of 1–element liftings of $\mathbf{X}_c^{n,n-d}$. These liftings correspond to certain affine arrangements of pseudohyperplanes in $\mathbb{R}^{n-d-1}$. In this section we make the connection with the second class of geometric objects explicit, that is, we characterize a class of arrangements of pseudohyperplanes in $\mathbb{R}^d$ corresponding to signotopes $\alpha \in S_{d+1}(n)$.

A *pseudohyperplane* H in $\mathbb{R}^d$ is a homeomorph of a hyperplane such that the two connected components of $\mathbb{R}^d \setminus H$ are homeomorphic to the d-ball. A set $\{H_1, \ldots, H_n\}$ of pseudohyperplane in $\mathbb{R}^d$ is an *arrangement of pseudohyperplanes* if for all j the set $\{H_i \cap H_j : i = 1, ..., j-1, j+1, ..., n\}$ is an arrangement of $n-1$ pseudohyperplanes in $H_j \cong \mathbb{R}^{d-1}$. We say d-arrangement to abbreviate for 'arrangement of pseudohyperplanes in $\mathbb{R}^d$'. A d-arrangement is *simple* if any set of $d+1$ pseudohyperplanes has empty intersection.

So far we have discussed arrangements of pseudolines which had been normalized by a marking face F and a specific labeling of the lines (increasing on a clockwise walk from $\overline{F}$ to F at infinity). For all arrangements of this section we assume that they are simple and that they are embedded in $\mathbb{R}^d$ in a normalized way as described in the next paragraph.

For $i = 1, ..., d-1$ let I_i be the $d-i$ dimensional space at infinity obtained by setting the last i coordinates equal to $-\infty$, i.e., with $x_d = -\infty, x_{d-1} = -\infty, \ldots, x_{d-i+1} = -\infty$ (if the reader feels uncomfortable with these 'spaces at infinity' he may assume that the arrangement is embedded in a d-dimensional unit hypercube and consider I_i as the side of this cube obtained by setting the last i coordinates equal to 0). We demand that the d-arrangement induces a $(d-i)$-arrangement with the same number of pseudohyperplanes on I_i. Moreover, the pseudohyperplanes are labeled by increasing x_1 coordinate at their intersection with I_{d-1}. We call an arrangement with these properties *normal*.

The intersection of every set of $d-1$ pseudohyperplanes of an arrangement $\mathcal{A}$ determines a line of the arrangement. If the arrangement is normal we consider these lines and the edges they support as oriented away from I_1 with expressions like 'behind', 'before', 'precedes' we refer to this orientation. A normal d-arrangement induces a sign function $f : \binom{[n]}{d+1} \to \{-,+\}$ by the following rule: Given $i_1 < i_2 < \ldots < i_{d+1}$ let $f(i_1, ..., i_{d+1}) = -$ iff on the intersection line of the pseudohyperplanes $h_{i_3}, ..., h_{i_{d+1}}$ the intersection with h_{i_1} comes before the intersection with h_{i_2}.

Hurrying ahead we define: A normal d-arrangement $\mathcal{A}$ is called a C_d-arrangement if the normal $(d-1)$-arrangement induced by $\mathcal{A}$ on I_1 corresponds to the minimal signotope $\alpha_0 \in S_d(n)$, the minimal signotope α_0 is the signotope with all signs $-$. It should be remarked that the arrangement corresponding to $\alpha_0 \in S_d(n)$ is the cyclic arrangement $\mathbf{X}_c^{n,d}$.

Theorem 19 *There is a bijection between C_d-arrangements with n pseudohyperplanes and signotopes in $S_{d+1}(n)$. The signotope corresponding to a C_d-arrangement $\mathcal{A}$ is the sign function of $\mathcal{A}$ as defined above.*

Proof. We use induction on d. Theorem 7 covers the case $d = 2$ and may serve as basis for the induction. For the induction step we also use that if (α, α') is a single step in $S_d(n)$ then the associated C_{d-1}-arrangements $\mathcal{A}$ and $\mathcal{A}'$ are related by a flip at a simplicial cell bounded by the hyperplanes corresponding to the unique d element set A with $\alpha(A) = -$ and $\alpha'(A) = +$.

For d dimensions we first consider normal arrangements of $d+1$ pseudohyperplanes labeled by the elements of $A = [d+1]$. Such an arrangement $\mathcal{A}$ has just one bounded cell which is a (pseudo)simplex. The set of bounded edges of $\mathcal{A}$ forms the skeleton graph of the simplex, i.e., a complete graph K_{d+1}. The vertex of this graph determined by the intersection of the pseudohyperplanes in $A^{\lfloor i \rfloor}$ will itself be denoted $A^{\lfloor i \rfloor}$.

Claim A. The orientation of lines induces an acyclic orientation on the graph of bounded edges of $\mathcal{A}$.

Let $A^{\lfloor i \rfloor}$, $A^{\lfloor j \rfloor}$ and $A^{\lfloor k \rfloor}$ be any three vertices of the graph. The three lines $A^{\lfloor i,j \rfloor}$, $A^{\lfloor i,k \rfloor}$, $A^{\lfloor j,k \rfloor}$ are supported by the plane $A^{\lfloor i,j,k \rfloor}$ which is a homeomorph of a disk D. The intersection of $A^{\lfloor i,j,k \rfloor}$ with I_1 corresponds to an interval on the boundary of D in which all three lines begin. Since lines and edges are oriented away from I_1 the orientation of the triangle with vertices $A^{\lfloor i \rfloor}$, $A^{\lfloor j \rfloor}$ and $A^{\lfloor k \rfloor}$ is acyclic. An orientation of the complete graph K_{d+1} with all triangles acyclic is acyclic. △

Claim B. For C_d-arrangements the orientation of K_{d+1} is either the transitive closure of $A^{\lfloor 1 \rfloor} \to A^{\lfloor 2 \rfloor} \to \ldots \to A^{\lfloor d+1 \rfloor}$ in which case the sign of the arrangement is $+$ or of $A^{\lfloor d+1 \rfloor} \to A^{\lfloor d \rfloor} \to \ldots \to A^{\lfloor 1 \rfloor}$ in which case the sign is $-$.

Since the graph is acyclic we can sweep arrangement $\mathcal{A}$ starting with I_1. Meaning, we find a sequence $s_0, s_1, \ldots, s_{d+1}$ of pseudohyperplanes such that they all share the pseudospere at infinity with $I_1 = s_0$ and between any two consecutive pseudohyperplanes s_i, s_{i+1} there is exactly one vertex of the arrangement. Since the arrangement is a C_d arrangement we know that the first vertex to be swept corresponds to a simplicial cell in the arrangement of the minimal element of $S_d(d+1)$. This arrangement has only two simplicial cells one bounded by the pseudohyperplanes in $A^{\lfloor 1 \rfloor}$ and the other by those in

$A^{\lfloor d+1 \rfloor}$. The arrangement induced on s_1 is thus obtained by flipping one of these cells. After this first flip one of the two branches of $S_d(d+1)$ which as we recall has the structure of $(2d+2)$-gon is determined. Playing with the bijection between the arrangements induced on the sweep-planes s_i and the corresponding signotopes we see that the sweep has to follow the choosen branch of $S_d(d+1)$. This results in one of the above orderings of the vertices of K_{d+1}. The statement about the sign of the arrangement follows from considering the orientation of the edge between $A^{\lfloor 1 \rfloor}$ and $A^{\lfloor 2 \rfloor}$. $\triangle$

From the previous claim we obtain generalized criteria for determining the sign of a $d+1$ element set A in a C_d-arrangement. Consider any two vertices $A^{\lfloor i \rfloor}$ and $A^{\lfloor j \rfloor}$ with $i < j$ of the arrangement induced by A. The sign of A is $+$ iff $A^{\lfloor i \rfloor}$ precedes $A^{\lfloor j \rfloor}$ on the line $A^{\lfloor i,j \rfloor}$.

With this at hand we can show monotonicity for the sign functions of a C_d-arrangement $\mathcal{A}$ with more then $d+1$ pseudohyperplanes: Let α be the sign function corresponding to $\mathcal{A}$ and let P be a $d+2$ element set of pseudohyperplanes. For $1 \leq i < j < k \leq d+2$ we have to show that $\alpha(P^{\lfloor i \rfloor}) = +$ and $\alpha(P^{\lfloor j \rfloor}) = -$ implies $\alpha(P^{\lfloor k \rfloor}) = -$ and $\alpha(P^{\lfloor i \rfloor}) = -$ and $\alpha(P^{\lfloor j \rfloor}) = +$ implies $\alpha(P^{\lfloor k \rfloor}) = +$. We only prove the first implication the other being similar. From $\alpha(P^{\lfloor i \rfloor}) = +$ we obtain that vertex $P^{\lfloor i,j \rfloor}$ precedes vertex $P^{\lfloor i,k \rfloor}$ on the line $P^{\lfloor i,j,k \rfloor}$. From $\alpha(P^{\lfloor j \rfloor}) = -$ we obtain that vertex $P^{\lfloor j,k \rfloor}$ precedes vertex $P^{\lfloor i,j \rfloor}$ on the line $P^{\lfloor i,j,k \rfloor}$. From transitivity $P^{\lfloor j,k \rfloor}$ precedes $P^{\lfloor i,k \rfloor}$ and hence $\alpha(P^{\lfloor k \rfloor}) = -$.

So far we have seen that the sign function of a C_d-arrangement of n pseudohyperplanes is a signotope in $S_{d+1}(n)$. Given a C_d-arrangement with signotope α the next thing to prove is the correspondence between simplicial cells in $\mathcal{A}$ and single steps involving α. For the first half note that a simplicial cell of $\mathcal{A}$ can be flipped leading to $\mathcal{A}'$. Since $\mathcal{A}'$ is a C_d-arrangement it has a corresponding signotope α'. Now compare the ordering of vertices on lines of $\mathcal{A}$ and $\mathcal{A}'$ to see that α and α' differ in just one sign. On the other hand, if α and α' only differ in the sign A then it is possible to show that for all i, j in $\mathcal{A}$ the two vertices $A^{\lfloor i \rfloor}$ and $A^{\lfloor j \rfloor}$ are adjacent along the line $A^{\lfloor i,j \rfloor}$. Therefore, the simplicial cell corresponding to A is not penetrated by any further pseudohyperplane.

Given any C_d-arrangement $\mathcal{A}$ we may move to any other C_d-arrangement (of same dimension with same number of pseudohyperplanes) using flips. This is due to the connectedness of $S_{d+1}(n)$ (Lemma 13). Therefore, the missing link for a complete proof is the existence of a single C_d-arrangement with n pseudohyperplanes. This can be provided by checking that the cyclic arrangements have the required properties. Here we indicate a construction which is similar in spirit to the construction of wiring diagrams as representatives of pseudolinearrangements:

Given $\alpha \in S_{d+1}(n)$ choose a chain $\beta_0 < \beta_1 < \ldots < \beta_{\binom{n}{d}}$ in $S_d(n)$ mapped by Π to α. By induction β_0 corresponds to a C_{d-1}-arrangement $\mathcal{B}_0$ of n pseudohyperplanes. Let A be the unique d-set with different sign in β_0 and β_1. We know that the pseudohyperplanes from A bound a simplicial cell in $\mathcal{B}_0$. Construct $\mathcal{B}_1$ by applying a simplicial-flip to this cell in $\mathcal{B}_0$. Repeate this to obtain a sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{\binom{n}{d}}$ of arrangements in $\mathbb{R}^{d-1}$ corresponding to $\beta_0, \beta_1, \ldots, \beta_{\binom{n}{d}}$. Introduce a new dimension x_d and place arrangement $\mathcal{B}_i$ in the affine $(d-1)$-dimensional space at $x_d = i$. The pseudohyperplane h_i of the arrangement $\mathcal{A}$ corresponding to α is obtained by properly interpolating between the ith pseudohyperplane in $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ for $j = 0, \ldots, \binom{n}{d} - 1$ and extending the ith pseudohyperplane of $\mathcal{B}_0$ and $\mathcal{B}_{\binom{n}{d}}$ to $x_d = -\infty$ and $x_d = \infty$ respectively. $\square$

Note that as consequence of Theorem 19 C_d-arrangements can be swept. This means that starting with a sweep-pseudohyperplane I_1 and always choosing a non-blocked vertex for the next step of the sweep the sweep never gets stuck. While this property is clearly shared by realizable arrangements there are reasons to believe that most higher dimensional arrangements can not be swept (e.g. the examples constructed by Richter-Gebert [17]). In fact it is not even known whether every d-arrangement of $n > d$ pseudo-hyperplanes contains a simplicial cell.

It would be desirable to extend the class of arrangements with at least some of the good properties of C_d-arrangements. One possible generalization would be to allow that the arrangement induced on I_1 is different, e.g., a different C-arrangement. On the combinatorial side this corresponds to a reorientation of $S_r(n)$, away from some $\alpha \in S_r(n)$ different from the minimal element. This approach has already been considered by Ziegler [22]. He shows that reorientations $S_r^\alpha(n)$ of $S_r(n)$, in general, behave less well. In particular he shows that while $S_3(5)$ is a lattice there is an α such that $S_3^\alpha(5)$ is not a lattice. Moreover, he shows that in some reorientations of $S_4(6)$ there are maximal chains of length less than $\binom{6}{4}$, i.e., in these reorientations single-step inclusion and inclusion lead to different order relations. With our final example we show that $S_3(6)$ also admits reorientations such that some maximal chains are not maximum, i.e., maximal chains of length less than $\binom{6}{3}$.

Example C. Consider the two zonotopal tilings of Figure 10. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the

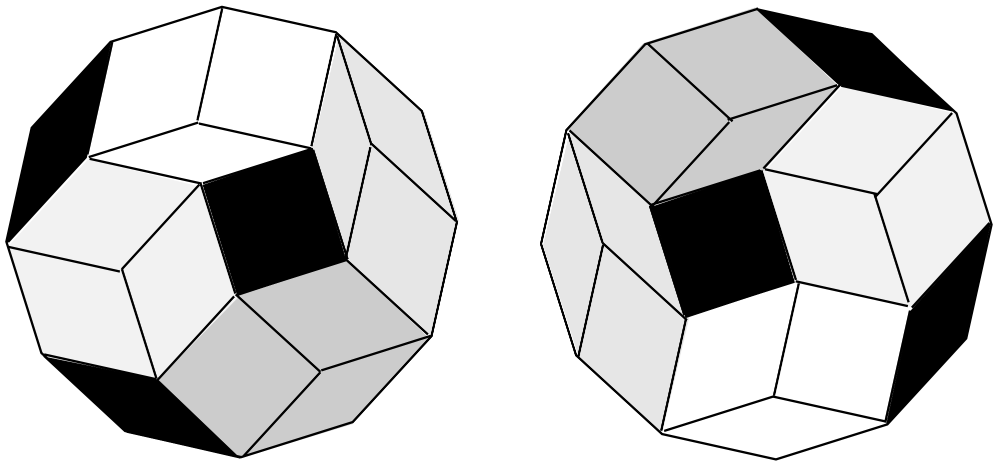


Figure 10: Zonotopal tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ with identical sets of triangular faces in the dual arrangements.

simple arrangements corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$. Both arrangements have exactly four triangular faces determined by the following sets of lines $\{1,3,5\}, \{1,4,6\}, \{2,3,4\}$ and $\{2,5,6\}$, moreover, the orientation of these triangles is the same in both arrangements. It follows that starting from $\mathcal{A}_1$ every possible triangular flip leads to an arrangement with more 3-element sets of lines being oriented different from their orientation in $\mathcal{A}_2$. Hence, if we orient $S_3(6)$ away from the signotope α_1 corresponding to $\mathcal{A}_1$ there is no single element step towards the signotope α_2 corresponding to $\mathcal{A}_2$. Hence, every chain from α_1 to the complement $\overline{\alpha_1}$ through α_2 has length $< \binom{6}{3}$. This example shows:

(1) Single step inclusion and inclusion are not identical for reorientations of $S_3(6)$ and hence $S_3(n)$ for all $n \geq 6$.

(2) An arrangement of pseudolines is not necessarily determined by the orientations of its triangular faces. Since the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are realizable the same holds for arrangements of lines.

Finally we remark that $\mathcal{A}_1$ is the arrangement given in [1] as a counterexample to Ringel's Conjecture about prescribable slopes.

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