RAMSEY THEORY FOR BINARY TREES AND THE SEPARATION OF TREE-CHROMATIC NUMBER FROM PATH-CHROMATIC NUMBER

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Abstract. We propose a Ramsey theory for binary trees and prove that for every r-coloring of "strong copies" of a small binary tree in a huge complete binary tree T, we can find a strong copy of a large complete binary tree in T with all small copies monochromatic. As an application, we construct a family of graphs which have tree-chromatic number at most 2 while the path-chromatic number is unbounded. This construction resolves a problem posed by Seymour.

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1. Introduction

Let G be a graph. A *tree-decomposition* of G is a pair $(T, \mathcal{B})$ where T is a tree and $\mathcal{B} = (B_t \mid t \in V(T))$ is a family of subsets of $V(G)$, satisfying:

(T1) for each $v \in V(G)$ there exists $t \in V(T)$ with $v \in B_t$; and for every edge $uv \in E(G)$ there exists $t \in V(T)$ with $u, v \in B_t$;
(T2) for each $v \in V(G)$, if $v \in B_t \cap B_{t''}$ for some $t, t'' \in V(T)$, and t' lies on the path in T between t and t'', then $v \in B_{t'}$.

Many researchers refer to the subset B_t as a *bag* and they consider B_t as an induced subgraph of G. With this convention, $|B_t|$ is just the number of vertices of G in the bag B_t, while $\chi(B_t)$ is the chromatic number of the induced subgraph of G determined by the vertices in B_t.

The quality of a tree-decomposition $(T, (B_t \mid t \in V(T)))$ is usually measured by its *width*, i.e. the maximum of $|B_t| - 1$ over all $t \in V(T)$. Then the *tree-width* of G is the minimum width of a tree-decomposition of G. In this paper we study the tree-chromatic number of a graph, a concept introduced by Seymour in [6]. The *chromatic number* of a tree-decomposition $(T, (B_t \mid t \in V(T)))$ is the maximum of $\chi(B_t)$ over all $t \in V(T)$. The *tree-chromatic number* of G, denoted by tree-$\chi(G)$, is the minimum chromatic number of a tree-decomposition of G. A tree-decomposition $(T, (B_t \mid t \in V(T)))$ is a *path-decomposition* when T is a path. The *path-chromatic number* of G, denoted by path-$\chi(G)$, is the minimum chromatic number of a path-decomposition of G. Clearly, for every graph G we have

$$\omega(G) \leqslant \text{tree-}\chi(G) \leqslant \text{path-}\chi(G) \leqslant \chi(G).$$

Furthermore, if $G = K_n$ is the complete graph on n vertices, then $\omega(G) = \chi(G) = n$, so all these inequalities can be tight. Accordingly, it is of interest to ask whether for consecutive parameters in this inequality, there is a sequence of graphs for which one parameter is bounded while the next parameter is unbounded.

In [6], Seymour shows that the classic Erdős construction [1] for graphs with large girth and large chromatic number yields a sequence $\{G_n : n \geqslant 1\}$ with $\omega(G_n) = 2$ and tree-$\chi(G_n)$ unbounded.

For an integer $n \geqslant 2$, the *shift graph* S_n is a graph whose vertex set consists of all closed intervals of the form $[a, b]$ where a, b are integers with $1 \leqslant a < b \leqslant n$. Vertices $[a, b]$, $[c, d]$ are adjacent in S_n when $b = c$ or $d = a$. As is well known (and first shown in [2]), $\chi(S_n) = \lceil \lg n \rceil$, for every $n \geqslant 2$. On the other hand, the natural path decomposition of S_n shows that path-$\chi(S_n) \leqslant 2$, for every $n \geqslant 2$, so as noted in [6], the family of shift graphs has bounded path-chromatic number and unbounded chromatic number.

Accordingly, it remains only to determine whether there is an infinite sequence of graphs with bounded tree-chromatic number and unbounded path-chromatic number. However, these two parameters appear to be more subtle in nature. As a first step, Huynh and Kim [4] showed that there is an infinite sequence $\{G_n : n \geqslant 1\}$ of graphs with tree-$\chi(G_n) \to \infty$ and tree-$\chi(G_n) <$ path-$\chi(G_n)$ for all $n \geqslant 1$.

In [6], Seymour proposed the following construction. Let T_n be the complete (rooted) binary tree with 2^n leaves. When y and z are distinct vertices in T_n, the path from y to z is called a "V" when the unique point on the path which is closest to the root of T_n is an intermediate point x on the path which is *strictly* between y and z. We refer to x as the *low point* of the V formed by y and z.

For a fixed value of n, we can then form a graph G_n whose vertices are the V's in T_n. We take V adjacent to V' in G_n when an end point of one of the two paths is the low point of the other. Clearly, $\omega(G_n) \leqslant 2$. Furthermore, it is easy to see that $\chi(G_n) \to \infty$ with n (we will say more about this observation later in the paper). Seymour [6] suggested that the family $\{G_n : n \geqslant 1\}$ has unbounded path-chromatic number.

To settle whether Seymour's intuition was correct, we needed to develop methods for working with arbitrary path decompositions of the graph G_n, extracting sufficient regularity to permit a detailed analysis of structural properties. Discovering regularity is a central theme in Ramsey theory, so it should not be surprising that developing Ramsey theoretic tools on binary trees will be a major component of this paper.

Although it is sometimes inadvisable to tell "how the story ends," we report that the Ramsey theoretic approach led us to find that graphs in the family $\{G_n : n \geqslant 1\}$ have path-chromatic number at most 2. On the one hand, it can be argued that no additional tools were required to discover this fact. One has to simply devise the appropriate path-decomposition for G_n at the outset.

However, our Ramsey theoretic approach allowed us to make the following minor modification to the construction. In the binary tree T_n, a subtree is called a "Y" when it has 3 leaves and the closest vertex in the subtree to the root of T_n is one of the three leaves. We then let H_n be the graph whose vertex set consists of the V's and Y's in T_n. Furthermore, Y is adjacent to Y' in H_n if and only if one of the two upper leaves of one of them is the lowest leaf in the other. Also, a Y is adjacent to a V if and only if one of the upper leaves in the Y is the low point of the V.

It is clear that tree-$\chi(H_n) \leqslant 2$. Using our Ramsey theoretic tools, we will then be able to show that path-$\chi(H_n) \to \infty$ with n, so that Seymour's question has been successfully resolved.

The rudiments of a Ramsey theoretic framework for binary trees were developed originally in connection with the question as to whether the local dimension of a poset is bounded in terms of the tree-width of its cover graph, and these tools are used in [5] to answer this question in the negative. However, the Ramsey theoretic tools we develop here are more comprehensive, and we believe they will be of independent interest.

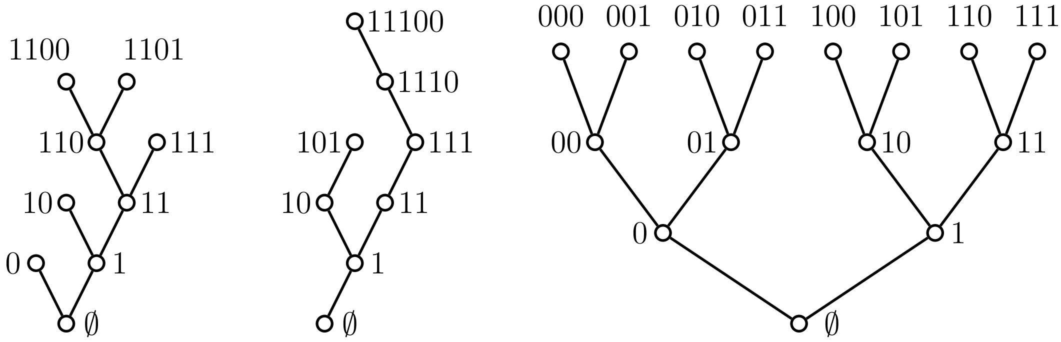

FIGURE 1. A Full Binary Tree, a Binary Tree, and a Complete Binary Tree of Order 3.

2. Ramsey Theory on Binary Trees

Although the concepts are included in elementary mathematics and computer science courses, we need some notation and terminology for working with binary trees. We view *binary trees* as posets. The elements of a binary tree T are in correspondence to binary strings such that with a string x in T all prefixes of x also correspond to elements of T. To simplify notation we frequently refer to elements of T via their corresponding strings, see Figure 1. There is a relation $y \leqslant x$ in T whenever y is a prefix of string x. The empty string corresponds to the least element (root) of T.

When a string is of modest length, we may write it explictly, e.g., $x = 01001101$ and we write xy to denote the concatenation of x and y. The *length of an element* of a binary tree is defined as the length of its string and the *length of binary tree* is the maximum length of an element of the tree. Note that the poset height of a binary tree of length n is $n + 1$.

For two elements $y < x$ in a binary tree T we say that x is in the *left subtree above* y in T if $y0$ is a prefix of x and we say y is in the *right subtree above* x in T when $y1$ is a prefix of x.

A binary tree T is *full* if for each $x \in T$ either x is a maximal element (leaf) of T or both the left and the right subtree are non-empty, i.e., either both $x0$ and $x1$ are not in T or both $x0$ and $x1$ are in T.

A binary tree is *complete of order* n if its elements are in correspondence to all binary strings of length at most n. Hence, if T is a complete tree[1] of order n then it has $2^{n+1} - 1$ elements, 2^n leaves and length n. In particular, a complete binary tree of order 0 is a one-point poset. For each $n \geqslant 0$, we define T_n to be a complete binary tree of order n.

For binary trees R, S, T with R being a subtree of T we say that R is a *strong copy* of S in T when there is a function $f : S \to R$ satisfying the following two requirements:

[1]The complete (rooted) binary tree we discussed in an informal manner in the opening section of this paper is just the cover graph of the poset T.

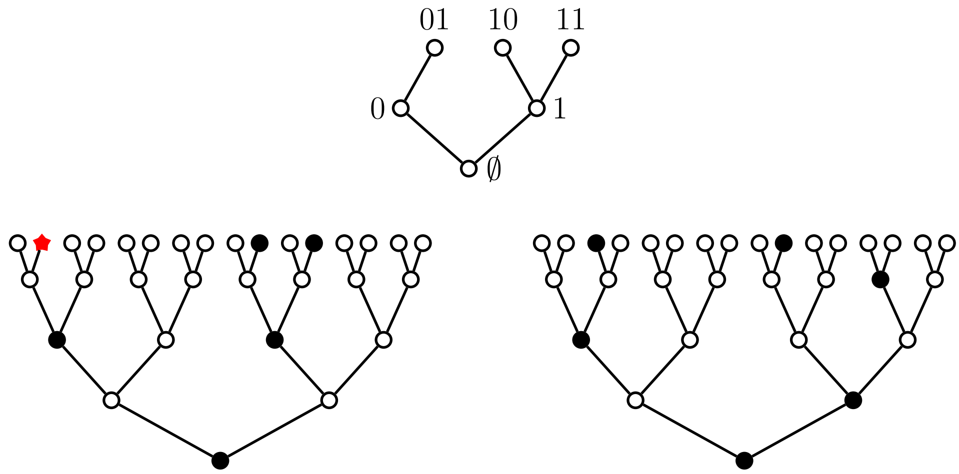

FIGURE 2. A binary tree and two copies of the tree within T_4. Only the right one is a strong copy.

(i) f is a poset isomorphism, i.e., f is a bijection and for all $x, y \in S$, $x \leqslant y$ in S if and only if $f(x) \leqslant f(y)$ in R;
(ii) for all $x, y \in S$ with $x < y$ in S, y is in the left (right) subtree above x in S if and only if $f(y)$ is in the left (right) subtree above $f(x)$ in T (see Figure 2).

As usual we let $[r]$ denote the set $\{1, 2, \ldots, r\}$. Also, an r-coloring of a set X is just a map $\phi : X \to [r]$. In some situations, we will consider a coloring ϕ using a set of r colors, but the set will not simply be the set $[r]$.

We begin with a result which can be considered as the analogue of the pigeon-hole principle for binary trees. However, note that the inductive structure of the argument is just the same as for the general form of Ramsey's theorem for graphs. In the statement of the theorem, the notation pt is an abbreviation for "point".

Theorem 2.1. *For every $r \geqslant 1$ and every $p \geqslant 0$ there is a least integer $n_0 = \mathrm{pt}(p; r)$ so that if $n \geqslant n_0$ and ϕ is an r-coloring of the elements of a complete binary tree T_n of order n, then there is some $\alpha \in [r]$ and a strong copy R of a complete binary tree T_p of order p in T_n such that ϕ assigns color α to every element in R.*

Proof. The argument proceeds by double induction, with the first induction on the number r of colors. For the second induction we look at sequence $(p_1, p_2, \ldots, p_r)$ of non-negative integers, and show that there is a least integer $n_0 = \mathrm{pt}(p_1, p_2, \ldots, p_r)$ so that if $n \geqslant n_0$ and ϕ is an r-coloring of the elements in a complete binary tree T_n of order n, then there is some $\alpha \in [r]$ and a strong copy R of a complete binary tree T_{p_α} of order p_α in T_n such that ϕ assigns color α to every element in R. The induction is on the sum $p_1 + p_2 + \cdots + p_r$.

The case when $r = 1$ is trivial, just set $\mathrm{pt}(p; 1) = \mathrm{pt}(p) = p$, for all p. If $r \geqslant 3$, then we note that:

$$\mathrm{pt}(p_1, p_2, \ldots, p_r) \leqslant \mathrm{pt}(\mathrm{pt}(p_1, p_2, \ldots, p_{r-1}), p_r).$$

Now consider the case where $r = 2$. As is the tradition in Ramsey theory, we now take color 1 to be "red" and color 2 to be "blue". Trivially, $\mathrm{pt}(0, p_2) = p_2$ since if we color any element in T_{p_2} red, we get a red copy of a complete binary tree of order 0 and if we color all elements blue, then we get a blue copy of a complete binary tree of order p_2. Similarly, $\mathrm{pt}(p_1, 0) = p_1$. Now, for $p_1, p_2 \geqslant 1$ we claim that:

$$\mathrm{pt}(p_1, p_2) \leqslant 1 + \max\{\mathrm{pt}(p_1 - 1, p_2), \mathrm{pt}(p_1, p_2 - 1)\}.$$

To see that this claim holds, let $n \geqslant 1 + \max\{\mathrm{pt}(p_1 - 1, p_2), \mathrm{pt}(p_1, p_2 - 1)\}$ and let ϕ be a red-blue coloring of T_n. Suppose first that ϕ assigns color red to the empty string in T_n. We then consider the left tree T_L above the empty string, noting that T_L is a complete binary tree of order $n - 1 \geqslant \mathrm{pt}(p_1 - 1, p_2)$. If there is a blue strong copy of a complete binary tree of order p_2 in this subtree, we are done. Otherwise, there is a red strong copy of a complete binary tree of order $p_1 - 1$ in T_L. Similarly, in the right tree T_R above the empty string in T_n, there is again a red strong copy of a complete binary tree of order $p_1 - 1$. Together with the empty string, we then have a red strong copy of a complete tree of order p_1 in T_n. The argument when ϕ assigns color blue to the empty string is symmetric. With this observation, the proof of the theorem is complete. □

Here is the general theorem we are about to prove.

Theorem 2.2. *Let (p, m, r) be a triple of integers with $p \geqslant m \geqslant 0$ and $r \geqslant 1$. Then there is a least positive integer $n_0 = \mathrm{Ram}(p, m; r)$ so that if $n \geqslant n_0$, Q is a full binary tree of length at most m and ϕ is an r-coloring of the strong copies of Q in a complete binary tree T_n of order n, then there is a color $\alpha \in [r]$ and a strong copy R of a complete binary tree T_p of order p in T_n such that ϕ assigns color α to every strong copy of Q contained in R.*

We note that Theorem 2.1 is the special case $m = 0$ of this result, i.e. $\mathrm{Ram}(p, 0; r) = \mathrm{pt}(p; r)$. As is often the case in Ramsey theory, we do not prove Theorem 2.2 directly. Instead, we have a second statement which can be viewed as a "bipartite" version. The two statements are then proved concurrently with an inductive argument which switches back and forth between the two.

When Q is a full binary tree and R is a strong copy of Q in T, we will say that R is a *bipartite copy of Q in T* when the least element of R is the least element of T. For a triple of integers (p, m, r) with $p \geqslant m \geqslant 0$ and $r \geqslant 1$ let $n_0 = \mathrm{BpRam}(p, m; r)$ be a least positive integer so that if $n \geqslant n_0$, Q is a full binary tree of length at most m and ϕ is an r-coloring of the bipartite copies of Q in a complete binary tree T_n of order n, then there is a color $\alpha \in [r]$ and a bipartite copy R of a complete binary tree T_p of order p in T_n such that ϕ assigns color α to every bipartite copy of Q in R.

With Theorem 2.1 as the basis of the induction, the following two claims imply Theorem 2.2.

Claim 1. For all $p > m \geqslant 0$ and $r \geqslant 1$, if $\mathrm{Ram}(p - 1, m; r')$ exists for all $r' \geqslant r$, then $\mathrm{BpRam}(p, m + 1; r)$ exists.

Claim 2. For all $m \geqslant 1$, if $\mathrm{BpRam}(p, m; r)$ exists for all pairs (r, p) with $r \geqslant 1$ and $p \geqslant m$, then $\mathrm{Ram}(p, m; r)$ exists for all pairs (r, p) with $r \geqslant 1$ and $p \geqslant m$.

Note that $\mathrm{BpRam}(p, 0; r) = p$ as there is only one bipartite copy of Q in any non-empty tree T.

We begin with the proof of Claim 1. Let $p > m \geqslant 0$ and $r \geqslant 1$. We will show that $\mathrm{BpRam}(p, m+1; r)$ exists and is at most $n_0 := 1+\mathrm{Ram}(p-1, m; r')$ where $r' = r \cdot 2^{2^{q+1}-1}$ and $q = \mathrm{Ram}(p-1, m; r)$.

Let Q be a full binary tree of length at most $m + 1$ and let ϕ be an r-coloring of a complete binary tree T_n of order $n \geqslant n_0$.

If Q is of length 0, then the statement holds as $n \geqslant n_0 = 1 + \mathrm{Ram}(p - 1, m; r') \geqslant 1 + (p - 1) = p$. If Q is of length at least 1, then the left subtree and the right subtree above the root in Q are non-empty and will be denoted Q_0 and Q_1, respectively. Then Q_0 and Q_1 have length at most $m - 1$.

In the tree T_n, the left (right) subtree above the root of T_n will be denoted F_0 (F_1, respectively). Of course, F_0 and F_1 are complete trees of order $n - 1$. Then let E_1 be any strong copy of T_q in F_1.

Consider a strong copy S of Q_0 in F_0. We define an r-coloring ϕ_S of the strong copies of Q_1 in E_1 in a quite natural manner: when S' is a strong copy of Q_1 in E_1, the elements in $\{\emptyset\} \cup S \cup S'$ form a bipartite copy of Q in T_n. We then set $\phi_S(S') = \phi(\{\emptyset\} \cup S \cup S')$.

Since $q \geqslant \mathrm{Ram}(p - 1, m; r)$, there is some $\alpha_S \in [r]$ and a strong copy E_S of T_{p-1} in E_1 such that ϕ_S assigns color α_S to every strong copy of Q_1 which is contained in E_S.

In turn, this process defines a coloring σ of the copies of Q_0 contained in F_0, i.e., we set $\sigma(S) = (\alpha_S, E_S)$. Note that $\sigma(S)$ takes one of at most $r \cdot 2^{2^{q+1}-1} = r'$ values (admittedly a rough estimate). Since $n - 1 \geqslant \mathrm{Ram}(p - 1, m; r')$, there is a color (β, R_1) used by σ and a subposet R_0 of F_0 so that R_0 is a strong copy of T_{p-1} and σ assigns color (β, R_1) to every strong copy of Q_0 contained in R_0.

The trees R_0 and R_1 together with the root of T_n form a subposet R of T_n which is a strong copy of T_p, the root of R is the root of T_n, and ϕ assigns color β to every bipartite copy of Q in R. This completes the proof of Claim 1.

Now we come to the proof of Claim 2. We first sketch the idea. Let $n \gg s \gg p$ be large enough and let ϕ be an r-coloring of the strong copies of Q in T_n. First we iteratively apply the bipartite version to obtain a strong copy of T_s in T_n such that for every element x in T_s all bipartite copies of Q in the subtree $T_s(x)$ rooted at x have the same color α_x. Assuming $s \geqslant \mathrm{pt}(p; r)$, then by Theorem 2.1 there is $\alpha \in [r]$ and a strong copy R of T_p in T_s such that $\alpha_x = \alpha$ for all elements x of R. It follows that all strong copies of Q in R have color α.

Now let $s = \mathrm{pt}(p; r)$ and define a sequence $q_0, q_1, \ldots, q_s$ in a reverse manner. First set $q_s = m$. If $0 < i \leqslant s$ and q_i has been defined, set $q_{i-1} = \mathrm{BpRam}(1 + q_i, m, r)$. We now show that $\mathrm{Ram}(p, m; r)$ exists and is at most q_0. Fix a full binary tree Q of length at most m. Let n be any integer with $n \geqslant 1 + q_0$ and let ϕ be an r-coloring of the strong copies of Q in T_n.

We iteratively construct binary trees S_i in T_n for $0 \leqslant i \leqslant s + 1$ such that

(i) S_i is a bipartite copy of a complete binary tree of order $i + q_i$ in T_n;
(ii) for each element x of length less than i in S_i there is a color $\alpha_x \in [r]$ such that all bipartite copies of Q in $S_i(x)$ have color α_x.

Let S_0 be a subtree of T_n consisting of all elements of length at most q_0 in T_n. Clearly S_0 is a bipartite copy of T_{q_0} in T_n. Since there is no element of length less than 0 in S_0 the second condition is void.

Now suppose that S_i has been defined. For an element x of length i in S_i let $S_i(x)$ be the subtree of S_i rooted at x. Since S_i has order $i + q_i$ we know that $S_i(x)$ is a complete binary tree of order q_i. Since $q_i = \mathrm{BpRam}(1 + q_{i+1}, m, r)$, there is a bipartite copy $B(x)$ of $T_{1+q_{i+1}}$ in $S_i(x)$ and some $\alpha_x \in [r]$ such that all bipartite copies of Q in $B(x)$ have color α_x. The binary tree S_{i+1} is obtained by replacing $S_i(x)$ by $B(x)$ in S_i for each x of length i.

In S_{s+1} all elements x of length at most s have their α_x fixed. Let S be a subtree of S_{s+1} consisting of all elements of length at most s. By construction S is a bipartite copy of T_s in T_n. It follows from the choice of s and Theorem 2.1 that there is some color $\alpha \in [r]$ and a subposet R of S such that R is a strong copy of T_p and $\alpha_x = \alpha$ for every element x in R. Clearly, this implies that ϕ assigns color α to every strong copy of Q contained in R. With this observation, the proof of Claim 2 is complete, and so is the proof of Theorems 2.2.

We have the following immediate corollary.

Corollary 2.3. *Let (p, m, r) be a triple of integers with $p \geqslant m \geqslant 0$ and $r \geqslant 1$. Then there is a least positive integer $n_0 = \mathrm{Ram}(p, m; r)$ so that if Q is a binary tree of length at most m and ϕ is an r-coloring of the strong copies of Q in a complete binary tree T_n of order n, then there is a color $\alpha \in [r]$ and a strong copy R of a complete binary tree T_p of order p in T_n such that ϕ assigns color α to every strong copy of Q contained in R.*

Proof. We note that if Q is a non-full binary tree of length m, then it is possible to add leaves to Q to obtain a full binary tree Q' of length m which contains Q as an induced subposet. It is easy to see that if we are given an r-coloring ϕ of the strong copies of Q in T_n, there is a natural way to extend ϕ to an r-coloring of the strong copies of Q' in T_n. We then apply Theorem 2.2. □

We pause here to comment that there are other formulations which lead to a Ramsey theory on binary trees. For example, we can weaken the requirement that R be a strong copy of Q and only require that R is isomorphic to Q as an ordered tree. However, since we have no application here for such variations, they will not be discussed further in this paper.

3. Separating Tree-chromatic Number and Path-chromatic Number

For the remainder of the paper, for a positive integer n, we let G_n be the graph of the V's in the complete binary tree T_n. Strictly speaking, a vertex V in G_n is a path which is determined by its two endpoints, but we find it convenient to specify V as a triple (x, y, z), where y and z are the endpoints of the path and x is the low point on the path. We view V as a triple and not a 3-element set so we can follow the convention that y is in the left tree above x and z is in the right tree above x. When $V_1 = (x_1, y_1, z_1)$ and $V_2 = (x_2, y_2, z_2)$ are vertices in G_n, we note that V_1 and V_2 are adjacent if and only if one of the following four statements holds: $z_1 = x_2$, $y_1 = x_2$, $y_2 = x_1$ or $z_2 = x_1$.

Also, for each $n \geqslant 1$, we let H_n be the graph of V's and Y's in T_n. Of course, G_n is an induced subgraph of H_n. Furthermore, the natural tree-decomposition of H_n shows that tree-$\chi(H_n) \leqslant 2$ for all $n \geqslant 1$.

Our goals for this section are to prove the following two theorems.

Theorem 3.1. path-$\chi(G_n) \leqslant 2$, *for all* $n \geqslant 1$.

Theorem 3.2. *For every* $r \geqslant 1$, *there is a positive integer* n *such that* path-$\chi(H_n) \geqslant r$.

We elect to follow the line of our research and prove the second of these two theorems first. In accomplishing this goal, we will discover a path-decomposition of G_n witnessing that path-$\chi(G_n) \leqslant 2$ for all $n \geqslant 1$.

Our argument for Theorem 3.2 will proceed by contradiction, i.e. we will assume that there is some positive integer r such that path-$\chi(H_n) \leqslant r$ for all $n \geqslant 1$. The contradiction will come when n is sufficiently large in comparison to r.

For the moment, we take n as a large but unspecified integer. Later, it will be clear how large n needs to be. We then take a path-decomposition of H_n witnessing that path-$\chi(H_n) \leqslant r$. We may assume that the host path in this decomposition is the set $\mathbb{N}$ of positive integers with i adjacent to $i+1$ in $\mathbb{N}$ for all $i \geqslant 1$. For each vertex v in H_n, the set of all integers i for which $v \in B_i$ is a set of consecutive integers, and we denote the least integer in this set as a_v and the greatest integer as b_v. Abusing notation slightly, we will denote this set as $[a_v, b_v]$, i.e., this interval notation identifies the integers $i \in \mathbb{N}$ with $a_v \leqslant i \leqslant b_v$. We point out the requirement that $[a_v, b_v] \cap [a_u, b_u] \neq \emptyset$ when v and u are adjacent vertices in H_n.

We may assume that $a_v < b_v$ for every vertex $v \in H_n$. Furthermore, we may assume that for each integer i, there is at most one vertex $v \in H_n$ with $i \in \{a_v, b_v\}$.

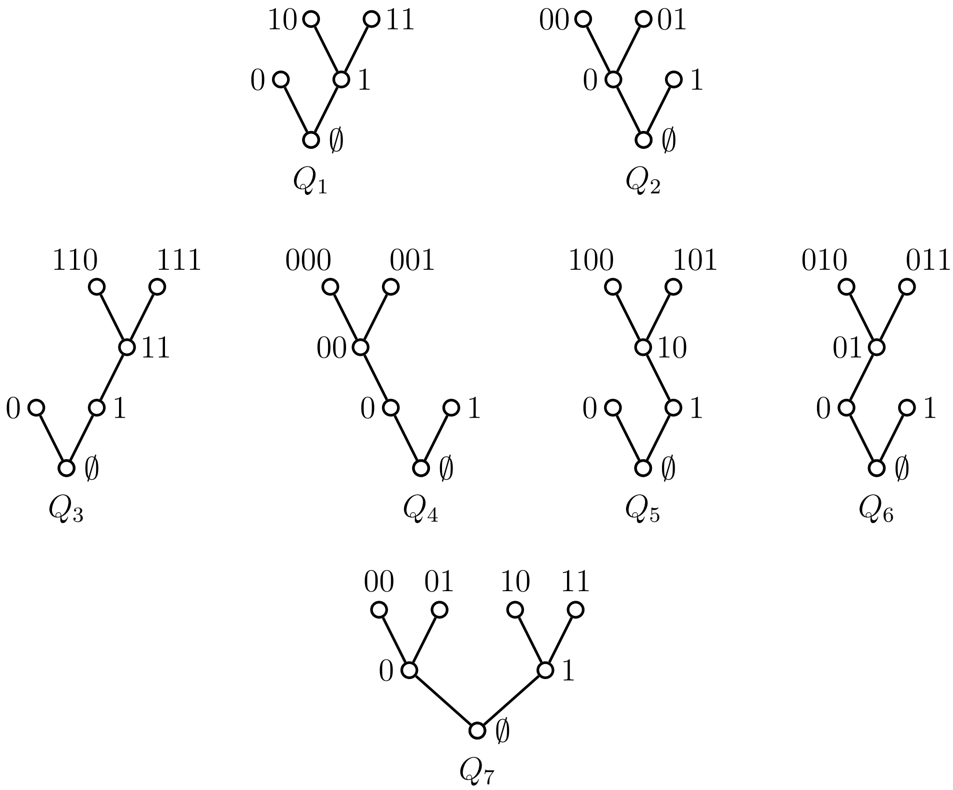

FIGURE 3. Applying Ramsey with Seven Full Binary Trees

For each $i \in \mathbb{N}$, we let $G_n(i)$ respectively denote the induced subgraph of G_n determined by those vertices $v \in G_n$ with $i \in [a_v, b_v]$. The graph $H_n(i)$ is defined analogously.

We pause here to point out an essential detail for the remainder of the proof. Since $\chi(H_n(i)) \leqslant r$ for all integers i, then for all $q > 2^r$, there is no positive integer i for which $H_n(i)$ contains the shift graph S_q as a subgraph.

To begin to make the connection with Ramsey theory, we observe that there is a natural 1–1 correspondence between V's in G_n and strong copies of T_1 in T_n. So in the discussion to follow, we will interchangeably view a vertex $V = (x, y, z)$ of G_n as a path in T_n and as a 3-element subposet of T_n forming a strong copy of T_1. Of course, we are abusing notation slightly by referring to T_n as a graph and as a poset, but by now the notion that as a graph, we are referring to the cover graph of the poset should be clear.

Now let (V_1, V_2) be an ordered pair of vertices in G_n. Referring to the binary trees in Figure 3, we consider 7 different ways this pair can appear in T_n:

(i) V_1 and V_2 are adjacent with $z_1 = x_2$. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_1.
(ii) V_1 and V_2 are adjacent with $y_1 = x_2$. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_2.

(iii) V_1 and V_2 are non-adjacent with x_2 in the right tree above z_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_3.
(iv) V_1 and V_2 are non-adjacent with x_2 in the left tree above y_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_4.
(v) V_1 and V_2 are non-adjacent with x_2 in the left tree above z_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_5.
(vi) V_1 and V_2 are non-adjacent with x_2 in the right tree above y_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_6.
(vii) V_1 and V_2 are non-adjacent and there is a vertex w in T_n so that x_1 is in the left tree above w while x_2 is in the right tree above w. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_7.

Now for the application of our Ramsey theoretic tools. Given a pair (V_1, V_2) of distinct vertices in G_n, there are 6 ways the intervals $[a_1, b_1]$ and $[a_2, b_2]$ can appear in the path-decomposition:

$a_1 < a_2 < b_1 < b_2$	Overlapping, moving right
$a_2 < a_1 < b_2 < b_1$	Overlapping, moving left
$a_1 < b_1 < a_2 < b_2$	Disjoint, moving right
$a_2 < b_2 < a_1 < b_1$	Disjoint, moving left
$a_1 < a_2 < b_2 < b_1$	Inclusion, second in first
$a_2 < a_1 < b_1 < b_2$	Inclusion, first in second

In the arguments to follow, we will abbreviate these 6 options as OMR, OML, DMR, DML, ISF and IFS, respectively.

We then define for each $i \in [7]$ a 6-coloring ϕ_i of the strong copies of Q_i in T_n. The colors will be the six labels $\{\text{OMR}, \text{OML}, \ldots, \text{IFS}\}$ listed above. When $i \in [7]$ and Q is a strong copy of Q_i, then Q is associated with a pair (V_1, V_2) of vertices from G_n. It is then natural to set $\phi_i(Q)$ as the label describing how the pair $([a_1, b_1], [b_1, b_2])$ of intervals are positioned in the path decomposition.

Now let p be a second large integer (p will be large relative to r but small relative to n), with $p > 3 \cdot 2^r$. Using Corollary 2.3 seven times, we may then assume that n is large enough to guarantee that there is a strong copy R of a complete binary tree T_p of order p in T_n and a vector $(\alpha_1, \alpha_2, \ldots, \alpha_7)$ of colors such that for each $i \in [7]$, ϕ_i assigns color α_i to all strong copies of Q_i in R. In the remainder of the argument, we will abuse notation slightly and simply consider that $R = T_p$.

Claim 1. α_1 is either OMR or OML.

Proof. A pair (V_1, V_2) of vertices in G_n associated with a strong copy of Q_1 in T_p is adjacent in G_n so that $[a_1, b_1]$ and $[a_2, b_2]$ intersect. So α_1 cannot be DMR or DML.

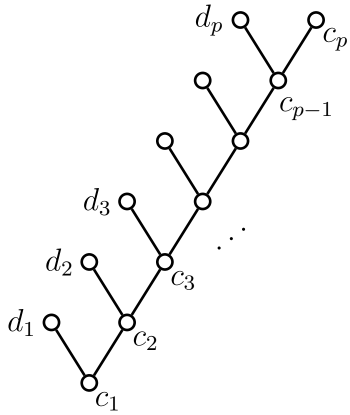

FIGURE 4. A Shift Graph S_p in G_n

We assume that α_1 is ISF and argue to a contradiction. The argument when α_1 is IFS is symmetric. Consider the subtree of T_p consisting of all non-empty strings for which each bit, except possibly the last, is a 1. We suggest how this subtree appears (at least for a modest value of p) in Figure 4.

For each interval $[i, j]$ with $1 \leqslant i < j \leqslant p$, we consider the vertex $V[i, j] = (c_i, d_i, c_j)$. Clearly, $V[i, j]$ is adjacent to $V[j, k]$ when $1 \leqslant i < j < k \leqslant p$, i.e., these vertices form the shift graph S_p.

Let $[a, b] = [a_{V[p-1,p]}, b_{V[p-1,p]}]$ be the interval for the vertex $V[p-1, p]$. We claim that $a \in [a_{V[i,j]}, b_{V[i,j]}]$ for each $V[i, j]$ with $1 \leqslant i < j \leqslant p-1$ This is immediate if $j = p-1$, since $\phi_1(V[i, p-1], V[p-1, p]) = \text{ISF}$, so $[a_{V[i,p-1]}, b_{V[i,p-1]}] \supseteq [a, b]$. Now suppose $j < p-1$. Then again $\phi(V[i, j], V[j, p-1]) = \text{ISF}$, so that in the path-decomposition we have $[a_{V[i,j]}, b_{V[i,j]}] \supseteq [a_{V[j,p-1]}, b_{V[j,p-1]}] \supseteq [a, b]$.

Now the induced subgraph $G_n(a)$ contains the shift graph S_p, and therefore $\chi(G_n(a)) \geqslant \chi(S_p) \geqslant \log(p) \geqslant r$. This is a contradiction. □

Without loss of generality, we take α_1 to be OMR, since if α_1 is OML, we may simply reverse the entire path-decomposition. To help keep track of the configuration information as it is discovered, we list this statement as a property.

Property 1. $\alpha_1 = \text{OMR}$, i.e., ϕ_1 assigns color OMR to a pair (V_1, V_2) of adjacent vertices in G_n when $z_1 = x_2$.

Although it may not be a surprise, once the color α_1 is set, colors $\alpha_2, \alpha_3, \ldots, \alpha_7$ are determined. In the discussion to follow, when we discuss a family $\{V_j : j \in [t]\}$ of V's in G_n, we will let $V_j = (x_j, y_j, z_j)$, and we will let $[a_j, b_j]$ be the interval in the path-decomposition corresponding to V_j, for each $j \in [t]$.

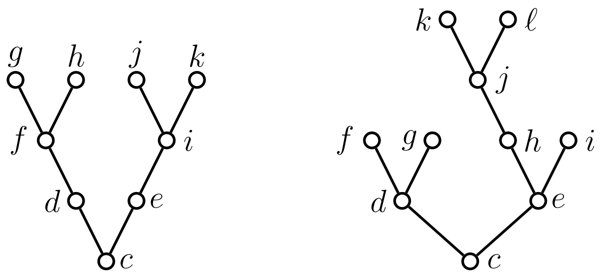

FIGURE 5. Two Useful Small Examples

Property 2. $\alpha_3 = \text{DMR}$, i.e., ϕ_3 assigns color DMR to a pair (V_1, V_2) of non-adjacent vertices in G_n when x_2 is in the right tree above z_1.

Proof. Let (V_1, V_2) be a pair of non-adjacent vertices in G_n with x_2 in the right tree above z_1. Then let w_3 be the string formed by attaching a 0 at the end of z_1, and set $V_3 = (z_1, w_3, x_2)$. Then V_3 is adjacent to both V_1 and V_2. Furthermore, $\phi_1(V_1, V_3) = \text{OMR}$ and $\phi_1(V_3, V_2) = \text{OMR}$. Accordingly, α_3 is either OMR or DMR. We assume that $\alpha_3 = \text{OMR}$ and argue to a contradiction.

Consider the shift graph used in the proof of Claim 1. Let $a = a_{V[p-1,p]}$ be the left endpoint of the interval for $V[p-1, p]$ in the path-decomposition. We claim that a is in the interval for $V[i, j]$ in the path-decomposition whenever $1 \leqslant i < j \leqslant p-1$. Again, this holds when $j = p-1$ since $\phi_1(V[i, p-1], V[p-1, p]) = \text{OMR}$. Also, when $j < p-1$, we have $\phi_3(V[i, j], V[p-1, p]) = \text{OMR}$, so that the interval for $V[i, j]$ in the path-decomposition also contains a. This again implies that $G_n(a)$ contains the shift graph S_p. The contradiction completes the proof. □

Property 3. $\alpha_2 = \text{OML}$, i.e., ϕ_2 assigns color OML to a pair (V_1, V_2) of adjacent vertices in G_n when $y_1 = x_2$. Also, $\alpha_4 = \text{DML}$, i.e., ϕ_4 assigns color DML to a pair (V_1, V_2) of non-adjacent vertices in G_n when x_2 is in the left tree above y_1.

Proof. We can repeat the arguments given previously to conclude that one of two cases must hold: Either (1) $\alpha_2 = \text{OMR}$ and $\alpha_4 = \text{DMR}$, or (2) $\alpha_2 = \text{OML}$ and $\alpha_4 = \text{DML}$. We assume that $\alpha_2 = \text{OMR}$ and $\alpha_4 = \text{DMR}$ and argue to a contradiction. Consider the binary tree contained in T_p as shown on the left side of Figure 5. Let $V_1 = (f, g, h)$, $V_2 = (i, j, k)$, $V_3 = (c, f, e)$ and $V_4 = (c, d, i)$.

Since $\phi_4(V_4, V_1) = \text{DMR}$, we know $b_4 < a_1$. Since $\phi_1(V_4, V_2) = \text{OMR}$, we know $a_2 < b_4$, so $a_2 < a_1$. Since $\phi_3(V_3, V_2) = \text{DMR}$, we know $b_3 < a_2$ so $b_3 < a_1$. But $\phi_2(V_3, V_1) = \text{OMR}$, which requires $a_1 < b_3$. The contradiction completes the proof of Property 3. □

Property 4. $\alpha_7 = \text{DMR}$, i.e., ϕ_7 assigns color DMR to a pair (V_1, V_2) of non-adjacent vertices in G_n when there is a vertex w in T_n such that x_1 is in the left tree above w while x_2 is in the right tree above w.

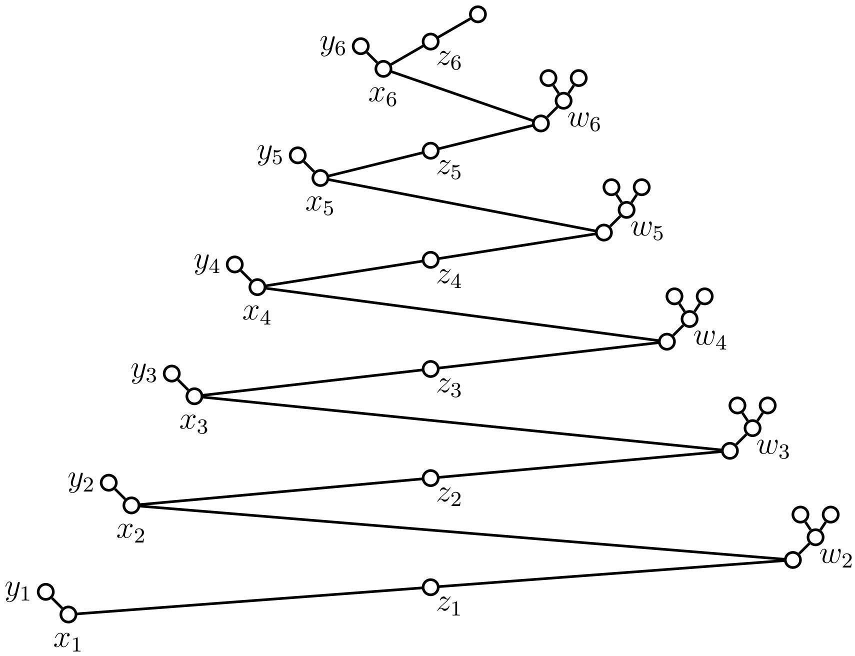

FIGURE 6. The Final Counter-Example

Proof. We again consider the binary tree shown on the left side of Figure 5. Again, we take $V_1 = (f, g, h)$ and $V_2 = (i, j, k)$. Noting that f is in the left tree above c and i is in the right tree above a, $\phi_7(V_1, V_2) = \alpha_7$.

Now let $V_5 = (c, d, e)$. Then $\phi_4(V_5, V_1) =$ DML and $\phi_3(V_5, V_2) =$ DMR. These statements imply $\alpha_7 =$ DMR. □

Property 5. $\alpha_5 = \alpha_6 =$ ISF, i.e., ϕ_5 assigns color ISF to a pair (V_1, V_2) of non-adjacent vertices in G_n when x_2 is in the left tree above z_1 and ϕ_6 assigns this pair color IFS when x_2 is in the right tree above y_1.

Proof. We prove that $\alpha_5 =$ ISF. The argument to show that $\alpha_6 =$ ISF is symmetric. Consider the binary tree shown on the right side of Figure 5. Let $V_1 = (c, d, e)$ and $V_2 = (j, k, l)$. Then j is in the left tree above e, so $\phi_5(V_1, V_2) = \alpha_5$.

Now set $V_3 = (d, f, g)$ and $V_4 = (e, h, i)$. We observe that $\phi_2(V_1, V_3) =$ OML, $\phi_7(V_3, V_2) =$ DMR, $\phi_1(V_1, V_4) =$ OMR and $\phi_4(V_4, V_2) =$ DML. Together, these statements imply $\alpha_5 =$ ISF. □

Up to this point in the proof, our entire focus has been on the V's in G_n. We now turn our attention to properties that the Y's in H_n must satisfy.

Consider the binary tree shown in Figure 6. Of course, we intend that this tree appear inside T_p. In our figure, the "size" of this construction is $m = 6$, but since $p > 3 \cdot 2^r$, we know $m > 2^r$. For each interval $[i, j]$ with $1 \leqslant i < j \leqslant m$, we let $Y[i, j]$ be the Y whose three leaves are x_i, x_j and w_j. Clearly, the family $\{Y[i,j] : 1 \leqslant i < j \leqslant m\}$ forms a copy of the shift graph S_m. To reach a final contradiction, it remains only to show that there is some integer $i \in \mathbb{N}$ for which all vertices in $\{Y[i,j] : 1 \leqslant i < j \leqslant m\}$ belong to $H_n(i)$.

For each $j \in [m]$, we let $V_j = (x_j, y_j, z_j)$, and as usual, we let $[a_j, b_j]$ be the corresponding interval for V_j in the path decomposition. By Property 2, we have $\alpha_3 = \text{DMR}$, so that:

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{m-1} < b_{m-1} < a_m < b_m.$$

For each $j = 2, 3, \ldots, m$, let $V'_j = (w_j, w_j0, w_j1)$, and we let $[a'_j, b'_j]$ be the corresponding interval in the path-decomposition. By Property 4, $\alpha_7 = \text{DMR}$ so that:

$$a'_m < b'_m < a'_{m-1} < b'_{m-1} < \cdots < a'_3 < b'_3 < a'_2 < b'_2.$$

Again, since $\alpha_7 = \text{DMR}$, we know that $a_m < b_m < a'_m < b'_m$.

Now consider a pair i, j with $1 \leqslant i < j \leqslant m$. The vertex $Y[i,j]$ is adjacent in H_n to both V_j and V'_j. This implies that the interval for $Y[i,j]$ must overlap both $[a_j, b_j]$ and $[a'_j, b'_j]$. However, this forces the interval for $Y[i,j]$ to contain $[b_m, a'_m]$. Therefore, $G_n(b_m)$ contains the shift graph S_m. With this observation, the proof of Theorem 3.2 is complete.

We now return to the task of proving Theorem 3.1, i.e., to the assertion that path-$\chi(G_n) \leqslant 2$ for all $n \geqslant 1$. Our proof for Theorem 3.2 suggests a natural way to define a path-decomposition of the graph G_n of V's in the binary tree T_n, one that satisfies all five properties we have developed to this point. We simply take a drawing in the plane of T_n using a geometric series approach. Taking a standard cartesian co-ordinate system in the plane, we place the root of T_n at the origin. If $m \geqslant 0$ and we have placed a string x of length m at (h, v), we set $\delta = 2^{-m}$ and place $x1$ and $x0$ at $(h + \delta, v + \delta)$ and $(h - \delta, v + \delta)$, respectively.

For each x in T_n, let $\pi(x)$ denote the vertical projection of x down onto the horizontal axis. In turn, for each $V = (x, y, z)$, we take $a_V = \pi(y)$ and $b_V = \pi(z)$. To illustrate this construction, we show in Figure 7 the interval $[a_V, b_V]$ corresponding to the vertex $V = (0, 00110, 010)$ in G_n.

Clearly, we may consider the host path P for the decomposition as consisting of all points on the horizontal axis of the form $\pi(x)$ where $x \in T_n$. Also, in the natural manner, $\pi(x)$ is adjacent to $\pi(x')$ in P when there is no string $x'' \in T_n$ with $\pi(x'')$ between $\pi(x)$ and $\pi(x')$.

So let $x_0 \in T_n$ and consider the bag $B_0 = B_{\pi(x_0)}$ consisting of all vertices $V = (x, y, z)$ in G_n with $\pi(y) \leqslant \pi(x_0) \leqslant \pi(z)$. We partition B_0 as $C_1 \cup C_2 \cup C_3$ where:

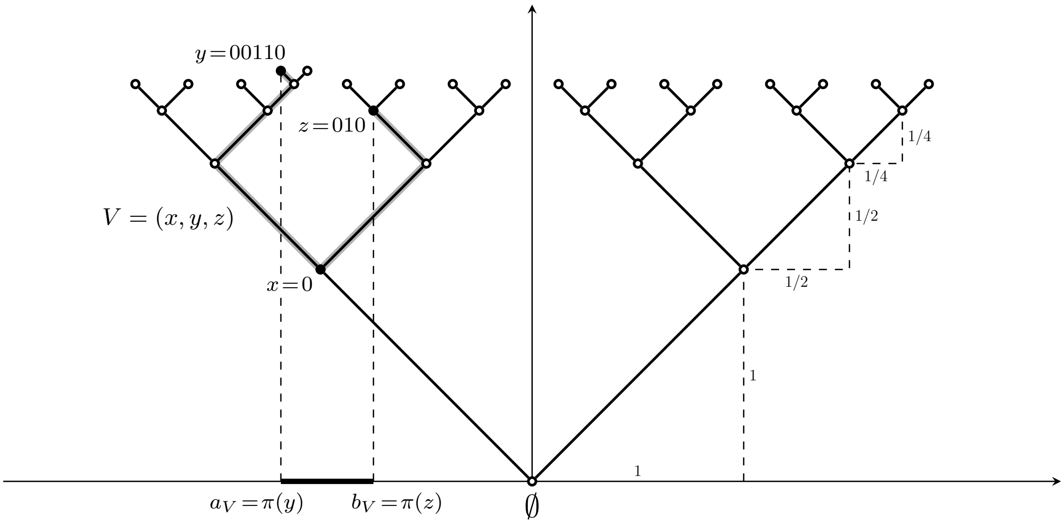

FIGURE 7. A Path-Decomposition of G_n

(i) A vertex $V = (x, y, z)$ of B_0 belongs to C_1 if $\pi(x) < \pi(x_0)$.
(ii) A vertex $V = (x, y, z)$ of B_0 belongs to C_2 if $\pi(x) > \pi(x_0)$.
(iii) A vertex $V = (x, y, z)$ of B_0 belongs to C_3 if $\pi(x) = \pi(x_0)$. In this case, $x = x_0$.

We now explain why C_1, C_2 and C_3 are independent sets in G_n. This is trivial for C_3. We give the argument for C_1, noting that the argument for C_2 is symmetric.

Suppose that V_1 and V_2 are adjacent vertices in C_1. If the pair (V_1, V_2) determines a strong copy of Q_1, then $\pi(z_1) = \pi(x_2) < \pi(x_0)$, which is a contradiction. On the other hand, if the pair (V_1, V_2) determines a strong copy of Q_2, then $y_1 = x_2$ so that $\pi(y_1) = \pi(x_2) < \pi(x_1) < \pi(x_0)$. Now the geometric series nature of the construction implies that $\pi(z_2) < \pi(x_0)$ which is again a contradiction.

With these observations, we have now proved that path-$\chi(G_n) \leqslant 3$ for all $n \geqslant 1$. This inequality is tight as evidenced by the following five elements of G_n which form a 5-cycle: $V_1 = (\emptyset, 0, 1)$, $V_2 = (1, 10, 11)$, $V_3 = (10, 100, 101)$, $V_4 = (101, 1010, 1011)$ and $V_5 = (1, 101, 11)$. Note that $\pi(101)$ is in $[a_i, b_i]$ for each $i \in [5]$.

Nevertheless, we are able to make a small but important change in the path-decomposition to obtain a decomposition witnessing that path-$\chi(G_n) \leqslant 2$. For the integer n, let $\varepsilon = 2^{-2n}$. Then for each vertex $V = (x, y, z)$ of G_n, we change the interval in the path decomposition for V from $[\pi(y), \pi(z)]$ to $[\pi(y) + \varepsilon, \pi(z) - \varepsilon]$. Our choice of ε guarantees that we still have a path-decomposition of G_n.

Again, we consider an element x_0 of T_n and the bag B_0 consisting of all $V = (x, y, z)$ with $\pi(y) \leqslant \pi(x_0) \leqslant \pi(z)$. As before, B_1, B_2 and B_3 are independent sets, although membership in these three sets has been affected by the revised path-decomposition. We claim that $B_1 \cup B_3)$ is also an independent set, so that the partition $B_0 = (B_1 \cup B_3) \cup B_2$ witnesses that path-$\chi(G_n) \leqslant 2$.

Suppose to the contrary that $V_1 \in C_1$ and $V_3 \in C_3$ with V_1 adjacent to V_3 in G_n. Clearly, this requires that (V_1, V_3) is associated with a strong copy of the binary tree Q_1 as shown in Figure 3. This implies that $z_1 = x_0$ so that $a_1 < \pi(x_1) < \pi(z_1) = \pi(x_0) - \varepsilon = b_1$. However, the assumption that $V_1 \in B_0$ which requires $a_1 \leqslant \pi(x_0) \leqslant b_1$. The contradiction completes the proof of Theorem 3.1.

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