On the Queue-Number of Partial Orders*

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Abstract

The queue-number of a poset is the queue-number of its cover graph viewed as a directed acyclic graph, i.e., the vertex order must be a linear extension of the poset. Heath and Pemmaraju conjectured that every poset of width w has queue-number at most w. Recently, Alam et al. constructed posets of width w with queue-number $w + 1$. Our contribution is a construction of posets with width w with queue-number $\Omega(w^2)$. This (asymptotically) matches the known upper bound.

1 Introduction

A *queue layout* of a graph consists of a total ordering on its vertices and a partition of its edge set into queues, i.e., no two edges in a single block of the partition are nested. The minimum number of queues needed in a queue layout of a graph G is its queue-number and denoted by $\mathsf{qn}(G)$.

To be more precise, let G be a graph and let L be a linear order of the vertices. A k-*rainbow* is a set of k edges $\{a_i b_i : 1 \le i \le k\}$ such that $a_1 < a_2 < \cdots < a_k < b_k < \cdots < b_2 < b_1$ in L. A pair of edges forming a 2-rainbow is said to be *nested*. A *queue* is a set of edges without nesting. Given G and L, the edges of G can be partitioned into k queues if and only if there is no rainbow of size $k + 1$ in L. The *queue-number* of G is the minimum number of queues needed to partition the edges of G over all linear orders L.

The queue-number was introduced by Heath and Rosenberg in 1992 [5] as a counterpart of book embeddings. Queue layouts were implicitly used before and have applications in fault-tolerant processing, sorting with parallel queues, matrix computations, scheduling parallel processes, and in communication management in distributed algorithm (see [3,5,7]). There is a rich literature exploring bounds on the queue-number of different classes of graphs [2,3,5,8].

In this note we study the queue-number of posets. This parameter was introduced in 1997 by Heath and Pemmaraju [4]. In the spirit of the older concept of the queue-number of directed acyclic graphs, it is required that a precedes b in a queue layout whenever there is a directed edge $a \to b$, i.e., the queue layout of a directed acyclic graph is a topological ordering and in the case of a poset, it is a linear extension.

A poset P is uniquely characterized by any digraph whose edge set is between the directed cover graph and the directed comparability graph of P. These two digraphs are respectively the transitive reduction and the transitive closure of any digraph representing P. In the context of drawings, embeddings and layouts, it is natural to work with the sparse cover graphs. For example a *diagram* of P is an upward drawing of the directed cover graph.

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The *queue-number* of P, denoted by $\mathsf{qn}(P)$, is the smallest k such that there is a linear extension L of P for which the resulting linear layout of the cover graph G_P contains no $(k+1)$-rainbow. Figure 1 shows an example.

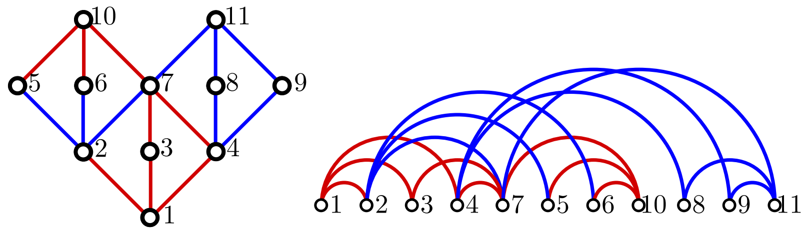

Figure 1 A poset of width 5 and a queue layout with 2 queues indicated by colors.

Clearly $\mathsf{qn}(G_P) \leq \mathsf{qn}(P)$, i.e., the queue-number of a poset is at least as large as the queue-number of its (undirected) cover graph. It was shown by Heath and Pemmaraju [4] that even for planar posets P there is no function f such that $\mathsf{qn}(P) \leq f(\mathsf{qn}(G_P))$. They also investigated the maximum queue-number of several classes of posets, in particular with respect to bounded width (the maximum number of pairwise incomparable elements) and height (the maximum number of pairwise comparable elements). In particular they gave a nice argument showing that $\mathsf{qn}(P) \leq \mathsf{width}(P)^2$ (see Proposition 1 below). The poset P of height 2 and width w whose cover graph is the complete bipartite graph $K_{w,w}$ attains $\mathsf{qn}(P) = \mathsf{width}(P)$. Actually, Heath and Pemmaraju conjectured that $\mathsf{qn}(P) \leq \mathsf{width}(P)$ for every poset P.

Knauer, Micek, and Ueckerdt [6] showed that the inequality $\mathsf{qn}(P) \leq \mathsf{width}(P)$ holds for all posets of width 2. Last year Alam et al. [1] constructed a non-planar poset of width 3 whose queue number is 4. They generalized the example and constructed for every $w > 3$ a poset P with $\mathsf{width}(P) = w$ and $\mathsf{qn}(P) = w+1$. A second contribution of Alam et al. consists in a slight improvement of the upper bound: They show $\mathsf{qn}(P) \leq (w-1)^2+1$ for all posets P of width at most w.

Our contribution is the following theorem.

▶ **Theorem 1.1.** *For every $w > 3$ there is a poset P_w of width w with*

$$\mathsf{qn}(P_w) \geq w^2/8.$$

These examples (asymptotically) match the upper bound. Besides yielding a strong improvement of the lower bound, we also believe that our construction is conceptually simpler than the examples provided by Alam et al. to disprove the conjecture of Heath and Pemmaraju.

As an open problem we promote the question whether the original conjecture holds for planar posets. In [6] it was shown that the queue-number of planar posets of width w is upper bounded by $3w-2$ and that there are such planar posets P with $\mathsf{qn}(P) = \mathsf{width}(P) = w$.

2 Preliminaries

Before getting serious with our construction, we revisit the nice upper bound argument of Heath and Pemmaraju. Let $P = (X, <)$ be a poset of width w. Dilworth's Theorem asserts that X can be decomposed into w chains of P.

▶ **Proposition 1** (Heath and Pemmaraju)**.** *For every poset P we have $\mathsf{qn}(P) \leq \mathsf{width}(P)^2$.*

Proof. Let $w = \mathsf{width}(P)$, let $C_1, \ldots, C_w$ be a chain partition, and let L be any linear extension of P. Partition the edges of the cover graph into w^2 sets $Q_{i,j}$ with $i, j \in [w]$ such that $(u, v) \in Q_{i,j}$ if $u \in C_i$ and $v \in C_j$. We claim that each $Q_{i,j}$ is a queue.

Let $a < b < c < d$ in L support a pair of nesting cover edges and suppose that both edges (a, d) and (b, c) belong to $Q_{i,j}$. By definition $a, b \in C_i$ and $c, d \in C_j$ and from the ordering in L we get $a < b$ and $c < d$ in P. Now we have $a < b$ and $b < c$ and $c < d$ in P whence the relation $a < d$ is implied by transitivity. This contradicts that (a, d) is a cover edge. ◀

In fact we have shown a much stronger statement: If P and a chain partition $C_1, \ldots, C_w$ are given, then there is a partition of the edges of the cover graph of P into parts $Q_{i,j}$ with $i, j \in [w]$ such that each $Q_{i,j}$ is a queue for every linear extension L of P.

2.1 Concepts needed for the construction

Let P be a poset. The *dual* of P, denoted $\bar{P}$, is the poset on the same ground set such that: $x < y$ in $P \iff y < x$ in $\bar{P}$.

A poset P is *2-dimensional* if and only if there are two linear extensions L_1 and L_2 such that: $x < y$ in $P \iff x < y$ in L_1 and L_2. Such a pair L_1, L_2 is called a *realizer* of P.

When drawing 2-dimensional posets, it is common to represent each element x by a point with coordinates (x_1, x_2) where x_1 is the position of x in L_1 and x_2 is the position of x in L_2.

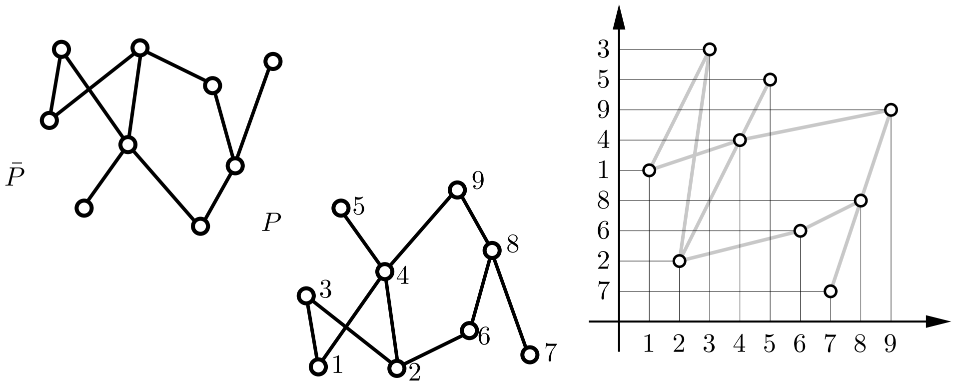

Figure 2 A poset P, its dual $\bar{P}$, and a 2-dimensional drawing of P.

3 Proof of Theorem 1.1

We define P_w recursively. For $w = 1, 2$ we let P_w be a w-chain, for $w \geq 3$, the construction of P_w is based on a copy of P_{w-2}, a reinforcement poset R_{w-2} of width $w-2$ with two linear extensions L_x and L_y, the duals $\bar{P}_{w-2}$ and $\bar{R}_{w-2}$ of P_{w-2} and R_{w-2}, and two additional points a and b. Due to space limitations, we refrain from giving a full formal description of the construction. Instead, we invite the reader to take a look at Figure 3, which shows a sketch of P_w. Note that the number $p(w)$ of vertices of P_w is given by the recursion $p(w) = 2p(w-2) + 6r(w-2) + 2$, where $r(w)$ is the number of vertices of R_w. The edges between X and R_{w-2} are given by L_x such that for each i the i-th element of the chain X is connected to the element of R_{w-2} which is at position i in L_x. The edges between Y and R_{w-2} are given by L_y in the same way.

It can be seen from the sketch that P_w is self-dual, the reflection $P_w \leftrightarrow \bar{P}_w$ has two fixed points a and b. This shows that when analyzing $\mathsf{qn}(P_w)$, we can restrict the attention to

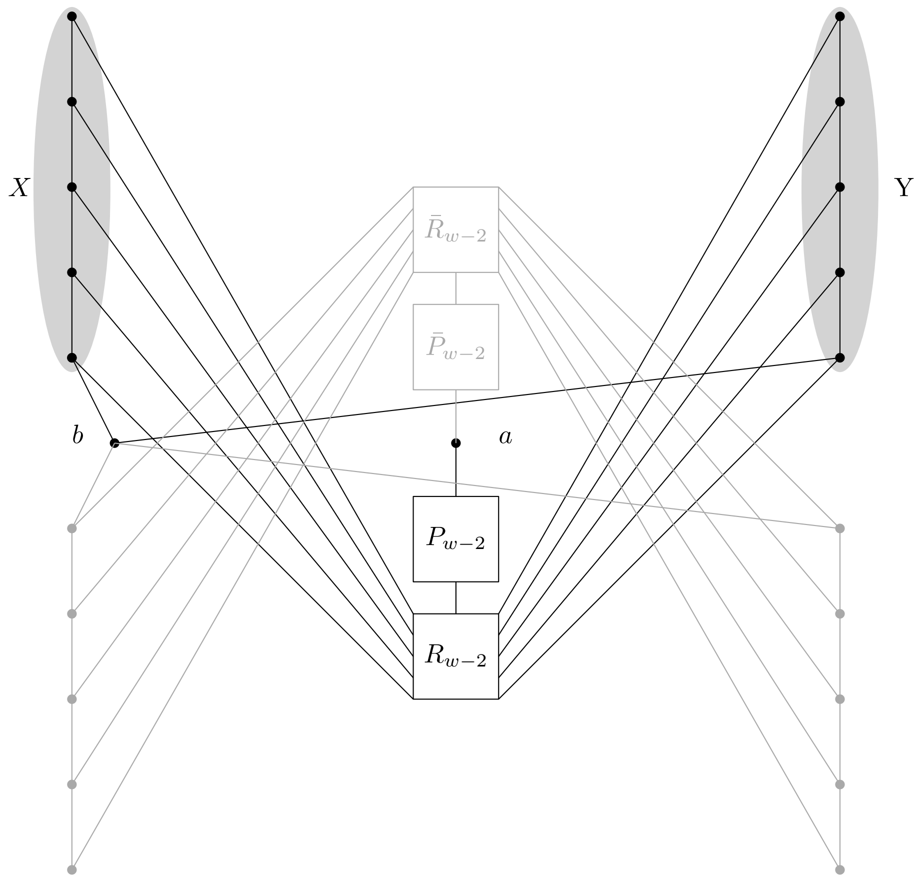

Figure 3 Recursive construction of P_w.

linear extensions of P_w which have a before b. With this assumption, a rainbow between R_{w-2} and either X or Y nests above each rainbow of P_{w-2}. If we let q_{w-2} be the size of a rainbow between R_{w-2} and either X or Y, then we have the recursion:

$$\mathsf{qn}(P_w) \geq \mathsf{qn}(P_{w-2}) + q_{w-2} \tag{1}$$

We think of this use of a self-dual construction as the *symmetry trick*. Constructions given in [6] (Proof of Prop. 2) and [1] (Proof of Thm. 4) also use a recursion based on two copies of the poset from the previous level of the recursion. This allows them to add an edge nesting over the rainbow from the previous level of the recursion.

Now suppose that for each width $u < w$, we choose the poset R_u to be an antichain of size u and the linear extensions L_x and L_y to be a realizer (think of L_x as the identity permutation and of L_y as its reverse). The Lemma of Erdős-Szekeres asserts that in every linear extension of R_u there is an increasing or a decreasing sequence of size at least $\lceil\sqrt{u}\rceil$, i.e., $q_u = \lceil\sqrt{u}\rceil$.

This value of q_u together with inequality (1) yields

$$\mathsf{qn}(P_w) \geq \sum_{u<w;\ u\equiv w(2)} \left\lceil\sqrt{u}\right\rceil \quad \in \Theta(w^{3/2}).$$

For the proof of the theorem we need a better construction for the reinforcement posets R_u. Such a construction will be provided in the proof of the following lemma[1].

▶ **Lemma 3.1.** *For each $u \geq 1$, there is a 2-dimensional poset R_u of width u with a realizer L_x, L_y, such that if L is a linear extension of R_u and d_x and d_y denote the maximum lengths of an increasing sequence of L which is decreasing in L_x and L_y respectively, then $d_x + d_y \geq u + 1$.*

The lemma says that we can assume the value $q_u = \lceil \frac{u+1}{2} \rceil$. With inequality (1) we get:

$$\mathsf{qn}(P_w) \geq \sum_{u<w;\ u\equiv w(2)} \left\lceil \frac{u+1}{2} \right\rceil$$

In the case w odd, $w = 2s+1$, we get $\mathsf{qn}(P_w) \geq \sum_{k=1}^{s} k = \binom{s+1}{2}$. In the case w even, $w = 2s$, we get $\mathsf{qn}(P_w) \geq \sum_{k=2}^{s} k = \binom{s+1}{2} - 1$. A simple computation shows that for $w \geq 4$ we get $\mathsf{qn}(P_w) \geq w^2/8$, independent of the parity of w. This completes the proof.

3.1 The construction of R_u for Lemma 3.1

The construction of R_u is again recursive. Let R_1 be a single point. Then clearly $d_x + d_y = 2$. For the construction of R_u for $u \geq 2$ we again use the symmetry trick. We take a series composition of $Q_1 + a + Q_2$ where Q_1 and Q_2 are two copies of R_{u-1} and compose it in parallel with a single element b. We remark that element a is here only for the sake of the exposition. The choice of the realizer L_x, L_y is depicted in Figure 4. Clearly $\mathsf{width}(R_u) = \mathsf{width}(R_{u-1}) + 1 = u$.

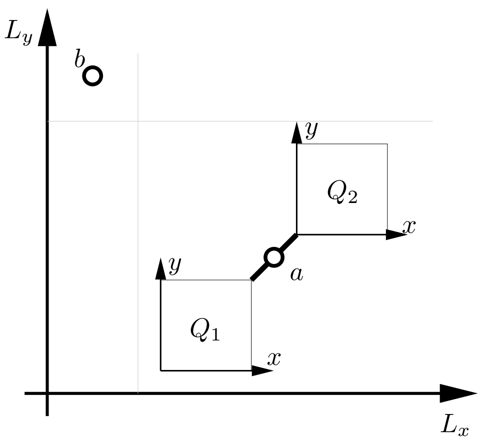

Figure 4 The recursive construction of R_u with its realizer L_x, L_y.

Let L be any linear extension of R_u. First suppose that $a < b$ in L. Let L' be the restriction of L to Q_1. By induction the lengths d'_x and d'_y of increasing sequences of L' which are decreasing in the two linear extensions of the realizer of Q_1 satisfy $d'_x + d'_y \geq u$. Since b precedes Q_1 in L_x and comes after Q_1 in L, we have $d_x \geq d'_x + 1$. Together with the trivial $d_y \geq d'_y$, we get $d_x + d_y \geq u + 1$.

[1] The lemma with a different proof was originally discovered by the first and the second author in joint research with Francois Dross, Piotr Micek, and Michał Pilipczuk.

If $b < a$ we consider Q_2. As before we get the two values d'_x and d'_y for the restriction L' of L to Q_2 and know by induction that $d'_x + d'_y \geq u$. The position of b relative to Q_2 in L and L_y shows that $d_y \geq d'_y + 1$, whence again $d_x + d_y \geq u + 1$. This completes the proof of the lemma.

4 Conclusion

We have made substantial progress in the understanding of queue-numbers of partial ordered sets. We take the opportunity to list and comment on open questions in the field.

- An obvious question is to ask for improved upper and lower bounds. More precisely, we now know that the growth rate of the queue-number of posets of width w is $(C + o(1))w^2$ for some constant C between 1/8 and 1. What is the precise value of constant C?
- What is the maximum queue-number of planer posets of width w? Knauer, Micek, and Ueckerdt [6] proved the lower bound w and the upper bound $3w - 2$.
- Heath and Pemmaraju [4] conjectured that planar posets on n elements have queue-number at most $\sqrt{n}$. The k antichain together with a matching up to a chain X and a matching down to a chain Y such that the chains represent a dual pair of linear extensions is a planar poset P with width $n = 3k$ elements and $\mathsf{qn}(P) = \sqrt{\lceil n/3 \rceil}$.
- In [6] is was shown that posets P of width 2 have $\mathsf{qn}(P) \leq 2$. In [1] it was shown that posets P of width 3 may have $\mathsf{qn}(P) \geq 4$ and satisfy $\mathsf{qn}(P) \leq 5$. Is 4 or 5 the best upper bound in this case?

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