Proportional Contact Representations of Planar Graphs [*]

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August 14, 2012

We study contact representations for planar graphs, with vertices represented by simple polygons and adjacencies represented by point-contacts or side-contacts between the corresponding polygons. Specifically, we consider proportional contact representations, where pre-specified vertex weights must be represented by the areas of the corresponding polygons. Natural optimization goals for such representations include minimizing the complexity of the polygons, and the unused area. We describe algorithms for proportional contact representations with optimal polygonal complexity for general planar graphs and planar 2-segment graphs, which include maximal outer-planar graphs and partial 2-trees.

1 Introduction

For both theoretical and practical reasons, there is a large body of work about representing planar graphs as *contact graphs*, where vertices are represented by geometrical objects with edges corresponding to two objects touching in some fashion. Typical classes of objects might be curves, line segments, or polygons.

In this paper we consider contact graphs with vertices represented by simple polygons in the plane with disjoint interiors, and adjacencies represented by point-contacts or side-contacts between corresponding polygons; see Fig. 1. In the weighted version of the problem, the input is not only a planar graph $G = (V, E)$ but also a weight function $w : V(G) \to R^+$ that assigns a weight to each vertex. A graph G admits a *proportional contact representation* with the

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weight function w if there exists a contact representation of G, where the area of the polygon for each vertex v of G is proportional to $w(v)$. Such representations have practical applications in cartography, VLSI Layout, and floor-planning.

Using adjacency of regions to represent edges in a graph can lead to a more compelling visualization than drawing a straight edge between two points [6]. In such representations of planar graphs it is desirable, for aesthetic, practical and cognitive reasons, to limit how complicated the polygons are. In practical areas such as VLSI layout, it is also desirable to minimize the unused area in the representation. With these considerations in mind, we study the problem of constructing proportional point-contact and side-contact representations of planar graphs with respect to the following parameters, partially taken from the cartography-oriented literature, e.g. [22, 31] :

- *complexity*: maximum number of sides in a polygon representing a vertex;

- *cartographic error:* $\max_{v \in V} |A(v) - w(v)|/w(v)$, where $A(v)$ is v's area, $w(v)$ its weight;

- *holes:* total area of the representation that is not used by a polygon and not adjacent to the unbounded face.

1.1 Related Work

Koebe's theorem [25] is an early example of point-contact representation and shows that a planar graph can be represented by touching circles. Any planar graph also has a contact representation, where all the vertices are represented by triangles [11] and with cubes in 3D [16]. Badent *et al.* [4] show that partial planar 3-trees and some series-parallel graphs also have contact representations with homothetic triangles. Recently, Gonçalves *et al.* [18] proved that any 3-connected planar graph and its dual can be simultaneously represented by touching triangles, and pointed out that 4-connected planar graphs also have contact representations with homothetic triangles.

While the above results deal with point-contacts, there is also related work on the problem of constructing side-contact representations. Gansner *et al.* [14] show that any planar graph G has a side-contact representation with convex hexagons. Moreover, they show that 6 sides are necessary if convexity is required. For maximal planar graphs, the representation obtained by the algorithm in [14] is hole-free. Buchsbaum *et al.* [6] give an overview on the state of the art concerning rectangle contact graphs. The characterization of graphs admitting a hole-free side-contact representation with rectangles was obtained by Koźmiński and Kinnen [26] or in the dual setting by Ungar [30]. There is a also a simple linear time algorithm for constructing triangle side-contact representations for outer-planar graphs [17].

Note that in all the contact representation results mentioned above, the areas of the circles or polygons are not considered. That is, these results deal with the unweighted version of the problem. Furthermore, previous works on side-contact representations rarely focused on the presence or absence of holes, or the actual area taken by such holes. In our work we take both the area of regions and the presence of holes into account. For example, we show that representations by triangles or any convex shapes are not possible for certain planar graphs with pre-specified weights.

Motivated by the application in VLSI layouts, contact representations of planar graphs with rectilinear polygons and no holes have also been studied. For example, Rahman *et al.* give an
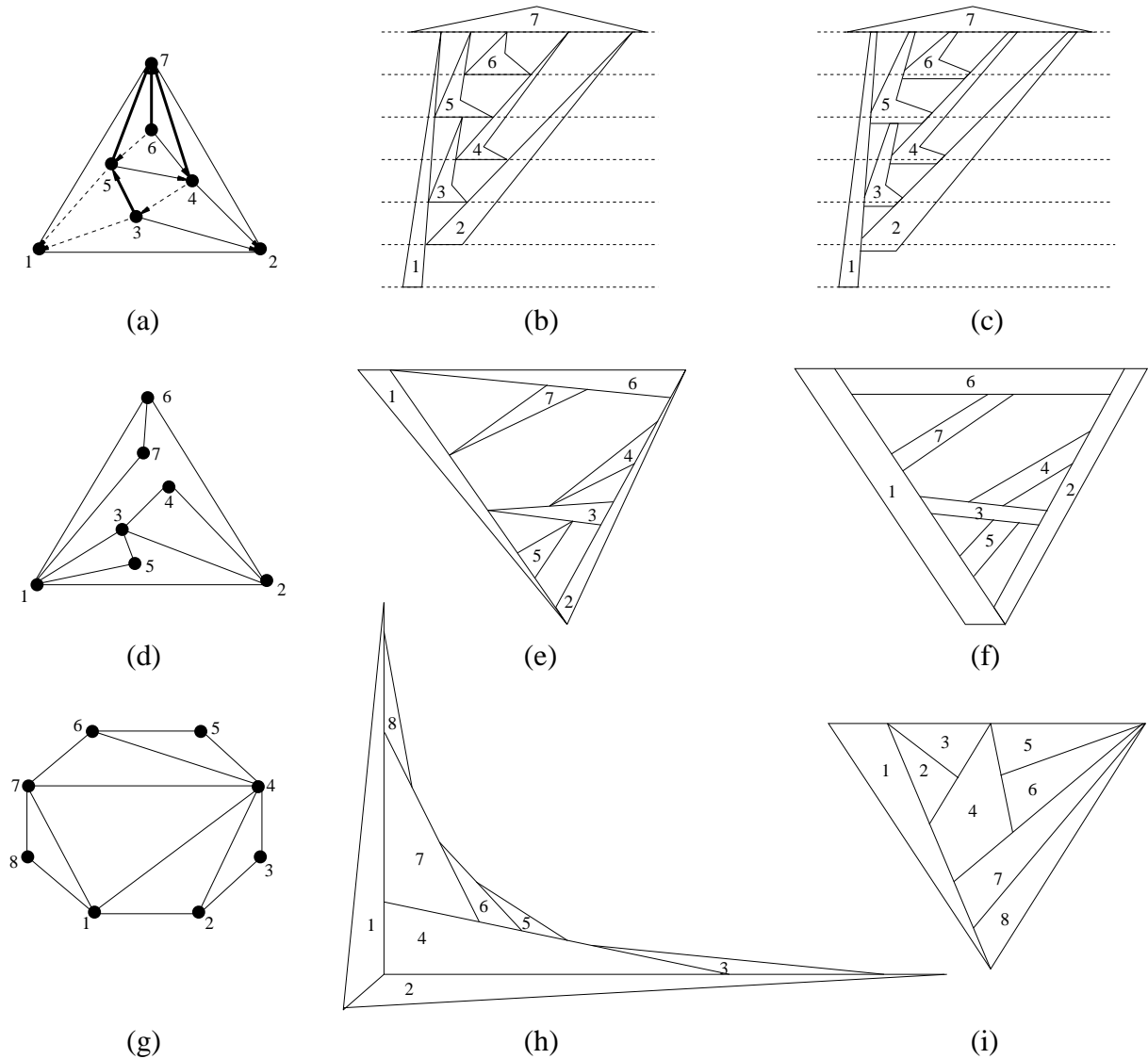
Figure 1: A general planar graph (a), its proportional point-contact representation with 4-sided non-convex polygons (b), and with its proportional side-contact representation with 7-sided polygons (c). A 2-tree (d), its proportional point-contact representation with triangles (e), and its proportional side-contact representation with trapezoids (f). A maximal outer-planar graph (g), its hole-free proportional side-contact representations with triangles (h), and with 4-sided convex polygons (i).

algorithm for hole-free proportional contact representation with 8-sided rectilinear polygons for a special class of plane graphs [28]. Another application of proportional contact representations can be found in *cartograms*, or value-by-area maps. Here, the goal is to redraw an existing geographic map so that a given weight function (e.g., population) is represented by the area of each country. Algorithms by van Kreveld and Speckmann [31] and Heilmann *et al.* [22] yield

representations for graphs obtained from geographic maps with rectangular polygons and with zero or small cartographic errors but occasionally they compromise either the number of sides or the adjacencies. De Berg *et al.* describe an algorithm for hole-free proportional contact representation with at most 40 sides for an internally triangulated plane graph G (and only 20 sides when G has four vertices on the exterior face and contains no separating triangles) [8]. This was later improved to 34 sides [24], then to 12 sides [5], and then to 10 sides [1]. It is known that 8 sides are sometimes necessary and always sufficient for the rectilinear umweighted case [32] and very recently, it was shown that that 8 sides are also sufficient for the rectilinear weighted case [3].

1.2 Our Results

In this paper we study the problem of proportional contact representation of planar graphs, with the goal to minimize the complexity of the polygons and the cartographic error. The main results in our paper are optimal (with respect to polygon complexity) algorithms for proportional contact representations for general planar graphs, and 2-segment graphs[1]. (G is a 2-segment graph if it can be represented by assigning interior-disjoint line segments to vertices such that line segments share a point if and only if the corresponding vertices are adjacent, and no 3 line segments share a point.) This class contains interesting subclasses such as partial 2-trees and outer-planar graphs. We say k-sided polygons are sometimes necessary and always sufficient for representations of a particular class of planar graphs when there is an algorithm to construct a representation for any graph of this class with k-sided polygons and there is at least one example of a graph in this class that requires a (non-degenerate) k-sided polygons for any representation. Specifically, we show that: (a) 4-sided polygons are sometimes necessary and always sufficient for a point-contact proportional representation of planar graphs; (b) triangles are necessary and sufficient for point-contact proportional representation of 2-segment graphs; (c) trapezoids are sometimes necessary and always sufficient for side-contact proportional representation of 2-segment graphs; (d) triangles are necessary and sufficient for hole-free side-contact proportional representation for maximal outer-planar graphs, while convex quadrilaterals are sometimes necessary and always sufficient if the outer-boundary has constant complexity. The main results are summarized in Table 1.

The rest of the paper is organized as follows. In Section 2, we present some terminology and background about canonical orders and Schnyder realizers. In Section 3 we prove that 4-sided non-convex polygons are always sufficient and sometimes necessary for proportional point-contact representation of planar graphs. In Section 4, we describe algorithms for proportional point-contact and side-contact representation of 2-segment graphs. Some of their subclasses allow stronger results than the general class, this includes partial 2-trees and maximal outer-planar graphs. In Section 5 we conclude with a brief discussion and some open problems.

2 Preliminaries

In a *point-contact representation* of a planar graph $G = (V, E)$, we construct a set P of closed simple interior-disjoint polygons with an isomorphism $\mathcal{P} : V \to P$, where for any two vertices $u, v \in V$, the boundaries of $\mathcal{P}(u)$ and $\mathcal{P}(v)$ touch through at least one *contact point* if and

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[1]The conference version of this paper [2] only had a construction with cartographic error for this class.

Class of Graphs	Convexity	Complexity Lower Bound	Complexity Upper Bound	Hole-Free	Type of Contact
Planar	no	4	4	no	point
2-segment graphs	yes	3*	3	no	point
2-segment graphs	yes	4**	4	no	side
Maximal outer-planar	yes	$3/4^\dagger$	$3/4^\dagger$	yes	side

Table 1: Entries in this table correspond to results proven in this paper, except one marked (*), which is trivial to see since any polygon with positive area requires at least three sides, and another marked (**), which follows from [17]. All the upper bound results here are obtained with algorithms that produce no cartographic error. Maximal outer-planar graphs marked with ($\dagger$) can be represented with convex quadrilaterals if the outer-boundary has constant complexity, or with triangles if the outer-boundary has linear complexity.

only if (u, v) is an edge. A *side-contact representation* of a planar graph is defined analogously, where instead of a contact point, we have a *contact side* between $\mathcal{P}(u)$ and $\mathcal{P}(v)$, which is a non-degenerate line segment in the boundary of both. A *hole* of such a contact representation is a non-empty region that does not belong to any $P(v)$ and is not part of the infinite region.

Let Γ be a contact (point-contact or side-contact) representation of G. We can then place a point p_v inside each $P(v)$ and for any edge (v, w) connect p_v and p_w via the contact point of $P(v)$ and $P(w)$. This gives a planar drawing of G. In this drawing, any face f of G contains inside either a point where all polygons of vertices on f meet, or one (or more) holes.

In the weighted version of the problem, the input also includes a weight function $w : V(G) \to R^+$ that assigns a positive weight to each vertex of G. We say that G admits a *proportional contact representation* with the weight function w if there exists a contact representation of G such that the area of the polygon for each vertex v of G is proportional to its weight $w(v)$. We define the *complexity of a polygon* as the number of sides it has. In this paper, we also consider a polygon with less than k sides to be a (degenerate) k-sided polygon for convenience.

A *plane graph* is a planar graph with a fixed embedding. A plane graph is *fully triangulated* or *maximally planar* if all its faces, including the outer-face, are triangles. Both the concept of "canonical order" [12] and "Schnyder realizer" [29] are defined for fully triangulated plane graphs in the context of straight-line drawings of planar graphs on an integer grid. We briefly review the two concepts below:

Let $G = (V, E)$ be a fully triangulated plane graph with outer-face u, v, w in clockwise order. Then G has a *canonical order* of the vertices $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$, $|V| = n$, which satisfies for every $4 \leq i \leq n$:

- The subgraph $G_{i-1} \subseteq G$ induced by $v_1, v_2, \ldots, v_{i-1}$ is biconnected, and the boundary of its outer-face is a cycle C_{i-1} containing the edge (u, v).

- The vertex v_i is in the exterior face of G_{i-1}, and its neighbors in G_{i-1} form an (at least 2-element) subinterval of the path $C_{i-1} - (u, v)$.

A *Schnyder realizer* of a fully triangulated graph G is a partition of the interior edges of G into three sets T_1, T_2 and T_3 of directed edges such that for each interior vertex v, the following conditions hold:

- v has out-degree exactly one in each of T_1, T_2 and T_3,

- the counterclockwise order of the edges incident to v is: entering T_1, leaving T_2, entering T_3, leaving T_1, entering T_2, leaving T_3.

The first condition implies that each T_i, $i = 1, 2, 3$ defines a tree rooted at exactly one exterior vertex and containing all the interior vertices such that the edges are directed towards the root. The following by now classical lemma shows a profound connection between canonical orders and Schnyder realizers.

Lemma 1 *Let G be a fully triangulated plane graph. Then a canonical order of the vertices of G defines a Schnyder realizer of G, where the outgoing edges of a vertex v are to its first and last predecessor (where "first" is w.r.t. the clockwise order around v), and to its highest-numbered successor.*

3 Proportional Point-Contact Representations

Recall that any planar graph can be represented by touching triangles; see Fig. 2(a)–(b). From this, it is very easy to create proportional point-contact representations: scale the representation such that the triangle $T(v)$ of vertex v has area at least $w(v)$, and then "carve away" a triangular part H of $T(v)$ near a corner to achieve the correct area; see Fig. 2(c). Now the new polygon has six corners with two of them overlapping with each other. This can be avoided by moving these two overlapped corners a small distance away from each other and changing the area of H accordingly; see Fig. 2(d). So we can easily achieve 6-sided representations.
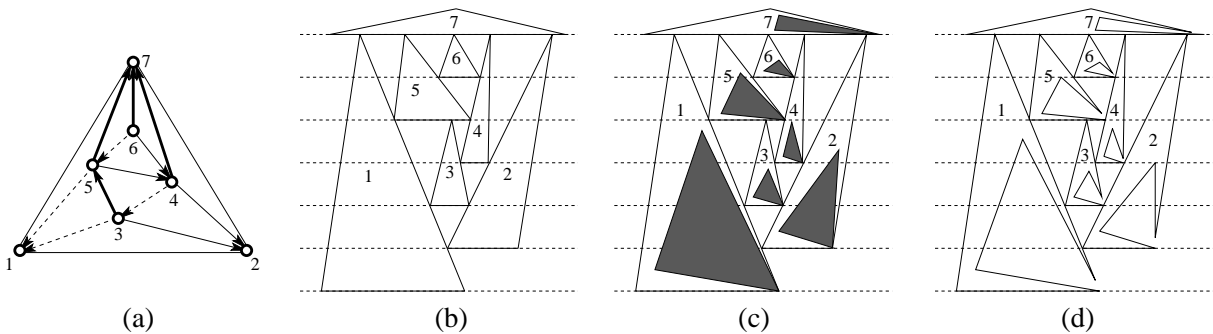

Figure 2: Construction of proportional point-ontact representations with 6-sided polygons.

In this section, we create point-contact representations with the optimal number of sides. Indeed, we show that 4-sided non-convex polygons always sufficient for a proportional contact representation of a planar graph. This is quite easy to do for 2-segment graphs (essentially by adding a small triangle at one end of the segment), but we show this here for *all* planar graphs.

We first describe an algorithm to obtain proportional point-contact representations of planar graphs using 4-sided non-convex polygons. We then show that there exists a planar graph with
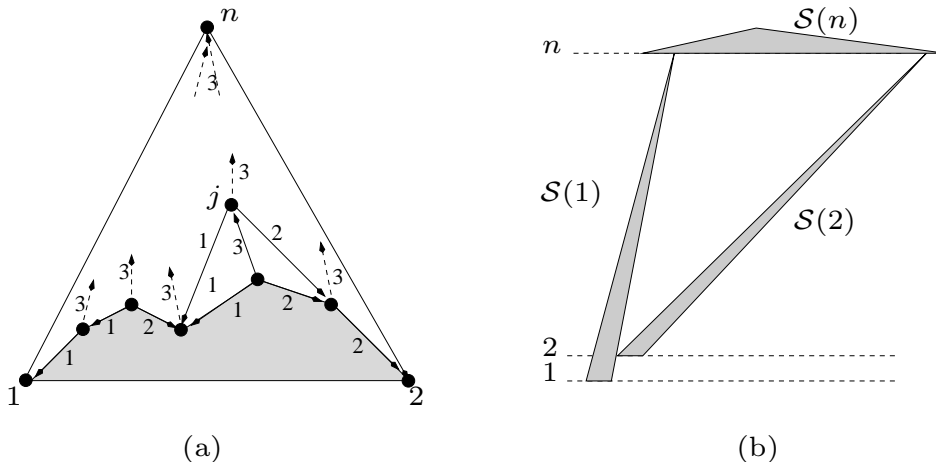
Figure 3: (a) The canonical order and T_i (marked by labels); (b) the placement of 1,2,n.

a given weight function that does not admit a proportional point-contact representation with convex polygons, thus making our 4-sided construction optimal.

Theorem 1 *Let $G = (V, E)$ be a planar graph and let $w : V \to R^+$ be a weight function. Then G admits a proportional point-contact representation with respect to w in which each vertex of V is represented by a 4-sided polygon. It can be found in linear time.[2]*

Proof. We first take a planar embedding of G and assume that it is fully triangulated, for if it is not, we can add dummy vertices (and edges to these vertices) to make it so, and later remove those dummy vertices from the obtained proportional contact representation.

Assume (after some scaling) that $w(v) \leq 1/n^2$ for all $v \in V$ and fix an arbitrary outer-face. We construct the drawing incrementally, following a canonical ordering $v_1, \ldots, v_n$. We prescribe what the polygon assigned to j looks like before even placing it (here and in the rest of the paper we use j as a shorthand for v_j). So let T_1, T_2, T_3 be the Schnyder realizer defined by the canonical ordering, where T_1 is rooted at 1, T_2 is rooted at 2 and T_3 is rooted at n; see Fig. 3. Let $\Phi_i(j)$ be the parent of j in tree T_i.

It is easy to show that $T_2^{-1} \cup T_1$ is an acyclic graph on the vertex set $V - \{n\}$, where T_2^{-1} is the tree T_2 with the direction of all its edges reversed. For every vertex $j \neq n$, let $\pi(j)$ be the index of j in a topological order of this graph. Then $n \geq \pi(\Phi_1(j)) > \pi(j) > \pi(\Phi_2(j)) \geq 1$. Now for every vertex $j \neq 1, 2, n$, we define the *spike* $\mathcal{S}(j)$ to be a 4-sided polygon with one reflex vertex. One segment (the *base*) is horizontal with y-coordinate j. Its length will be determined later, but it will always be at least $2/n^2 \geq 2w(j)$. From the left endpoint of the base, the spike continues with the *upward segment*, which has slope $\pi(j)$ and up to its *tip* which has y-coordinate $y = \Phi_3(j)$. Next comes the *downward segment* until the reflex vertex, and from there to the right endpoint of the base; see Fig. 4(a). The placement of the reflex vertex is arbitrary, as long as the resulting shape has area $w(j)$ and the down-segment has positive slope. Note that since the base has length $\geq 2w(j)$ and y-coordinate j, the reflex vertex will have y-coordinate at most
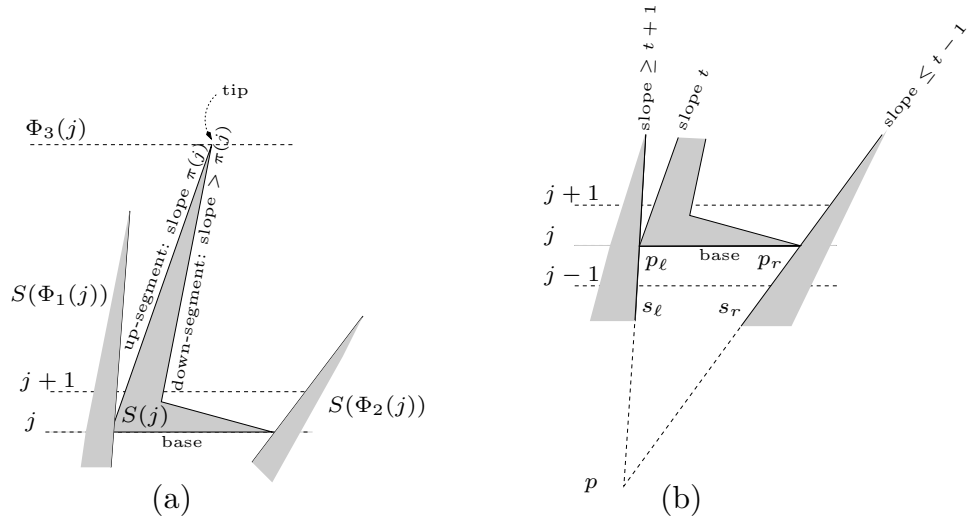

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Figure 4: (a) Adding j; (b) computing the width of the base.

$j + 1$. We first place $1, 2, n$, and then add $3, \ldots, n-1$ (in this order):

- Vertex 1 is represented by a triangle $\mathcal{S}(1)$ whose base has length $2w(1)/(n-1)$, placed arbitrarily with y-coordinate 1. The tip of $\mathcal{S}(1)$ has y-coordinate n.

- Vertex 2 is represented by a triangle $\mathcal{S}(2)$ whose base has length $2w(2)/(n-2)$, placed at y-coordinate 2 and with its left endpoint abutting $\mathcal{S}(1)$. The tip of $\mathcal{S}(2)$ has y-coordinate n.

- Vertex n is represented by a triangle whose base is at y-coordinate n and long enough to cover the tips of $\mathcal{S}(1)$ and $\mathcal{S}(2)$. We choose the height of $\mathcal{S}(n)$ such that the area is correct.

We maintain the following invariant: For $j \geq 2$, after vertex j has been placed, the horizontal line with y-coordinate $j+1$ intersects only the spikes of the vertices on the outer-face of G_j, and in the order in which they occur on the outer-face.

To place $j \geq 3$, we place the base of $\mathcal{S}(j)$ with y-coordinate j, and extend it from the down-segment of $\Phi_1(j)$ to the up-segment of $\Phi_2(j)$. Recall that $\Phi_2(j)$ and $\Phi_1(j)$ are exactly the first and last predecessor of j, and $j = \Phi_3(i)$ for all other predecessors $i \neq j$. Hence $\mathcal{S}(j)$ touches $\mathcal{S}(\Phi_1(j))$ and $\mathcal{S}(\Phi_2(j))$ at the ends of the base, and all other predecessors i of j have their tips at the base. Note that this creates a contact between j and all its predecessors, as desired. The rest of $\mathcal{S}(j)$ is then as described above. It is easy to verify the invariant, and therefore $\mathcal{S}(j)$ does not intersect any other spikes.

To see that the base of $\mathcal{S}(j)$ has length $\geq 2/n^2$ as required, let p_ℓ and p_r be its left and right endpoints, and s_ℓ and s_r be the other segments containing them. Imagine that we extend s_ℓ and s_r until they meet in a point p. Since s_r contains a point with y-coordinate $\leq j-1$ (at the base of $\mathcal{S}(\Phi_2(j))$), triangle $\Delta\{p, p_\ell, p_r\}$ has height $h \geq 1$; see Fig. 4.

Let $t = \pi(v_j)$ be the slope of the up-segment of $\mathcal{S}(v_j)$. Since $\pi(\Phi_2(v_j)) < \pi(v_j) = t$, we have that s_r has slope at most $t-1$ and $x(p_r) \geq x(p) + \frac{h}{t-1}$. On the other hand, the slope of s_ℓ

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is positive by construction, and must exceed the slope of the up-segment of $\Phi_1(v_j)$, which has slope $\pi(\Phi_1(v_j)) > \pi(v_j) = t$. So s_ℓ has slope $\geq t + 1$ and $x(p_\ell) \leq x(p) + \frac{h}{t+1}$. Therefore,

$$x(p_r) - x(p_\ell) \geq \frac{h}{t - 1} - \frac{h}{t + 1} = \frac{h((t + 1) - (t - 1))}{t^2 - 1} \geq \frac{2h}{t^2} \geq \frac{2}{n^2}$$

as desired. Therefore the base of $\mathcal{S}(j)$ is wide enough, which shows the correctness of our construction.

It remains to analyze the run-time of the algorithm implicit in our constructive proof. Computing the Schnyder decomposition can be done in linear time. We claim that processing vertex j also takes constant time; the claim then follows. Note that the base of $S(j)$ is already fixed when handling j, since $\Phi_1(j)$ and $\Phi_2(j)$ are placed already, and we only need to compute the intersection of their polygons with the horizontal line with y-coordinate j. This also fixes the tip of $S(j)$. All that remains to do is hence to find an appropriate point r for the reflex vertex. Let ℓ be the line from the tip to the right end of the base of $S(j)$, and let C be the convex hull of $S(j)$ (i.e., the triangle defined by the tip and the base of $S(j)$.) If C has area A, then r must have height $2(A - w(v))/\|\ell\|$ over ℓ. Draw the line ℓ' parallel to ℓ at that distance. Also draw the vertical line ℓ'' through the tip of $S(j)$. Any point that is on ℓ', to the left of ℓ'' and inside C is suitable for r, and such a point exists by the above discussion of correctness. Hence finding r takes a constant number of arithmetic operations, and the algorithm to find the contact representation takes linear time. $\qquad\square$

Our construction used non-convex shapes. Lemma 2 shows that this is sometimes required.

Lemma 2 *There exists a planar graph and a weight function such that the graph does not admit a proportional point-contact representation, with respect to the weight function, with convex shapes for all vertices.*

Proof. We aim to show that the graph in Fig. 5(a) has no proportional representation with convex polygons if the small vertices (a_0, a_1, a_2, b) have weight δ and the larger vertices (c_0, c_1, c_2, d) have weight $D > 3\delta$. Assume for contradiction that we had such a representation. Note that this graph is 3-connected and all faces of this graph are isomorphic (even when taking vertex weights into account), so all planar embeddings of it are equivalent. We may assume therefore that d is incident to the outer-face. We will focus now on the sub-graph defined by a_0, a_1, a_2 and its interior. Fig. 5(b) and (c) illustrate the notation for the following argument.

For $i = 0, 1, 2$, let p_i be a contact point between $\mathcal{P}(a_i)$ and $\mathcal{P}(a_{i+1})$. (All additions and subtractions in this proof are modulo 3.) Let $T_0 = \Delta\{p_0, p_1, p_2\}$ be the triangle spanned by p_0, p_1 and p_2. Further, let q_i be a point of contact between $\mathcal{P}(a_i)$ and $\mathcal{P}(b)$. Let L_i be the line parallel to $p_{i-1}p_i$ that passes through q_i, and let H_i be the half-space supported by L_i that contains $p_{i-1}p_i$. The lines L_0, L_1, L_2 define a triangle T_1 with corners p_0', p_1', p_2' where p_i' is the corner of T_1 corresponding to the corner p_i of T_0.

Observe that triangle $\Delta\{p_0, p_1, q_1\}$ is a subset of $\mathcal{P}(a_1)$ by convexity, so it has area at most δ. The trapezoid $T_0 \cap H_1$ has less than twice the area of $\Delta\{p_0, p_1, q_1\}$, so it has area at most 2δ. Since the triangle $\Delta\{p_1, q_2, q_1\}$ has to accommodate $\mathcal{P}(c_0)$ it has area at least $D > 3\delta$. This implies that $q_2 \notin T_0 \cap H_1$. Analogous arguments show that for any $i \neq j$, q_i is not inside $T_0 \cap H_j$. Hence, q_i is on L_i between p_i' and p_{i-1}'. The generic situation is illustrated in Fig. 5(c).

Now consider the triangle $\Delta\{p_1, p_1', q_1\}$: it has the same height and a base that is no larger than that of $\Delta\{p_0, p_1, q_1\}$, so the area of $\Delta\{p_1, p_1', q_1\}$ is at most δ. Similarly, one can show that triangle $\Delta\{p_1, q_2, p_1'\}$ has area at most δ.
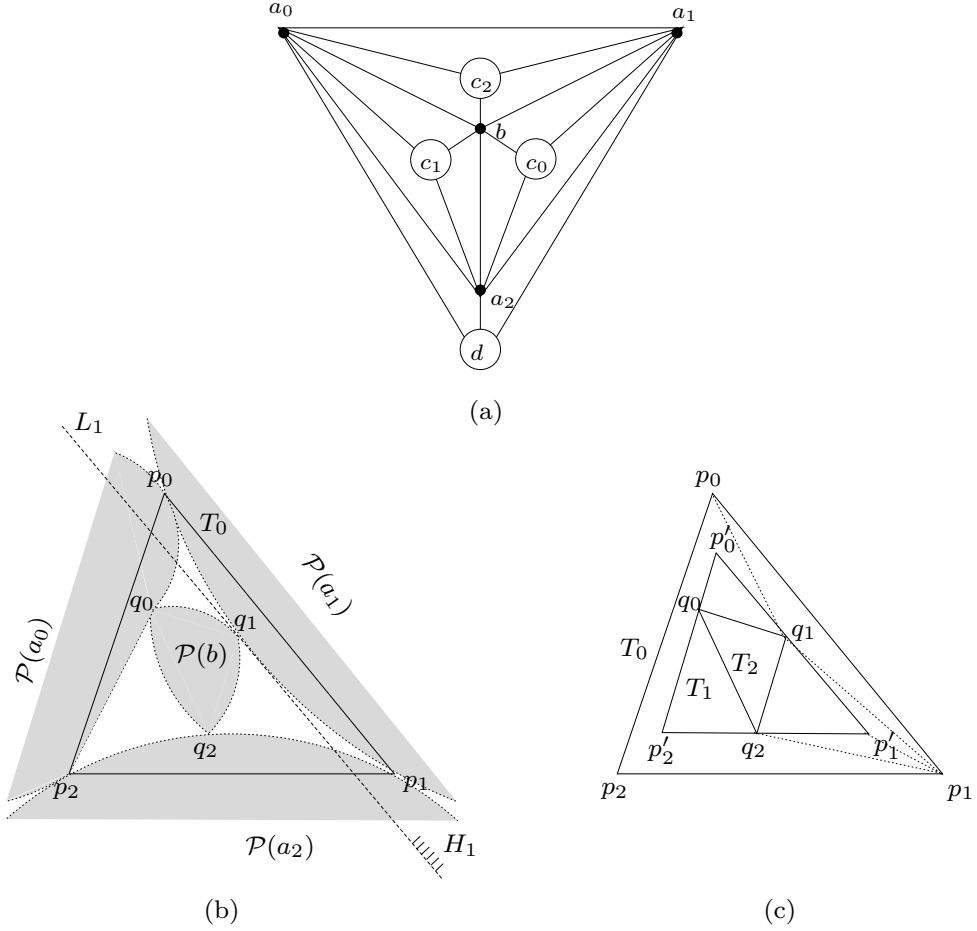
Figure 5: A graph G that does not have a proportional contact-representation with convex shapes.

Since the triangle $\Delta\{p_1, q_2, q_1\}$ contains $\mathcal{P}(c_0)$ it has area at least D. Therefore triangle $\Delta\{p_1', q_2, q_1\} = \Delta\{p_1, q_2, q_1\} - \Delta\{p_1, q_2, p_1'\} - \Delta\{p_1, p_1', q_1\}$ has area at least $D - \delta - \delta > \delta$ (by choice of $D > 3\delta$.) Similarly, one can show that triangle $\Delta\{p_2', q_0, q_2\}$ and triangle $\Delta\{p_0', q_1, q_0\}$ have area strictly greater than δ.

Define T_2 to be the triangle $\Delta\{q_0, q_1, q_2\}$, and observe that $T_2 \subseteq \mathcal{P}(b)$, and hence T_2 has area at most δ. But now we have a triangle T_2 of area at most δ that is circumscribed by a triangle T_1 such that the three triangles of $T_1 - T_2$ each have area strictly greater than δ. This is impossible due to a classic geometric result, which states that when a triangle T_2 is inscribed in another triangle T_1, then the area of T_2 is at least as much as the minimum of the areas of the three triangles in $T_1 - T_2$. For details, see e.g. [13]. □

Lemma 2 implies that 3-sided polygons are not always sufficient for proportional contact representations of planar graphs. On the other hand, Theorem 1 implies that any planar graph has a proportional contact representation with any given weight function on the vertices so that each of the vertices is represented by a non-convex 4-sided polygon. Summarizing these two results we have the following theorem.

Theorem 2 *4-sided non-convex polygons are always sufficient and sometimes necessary for proportional point-contact representation of a planar graph with a given weight function on the vertices.*

4 Subclasses of Planar Graphs with Convex Representations

In this section we address the problem of proportional contact representations with convex polygons of low complexity. The lower bound in Lemma 2 shows that for some planar triangulations, the complexity in any proportional contact representation must be at least 4 and the polygons must be non-convex. We hence focus on planar graphs with fewer edges.

We first give some constructions for so-called 2-segment graphs and then discuss what these graphs are and which well-known subclasses of planar graphs (such as series-parallel graphs and triangle-free planar graphs) fall into them. Finally we give an entirely different construction for maximal outer-planar graphs, which (as opposed to all previous constructions) gives hole-free representations.

4.1 2-segment graphs

Call a planar graph a *2-segment graph* if it can be represented by assigning interior-disjoint line segments to vertices such that two line segments share a point if and only if the corresponding vertices are adjacent, and no 3 line segments share a point; see Fig. 6(a)–(b).

Given a 2-segment representation Γ of a graph G, we can easily construct a side-contact representation for G by giving an arbitrary thickness to each segment of Γ. This is a side-contact representation and uses convex shapes. It also seems intuitive that we can choose the thickness suitable so that weights of vertices are respected. However, choosing the thickness is non-trivial for two reasons: we must be careful not to make segments too thick (and hence created unwanted adjacencies), and thickened segments may overlap, and removing such overlap might create error unless we are very careful about how we thicken the segments.

Let Γ be a 2-segment representation of G, with vertex v represented by line segment $\ell(v)$. After possible rotation, we may assume that Γ has no horizontal line segment, hence every line segment has a well-defined *left* and *right side* and *top* and *bottom endpoint*. After lengthening segments, if necessary, we may also assume that no two line segments end at the same point. So that if two line segments share a point, then one of them ends on either the right or the left side of the other.

Lemma 3 *There exists an order $v_1, \ldots, v_n$ of the vertices in G such that for any i, the right side of $\ell(v_i)$ contains only ends of neighbors v_j with $j > i$.*

Proof. It is enough to show that there is an "unobstructed" segment ℓ, i.e., a segment for which no other segment ends on the right side. We then set v_n to be the vertex belonging to ℓ, and obtain the complete ordering by induction after removing ℓ.

To find an unobstructed segment look at the scene from $(\infty, 0)$ and order the visible ends of segments by increasing y-coordinate. This yields a sequence of top and bottom ends of segments. The first point in the sequence is a bottom endpoint and the last one is a top endpoint. Hence there are two consecutive endpoints p_1, p_2 for which p_1 is a bottom endpoint and p_2 a top

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[2]A point is *visible* from $(\infty, 0)$ when the line segment between these two points does not cross any segments.
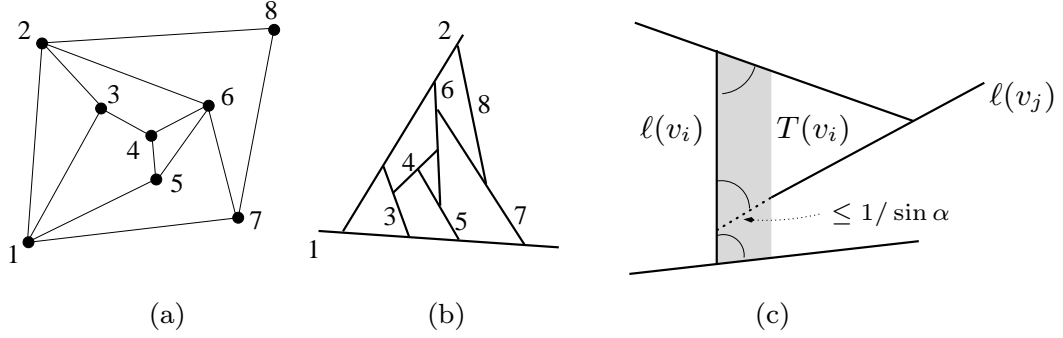
Figure 6: (a) A 2-segment graph; (b) a representation with line segments; the numbers indicate a suitable vertex order; (c) converting $\ell(v_i)$ into a trapezoid $T(v_i)$ and clipping $\ell(v_j)$. The indicated angles are all at least α, hence the off-set is at most 1, and at most $\frac{1}{\sin \alpha}$ is cut off $\ell(v_j)$.

endpoint. Since the segments in Γ do not cross, this pair of points is the pair of endpoints of an unobstructed segment. $\qquad\square$

Theorem 3 *Let $G = (V, E)$ be a 2-segment graph and let $w : V \to R^+$ be a weight function. Then G admits a proportional point-contact representation with respect to w in which each vertex of V is represented by a trapezoid. Given a 2-segment representation of G, such a contact representation can be found in $O(n \log n)$ time.*

Proof. As before we presume that no line segment is horizontal and no two line segments end in a point. Let $\delta > 0$ be the minimum feature size of this segment representation, i.e., the smallest distance between a line segment ℓ and an endpoint of another line segment that does not end at ℓ. Next, let α be the smallest angle between two line segments where one ends at the other. Scale the entire drawing, if needed, such that $\delta > 2$ and $||\ell(v)|| \geq w(v) + \frac{2}{\sin \alpha} + \frac{1}{\tan \alpha}$ for all vertices v.

Compute the sequence $v_1, \ldots, v_n$ of vertices such that the right side of $\ell(v_i)$ contains only endpoints of $\ell(v_j)$ for $j > i$. We thicken vertices in order $v_1, \ldots, v_n$. At the time of handling v_i, we have a contact representation where each v_h, $h < i$ is represented as a trapezoid $T(v_h)$ of area $w(v_h)$, while each v_j, $j \geq i$ is represented as a line segment that is a part of $\ell(v_j)$ (but may have been shortened a bit). We will guarantee in the following that $\ell(v_j)$ has been shortened by at most $\frac{1}{\sin \alpha}$ at each end, so that it still has length at least $w(v_j) + \frac{1}{\tan \alpha}$.

To thicken v_i, off-set $\ell(v_i)$ by moving a copy of it to the right in parallel while shortening/lengthening it so that it still touches the same segments/trapezoids as it did before. We choose the distance d for off-setting such that the trapezoid $T(v_i)$ between the off-set line and $\ell(v_i)$ has area $w(v_i)$. In particular, observe that $T(v_i)$ has a base of length $||\ell(v_i)|| \geq w(v_i) + \frac{1}{\tan \alpha}$ and the angles at the base are at least α. Then the length of the segment parallel to $\ell(v_i)$ is at least $(||\ell(v_i)|| - 2\frac{d}{\tan \alpha})$ and the area of the newly formed trapezoid is $d(||\ell(v_i)|| - \frac{d}{\tan \alpha}) \geq d(w(v_i) + \frac{1}{\tan \alpha} - \frac{d}{\tan \alpha})$. Therefore, the required off-set d is at most $1 < \delta/2$, which implies that in the final representation no trapezoids intersect unless their line segments touched.

This yields the desired contact representation, except that a line segment $\ell(v_j)$ that ended on the right side of $\ell(v_i)$ now intersect $T(v_i)$. By the chosen vertex order, $j > i$, $\ell(v_j)$ has not been thickened into a trapezoid yet. We clip $\ell(v_j)$ so that it now ends on the right side of $T(v_i)$.

Since $T(v_i)$ had height at most 1, and $\ell(v_j)$ attaches to $\ell(v_i)$ with angle at least α, this clips at most $\frac{1}{\sin\alpha}$ off $\ell(v_j)$ as desired. No segments can attach at $\ell(v_j)$ in the clipped-off part, since it is all within distance $1 < \delta/2$ of $\ell(v_i)$. So we obtain the contacted representation with $v_1, \ldots, v_i$ thickened, and the entire proportional contact representation can be built by induction.

All operations used to compute this contact representation take constant time per vertex, except for the computation of δ. It is easy to find the two closest contacts of a 2-segment representation in linear time. In general, however, the feature size may be determined by the distance from the end of one segment to an interior point of another segment. Determining these distances in cases where we have non-convex faces in the representation is more intricate can be done with a sweep-line algorithm in $O(n \log n)$ time.[3] □

Theorem 4 *4-sided convex polygons are always sufficient and sometimes necessary for proportional side-contact representation of a 2-segment graph with a given weight function on the vertices. Given a 2-segment representation of G, such a contact representation can be found in $O(n \log n)$ time.*

Proof. The sufficiency and the running time for the algorithm (assuming that a 2-segment representation is given) have been discussed before.

To establish necessity, consider the graph $K_{2,5}$. This is a 2-segment graph. In this graph two vertices have five common neighbors, but as was proved in [17], in any side-contact representation with triangles, any pair of vertices has at most four common neighbors. Hence this graph has no side-contact representation with triangles, let alone one that respects the weights. Another, smaller, example consists of the graph obtained from $K_{2,4}$ by adding an edge between the vertices v_1, v_2 of the partition of size two. This graph is a 2-segment graph. The two vertices v_1, v_2 have four common neighbors, but as was proved in [17], in any side-contact representation with triangles, any pair of adjacent vertices has at most three common neighbors. So again this graph has no side-contact representation with triangles. □

If we switch from side-contact representations to point-contact representations, however, we can reduce the complexity of the regions from four to three. Specifically, we can replace line-segments by triangles.

The construction of the triangle $P(v)$ of a vertex v can be done very much like the construction of the trapezoids in the previous proof. Instead of taking a parallel shift of the segment $\ell(v)$ we now fix the lower endpoint and rotate a copy of $\ell(v)$ clockwise, lengthening/shortening it as needed to maintain contact with the neighbor, until the area of the triangle between the two copies of $\ell(v)$ is $w(v)$. Then clip neighbors that ended on the right side of $\ell(v)$ as before. Note that this construction guarantees that at least one half of the edges of G are realized by side contacts.

Theorem 5 *Triangles are always sufficient (and of course necessary) for a proportional point-contact representation of a 2-segment graph with a given weight function on the vertices. Given a 2-segment representation of G, such a contact representation can be found in $O(n \log n)$ time.*

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[3] A lower bound δ' for the minimum feature size can be computed in linear time using Chazelle's triangulation algorithm for polygons [7]. Since Chazelle's algorithm is rather complex and the resulting δ' can be arbitrarily smaller than the true δ we stick to the $O(n \log n)$ time bound.

4.1.1 Constructing 2-segment representations

The constructions for Theorem 4 and 5 are based on a 2-segment representation. In this subsection we review the characterization of 2-segment graphs and discuss issues of constructing such a representation.

Thomassen presented the characterization (Theorem 6) of 2-segment graphs at Graph Drawing 1993 but never published his proof. A proof of the theorem is part of [10].

The characterization theorem together with results by Lee and Streinu imply that one can test in quadratic time whether a given graph is a 2-segment graph. In contrast, Hliněný [23] showed that the recognition of general contact graphs of segments is NP-complete.

Theorem 6 *[10] A planar graph $G = (V, E)$ is a 2-segment graph if and only if it is $(2,3)$-sparse, i.e., for any $W \subseteq V$ the set $E[W]$ of edges induced by W must satisfy $|E[W]| \leq 2|W| - 3$.*

The necessity is quite straightforward to see. Let $\mathcal{S}$ be the set of segments of a 2-segment representation of G. For $W \subset V$ let X_W be the set of end-points of segments in $\mathcal{S}$ corresponding to vertices of W. Since we have a 2-segment representation we may assume that $|X_W| = 2|W|$. There is an injection ϕ from edges in $E[W]$ to points in X_W, but points belonging to the convex hull of X_W cannot be in the image of ϕ. Since the convex hull contains at least three points we get $|E[W]| \leq |X_W| - 3 = 2|W| - 3$. So if G is a 2-segment graph, then it is $(2,3)$-sparse.

Below we give a new proof of the sufficiency, which has three advantages: (a) It is shorter and more direct than the proof in [10], (b) it uses an interesting detour into rigidity theory to prove the result, and most importantly (c) it is constructive and allows us to construct a 2-segment representation in quadratic time.

Theorem 7 *Given a planar graph G, we can test in quadratic time whether it is a 2-segment graph, and if so, construct a 2-segment representation.*

Proof. We will give an algorithm that either detects that G is not $(2,3)$-sparse (which by Theorem 6 means it is not a 2-segment graph), or construct a 2-segment representation of G in quadratic time. This implies that every $(2,3)$-sparse graph is a 2-segment graph, i.e., the sufficiency for Theorem 6.

We need some prerequisites. A *Laman graph* (also called a $(2,3)$-*tight graph*) is a $(2,3)$-sparse graph with the maximum number $(2n - 3)$ of edges. Laman graphs are of interest in rigidity-theory, see e.g. [15, 19]. Laman graphs admit a *Henneberg construction*, i.e., an ordering $v_1, \ldots, v_n$ of the vertices such that if G_i is the graph induced by the vertices $v_1, \ldots, v_i$ then G_3 is a triangle and G_i is obtained from G_{i-1} by one of the following two operations:

(**H$_1$**) Choose two vertices x, y from G_{i-1} and add v_i together with the edges (v_i, x) and (v_i, y).

(**H$_2$**) Choose an edge (x, y) and a third vertex z from G_{i-1}, remove (x, y) and add v_i together with the three edges (v_i, x), (v_i, y), and (v_i, z).

In [20] it is shown that planar Laman graphs admit a planar Henneberg construction, in the sense that the graph is constructed together with a plane straight-line embedding and vertices stay at their position once they have been inserted.

Now let G be a planar graph. First apply the algorithm by Lee and Streinu [27] to test in quadratic time whether G is a $(2,3)$-sparse graph. If not, then we are done, so presume in the following that G is $(2,3)$-sparse.
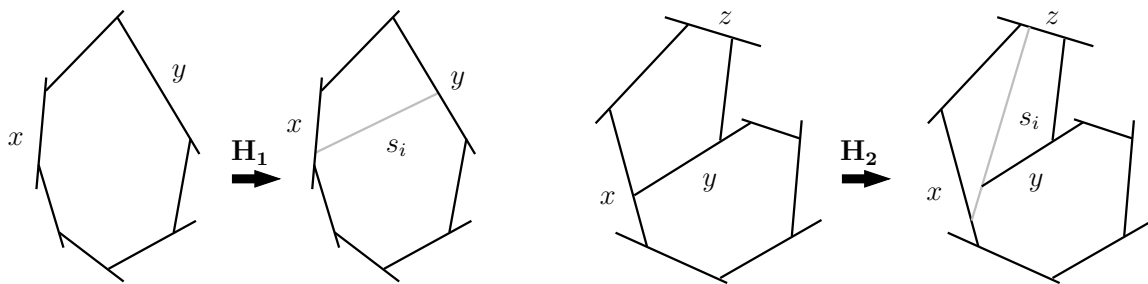
Figure 7: The addition of segment s_i.

Claim: We can add edges to G such that the resulting graph G' is a planar Laman graph.
Proof of Claim: Find the *components* of G, which are the maximal subgraphs that are $(2,3)$-tight. From [27], it is known that components are a partition of the edges and two components share at most one vertex. If G has only one component, then G is a Laman graph and we are done. Otherwise, find a face f with three consecutive vertices v_1, v_2, v_3 such that edges (v_1, v_2) and (v_2, v_3) belong to different components C_1 and C_2. Then no component C contains both v_1 and v_3, otherwise $C \cup \{v_2\}$ would be an even bigger $(2,3)$-tight graph, contradicting the definition of component. Therefore, the pair (v_1, v_3) is not an edge of the graph. Add edge (v_1, v_3); this maintains planarity. Also, the resulting graph is again $(2,3)$-sparse since the endpoints of the new edge did not reside within one component. Finally, the components of the resulting graph are the same as before, except that (as a simple counting-argument shows) $C_1 \cup C_2 \cup (v_1, v_3)$ becomes one new component. Hence the new graph has fewer components and the claim follows by induction.

Observe that the edges in the above claim can be found in quadratic time: We once compute components with the algorithm of [27], and then spend at most $O(n)$ time per added edge to find the edge to add and to update components.

So in the following we will create a 2-segment representation of the planar supergraph G' that is a Laman-graph; we can obtain one for G from it by retracting segments at the added edges. Since G' is a Laman-graph, it has a Henneberg sequence $G_3, \ldots, G_n$ (which we can find with the algorithm of Lee and Streinu in quadratic time.) We build the 2-segment representation following this sequence. Starting from three pairwise touching segments representing G_3, we add segments one by one. For the induction we need the invariant that after adding the ith segment s_i we have a 2-segment representation of G_i for which all *cells* (connected components that are not on the infinite face) are convex. Moreover, there is a correspondence between the cells and the interior faces of G_i which preserves edges, i.e., if (x, y) is an edge of the face, then one of the corners of the corresponding cell is a contact between s_x and s_y. Fig. 7 indicates how to add segment s_i in the cases where v_i is added by $\mathbf{H_1}$, resp. $\mathbf{H_2}$. It is easy to see that the invariant for the induction is maintained.

Directly from the construction it should be clear that we can find the 2-segment representation (given the Henneberg sequence) in linear time. So the running time is dominated by making the graph into a planar Laman graph and finding the Henneberg sequence, which takes $O(n^2)$ time. $\qquad \square$

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4.1.2 Subclasses of 2-segment graphs

Planar and triangle-free

Let G be planar and triangle-free. Then $m \leq 2n - 4$ by the usual counting-argument using Euler's formula and since every face has at least 4 edges on it. By Theorem 6, hence G is a 2-segment graph. (This was already known by a direct construction that uses only three slopes, see [9].)

Planar bipartite

Planar bipartite graphs are a subclass of planar triangle-free graphs, in particular they are 2-segment graphs. In fact, the segments can be restricted to be horizontal or vertical [21], and the segments can be found in linear time and have minimum feature size 1. Hence for planar bipartite graphs, we can construct in linear time proportional side-contact representations with trapezoids. In fact, the trapezoids used in such a representation are rectangles. On the other hand, side-contact representations with triangles are not possible since $K_{2,5}$ does not have one, as discussed in Theorem 4.

Planar 4-connected 3-colorable

Any 4-connected 3-colorable planar graph is also a 2-segment graph [10]. By Theorem 7 we can find such a 2-segment representation, and hence the proportional contact representations in quadratic time.

Planar 2-shellable

A graph G is *2-shellable* if G has a vertex order $v_1, \ldots, v_n$ such that for $i \geq 3$ vertex v_i has at most two neighbors in $v_1, \ldots, v_{i-1}$. (Other names in the literature for such graphs include *2-degenerate graphs*, *2-strippable graphs*, and *2-regular acyclic orientable graphs*.). From the definition it follows that a 2-shellable graph has at most $2n - 3$ edges. Since the property is hereditary, the class is $(2, 3)$-sparse. By Theorem 7 we can find a 2-segment representation, and hence the proportional contact representations of 2-shellable planar graphs in quadratic time.

Partial 2-trees

A *2-tree* is defined as follows: It is either an edge or a graph G with a vertex v of degree two in G such that $G - v$ is a 2-tree and the neighbors of v are adjacent. A *partial 2-tree* is a subgraph of a 2-tree. Every partial 2-tree is planar. Partial 2-trees are the same as series-parallel graphs and include all outer-planar graphs.

Partial 2-trees are 2-shellable, hence 2-segment graphs. However, we can construct the 2-segment representation for them more efficiently.

Let G be a partial 2-tree. It is well-known that we can find in linear time a supergraph G' of G that is a 2-tree, and with it an *elimination order* $v_1, \ldots, v_n$, where for any $i \geq 3$, vertex v_i has exactly two earlier neighbors and they are adjacent. In the following we review the very simple construction of a 2-segment representation of G'.
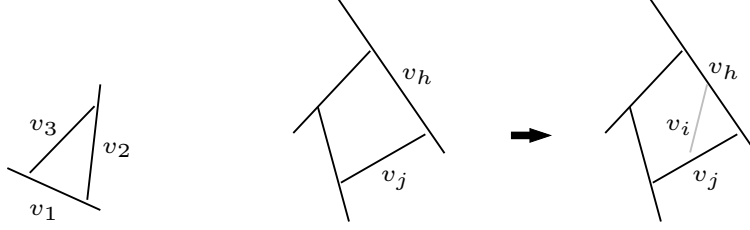
Figure 8: The 2-segment representation for 2-trees.

Lemma 4 *Let G' be a 2-tree with vertex elimination order $v_1, \dots, v_n$. Then G' has a 2-segment representation with convex interior faces and positive feature size. Moreover, for any i line segment $\ell(v_i)$ ends at the line segments of the predecessors of v_i. It can be found in linear time.*

Proof. Vertices $\{v_1, v_2, v_3\}$ form a triangle and it is easy to find three line segments for them that satisfy the claim. See Fig. 8. Now consider v_i, $i \geq 4$ and presume we found line segments for $v_1, \dots, v_{i-1}$ already. Let v_h, v_j be the predecessors of v_i, $h < j$. By definition of a 2-tree edge (v_j, v_h) exists, and by our invariant $\ell(v_j)$ ends at $\ell(v_h)$. Cut off a small triangle near the contact point of $\ell(v_j)$ and $\ell(v_h)$ and assign the segment to v_i; this satisfies all requirements. $\square$

So for every partial 2-tree G, we can find a supergraph that is a 2-tree, find its 2-segment representation in linear time, retract segments to obtain one for G (and at the same time compute the minimum feature size), and then apply the above constructions. So every partial 2-tree has a proportional side-contact representation with trapezoids and a proportional point-contact representation with triangles, and they can be found in linear time.

On the other hand, side-contact representations with triangles are not possible since $K_{2,4}$ with an added edges is a 2-tree, but does not have one as discussed in Theorem 4.

4.2 Maximal Outer-planar Graphs

In this section, we study maximal outer-planar graphs, i.e., planar graphs whose outer-face is a cycle and all interior faces are triangles. These are 2-trees, so the results from the previous subsection apply, but with a different construction we can generate *hole-free* side-contact representations. First we show how to generate hole-free proportional side-contact representations using quadrilaterals so that the entire representation fits inside a triangle, that is, the outer-boundary has constant complexity. At the cost of an outer-boundary of high complexity, we can construct hole-free proportional side-contact representations using only triangles. We also show that the use of triangles might also require a boundary of linear complexity in the size of the graph.

Let G be a maximal outer-planar graph. For any two vertices u, v, denote by $G(u, v)$ the graph induced by the vertices that are between u to v (ends excluded) while walking along the outer-face in counterclockwise order, and let $w(G(u, v))$ be the sum of the weights of all these vertices; see Fig. 9. Define an *aligned triangle* to be one with horizontal base and tip below the base. This naturally defines a *left* and *right side* of the triangle. The next lemma shows that an outer-planar graph can be represented inside *any* aligned triangle of suitable area.
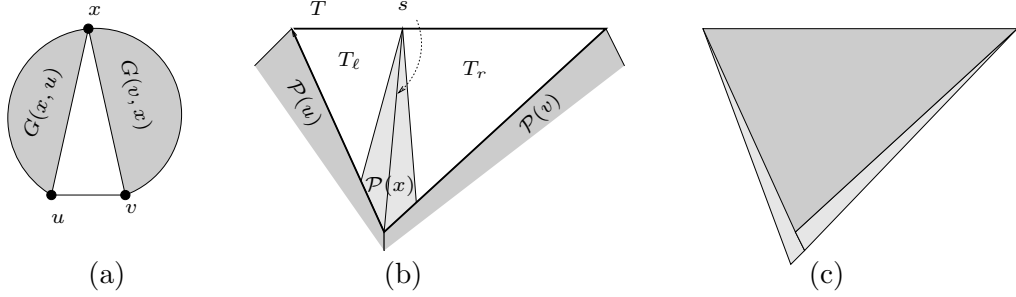
Figure 9: The construction for maximal outer-planar graphs: (a) the graph; (b) splitting triangle T suitably; (c) adding u and v in the outer-most recursion.

Lemma 5 *Let $G = (V, E)$ be a maximal outer-planar graph and let $w : V \to R^+$ be a weight-function. Then for any aligned triangle T of area $w(G(v, u))$, there exists a hole-free proportional side-contact representation of $G(v, u)$ inside T such that the left [right] side of T contains segments of the neighbors of u [v] and of no other vertices. It can be found in linear time.*

Proof. We proceed by induction on the number of vertices in G. In the base case, G is a 3-cycle $\{u, v, x\}$. Use T itself to represent x; this satisfies all conditions.

In the inductive step, let x be the unique common neighbor of u and v. Divide T with a segment s from the tip to the base such that the region T_ℓ left of s has area $w(G(x, u)) + \frac{1}{2}w(x)$, and the region T_r right of ℓ has area $w(G(v, x)) + \frac{1}{2}w(x)$. Cut off triangles of area $\frac{1}{2}w(x)$ each from the tips of T_ℓ and T_r; the combination of these two triangles forms a convex quadrilateral of area $w(x)$ which we use for x; see Fig. 9. Recursively place $G(x, u)$ and $G(v, x)$ (if non-empty) in the remaining triangles of T; it is easy to verify that these have the correct area, which yields the desired side-contact representation.

As for the linear time, this can be achieved with a 2-pass approach. In the first pass, split $G(u, v)$ into graph $G(v, x)$ and $G(x, u)$, and so on recursively until all graphs are triangles. While returning from the recursing, we can hence compute $w(G(y, z))$ for all those subgraphs $G(y, z)$ where it will be needed later (which is exactly those subgraphs where (y, z) is an edge not on the outer-face, hence $O(m)$ many.) This takes linear time in total. With these values readily available, computing the positions of corners of $P(x)$ involves only elementary arithmetic operations and takes constant time, hence the algorithm has linear run-time overall. □

Apply this lemma for an arbitrary edge (u, v) on the outer-face of a maximal outer-planar graph G and an arbitrary triangle T with area $w(G(v, u))$. We can then add triangles for u and v to it to complete the drawing into a contact representation of G; see Fig. 9(c). So we obtain:

Corollary 8 *Let $G = (V, E)$ be a maximal outer-planar graph and let $w : V \to R^+$ be a weight function. Then G admits a hole-free proportional side-contact representation where vertices are represented by triangles or convex quadrilaterals and the outer boundary is a triangle. It can be found in linear time.*

Next, we restrict ourselves to representations with triangles.

Lemma 6 *Let $G = (V, E)$ be a maximal outer-planar graph and (u, v) an edge on the outer-face of G, with u before v in counterclockwise order. Let $w : V \to R^+$ be a weight-function. Then*
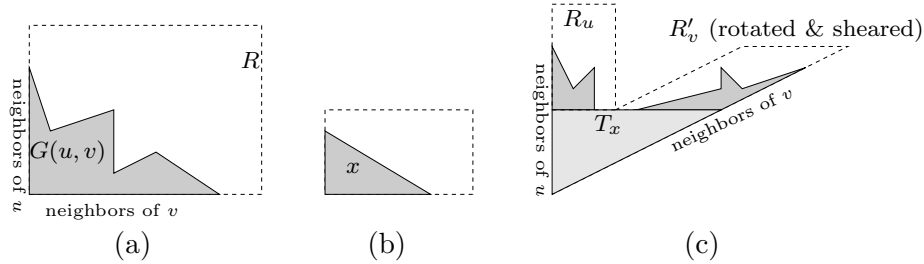
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Figure 10: The construction for maximal outer-planar graphs: (a) requirements on drawing; (b) the base case; (c) combining the drawings.

there exists a hole-free proportional side-contact representation of $G(v, u)$ with triangles that can be placed inside an axis-aligned rectangle R such that:

(i) *From the bottom left corner of R upward, we encounter boundaries of all neighbors of u (in order), followed by unused space.*

(ii) *From the bottom left corner of R rightward, we encounter boundaries of all neighbors of v (in order), followed by unused space.*

It can be found in linear time.

Proof. The idea for the construction is illustrated in Fig. 10. We proceed by induction on the number of vertices in G. In the base case, G is a 3-cycle $\{u, v, x\}$. Represent x as a cut-off corner (of appropriate area) of an axis-aligned rectangle; this satisfies both conditions of the lemma; see Fig. 10(b).

In the inductive step, let x be the unique common neighbor of u and v. Recursively draw $G(x, u)$ and $G(v, x)$ inside axis-aligned rectangles R_u and R_v. Rotate the drawing inside R_v such that the neighbors of x are now on the bottom side of R_v, while the neighbors of v are on the right side of v.

The crucial operation to apply now is a *shear*. A *horizontal shear* maps a point (x, y) to point $(x + k \cdot y, y)$ for some constant k. A shear preserves straight lines and areas (but it changes angles.)

We apply a horizontal shear to R_v for some positive k, turning R_v into a parallelogram R_v' whose slope depends on k. We chose k such that the area of x is correct in the resulting drawing. Specifically, consider the triangle T_x formed by the extension of the left side of R_u, the bottom sides of R_u and R_v' (placed next to each other) and the extension of the right side of R_v'; see Fig. 10(c).

If we choose $k = 0$ (i.e., no shear has been applied), then T_x has infinite area. If we choose $k = \infty$ (i.e., R_v is flattened into horizontal ray), then T_x has zero area. By the intermediate value theorem, there exists some horizontal shear such that the area of T_x is exactly the weight of x, and we use this shear. (In fact, the correct value can easily be computed; it needs to be such that the slanted edge has slope $s = 2A(x)/b$, where b is the length of the base of T_x.) Then use T_x to represent vertex x.

Consider the two rays that emanate from the tip of T_x. Along the vertical ray, we encounter (in order) first the boundary of x, then all other neighbors of u (which were on R_u), and finally free space. Hence we encounter all neighbors of u in order. Similarly we encounter all neighbors

of v in order along the other ray. The drawing then satisfies all conditions, except that it is contained inside a triangle, rather than a rectangle. But this is easily fixed with a vertical shear that maps (x, y) to $(x, y - s \cdot x)$, where $s = 2A(x)/b$ is the slope of the slanted edge of the triangle. The drawing is then contained in a rectangle as desired.

To achieve linear run-time, we use a 2-pass approach. In the first pass, we break each graph $G(u, v)$ into subgraphs $G(x, u)$ and $G(v, x)$ and recurse. When returning from the recursion, each subgraph reports the enclosing rectangle that can be achieved, but does not actually have final coordinates yet. Instead, $G(u, v)$ stores what translations and shares need to be applied to the two subgraphs $G(x, u)$ and $G(v, x)$, which in turn store which translations and shears must be applied inside, and so on.

On the second pass, we compute final coordinates, by combining all the translations and shears to be applied in the subgraph into one affine transformation, applying it for x to compute the final coordinates of $P(x)$, and recursing in the subgraphs. This computes all final coordinates in linear time. $\qquad\square$

Let G be a maximal outer-planar graph. Apply Lemma 6 for an arbitrary edge (u, v) on the outer-face of G. We can then add triangles for u and v to it to complete the drawing into a contact representation of G. We thus obtain the following result.

Corollary 9 *Let $G = (V, E)$ be a maximal outer-planar graph and let $w : V \to R^+$ be a weight function. Then G admits a hole-free proportional side-contact representation where vertices are represented by triangles.*

Note that in this construction, even though each vertex is represented by a triangle, the outer-face may have complexity $\Omega(n)$. Moreover, in some cases this high complexity is unavoidable, even for unweighted contact representations, as shown in the following lemma.

Lemma 7 *There exists a maximal outer-planar graph with n vertices for which any hole-free side-contact representation with triangles requires $\Omega(n)$ sides on the outer-face.*

Proof. Consider the *snowflake graph* $G = G_k$ with $n = 3 \cdot 2^k$ vertices, which is an outer-planar 2-tree obtained from a triangle by repeatedly walking around the outer-face and adding a vertex of degree 2 at each edge; see Fig. 11(a). The vertices added in the last round form an independent set S of $n/2$ vertices such that each vertex of S has degree two.

Assume we have a contact representation Γ of G that has no holes and uses triangles. The n triangles then have $3n$ corners. Of these $3n$ corners, at least $2n - 4$ must have their tip at a point that is not on the outer-boundary of Γ. This holds since each of the $n - 2$ inner faces of G corresponds to a point where three polygons meet in Γ (recall that there are no holes.) Of these three polygons, at least two have a strictly convex angle and hence a corner; see Fig. 11(b).

Also, each vertex v in S has at least two of its corners on the outer-boundary of Γ. As $\deg(v) = 2$, at most two sides of v can be shared with neighbors of v, and one entire side of v (and hence the two corners at its ends) belongs to the outer-boundary of Γ.

We now have that at least $2n - 4$ corners of triangles are not on the outer-boundary and $2|S| = n$ corners of triangles of S are on the outer-boundary. This leaves at most 4 corners that could be on the outer-boundary and belong to a vertex not in S. Put differently, there are $n - |S| - 4$ vertices w that are not in S and do not have a corner on the outer-boundary.

Consider one such vertex $w \notin S$, and let $v_1, v_2 \in S$ be its neighbors on the outer-boundary. Vertex w must have at least a point on the outer-boundary, otherwise v_1 and v_2 would be adjacent. So some point on one side s of w belongs to the outer-boundary, but the corners at

the ends of s do not belong to the outer-boundary; ; see Fig. 11(c). Therefore the outer-boundary must touch and then leave s, which means it has a reflex vertex somewhere along s.

As a result, the outer-face has at least $n - |S| - 4 \geq n/2 - 4$ reflex vertices, which completes the claim. $\qquad\square$
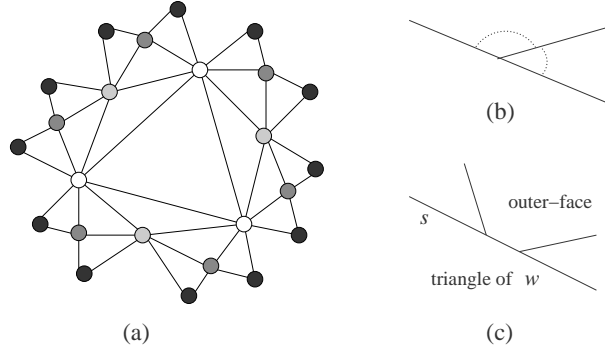

Figure 11: (a) The snowflake graph G_3; (b)–(c) illustration for the proof of Lemma 7.

Lemma 7 shows that proportional side-contact representations with triangles requires the outer-boundary to have linear complexity for some maximal outer-planar graphs. Corollary 8 shows that we can also have a triangle as the outer-boundary, but at the cost of using convex quadrilaterals for vertices. These results are summarized in the last theorem.

Theorem 10 *Convex quadrilaterals are always sufficient and sometimes necessary for hole-free proportional side-contact representations of maximal outer-planar graphs when an outer-boundary of constant complexity is required. Hole-free proportional contact representation with triangles can also be computed but at the expense of linear complexity of the outer boundary. Both types of contact representations can be computed in linear time.*

5 Conclusion and Open Problems

We described algorithms for proportional point-contact and side-contact representations of planar graphs, 2-segment graphs, outer-planar graphs, and 2-trees. We focused on the complexity of the polygons representing vertices, and provided bounds on this complexity that are tight, for a variety of graph classes and drawing models.

However, many problems still remain open. What is the complexity of side-contact proportional representations of maximal planar graphs? We can achieve 7-sided polygons easily (essentially by cutting the convex corners of the 4-sided spikes), but can we do better? Likewise, what is the complexity for hole-free proportional representations of maximal planar graphs? Here, a bound of 8 is known (and the polygons are orthogonal) [3], but can we do better if polygons need not be orthogonal?

References

[1] M. J. Alam, T. C. Biedl, S. Felsner, A. Gerasch, M. Kaufmann, and S. G. Kobourov. Linear-time algorithms for hole-free rectilinear proportional contact graph representations. In *International Symposium on Algorithms and Computation(ISAAC'11)*, volume 7074 of *Lecture Notes in Computer Science*, pages 281–291. Springer, 2011.

[2] M. J. Alam, T. C. Biedl, S. Felsner, M. Kaufmann, and S. G. Kobourov. Proportional contact representations of planar graphs. In *Graph Drawing (GD'11)*, volume 7034 of *Lecture Notes in Computer Science*, pages 26–38. Springer, 2012.

[3] M. J. Alam, T. C. Biedl, S. Felsner, M. Kaufmann, S. G. Kobourov, and T. Ueckerdt. Computing cartograms with optimal complexity. In *Symposium on Computational Geometry (SoCG'12)*, 2012 (to appear).

[4] M. Badent, C. Binucci, E. D. Giacomo, W. Didimo, S. Felsner, F. Giordano, J. Kratochvíl, P. Palladino, M. Patrignani, and F. Trotta. Homothetic triangle contact representations of planar graphs. In *Canadian Conference on Computational Geometry (CCCG'07)*, pages 233–236, 2007.

[5] T. C. Biedl and L. E. R. Velázquez. Orthogonal cartograms with few corners per face. In *Workshop on Algorithms and Data Structures (WADS'11)*, volume 6844 of *Lecture Notes in Computer Science*, pages 98–109. Springer, 2011.

[6] A. L. Buchsbaum, E. R. Gansner, C. M. Procopiuc, and S. Venkatasubramanian. Rectangular layouts and contact graphs. *ACM Transactions on Algorithms*, 4(1), 2008.

[7] B. Chazelle. Triangulating a simple polygon in linear time. *Discrete & Computational Geometry*, 6:485–524, 1991.

[8] M. de Berg, E. Mumford, and B. Speckmann. On rectilinear duals for vertex-weighted plane graphs. *Discrete Mathematics*, 309(7):1794–1812, 2009.

[9] N. de Castro, F. J. Cobos, J. C. Dana, A. Márquez, and M. Noy. Triangle-free planar graphs and segment intersection graphs. *Journal of Graph Algorithms and Applications*, 6(1):7–26, 2002.

[10] H. de Fraysseix and P. O. de Mendez. Representations by contact and intersection of segments. *Algorithmica*, 47(4):453–463, 2007.

[11] H. de Fraysseix, P. O. de Mendez, and P. Rosenstiehl. On triangle contact graphs. *Combinatorics, Probability and Computing*, 3:233–246, 1994.

[12] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[13] H. Debrunner. Aufgabe 260. *Elemente der Mathematik*, 12, 1957.

[14] C. A. Duncan, E. R. Gansner, Y. F. Hu, M. Kaufmann, and S. G. Kobourov. Optimal polygonal representation of planar graphs. *Algorithmica*, 63(3):672–691, 2012.

[15] S. Felsner. *Geometric Graphs and Arrangements*. Advanced Lectures in Mathematics. Vieweg Verlag, 2004.

[16] S. Felsner and M. C. Francis. Contact representations of planar graphs with cubes. In *Symposium on Computational Geometry (SoCG'11)*, pages 315–320, 2011.

[17] E. R. Gansner, Y. Hu, and S. G. Kobourov. On touching triangle graphs. In *Graph Drawing (GD'10)*, volume 6502 of *Lecture Notes in Computer Science*, pages 250–261. Springer, 2010.

[18] D. Gonçalves, B. Lévêque, and A. Pinlou. Triangle contact representations and duality. In *Graph Drawing (GD'10)*, volume 6502 of *Lecture Notes in Computer Science*, pages 238–249. Springer, 2010.

[19] J. Graver, B. Servatius, and H. Servatius. *Combinatorial Rigidity*. Graduate Studies in Mathematics. American Mathematical Society, 1993.

[20] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. L. Souvaine, I. Streinu, and W. Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Computational Geometry*, 31(1-2):31–61, 2005.

[21] I. Hartman, I. Newman, and R. Ziv. On grid intersection graphs. *Discrete Mathematics*, 97:41–52, 1991.

[22] R. Heilmann, D. A. Keim, C. Panse, and M. Sips. Recmap: Rectangular map approximations. In *IEEE Symposium on Information Visualization (INFOVIS'04)*, pages 33–40, 2004.

[23] P. Hlinený. Contact graphs of line segments are NP-complete. *Discrete Mathematics*, 235(1–3):95–106, 2001.

[24] A. Kawaguchi and H. Nagamochi. Orthogonal drawings for plane graphs with specified face areas. In *International Conf. Theory and Applications of Models of Computation (TAMC'07)*, volume 4484 of *Lecture Notes in Computer Science*, pages 584–594, 2007.

[25] P. Koebe. Kontaktprobleme der konformen Abbildung. *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig. Math.-Phys. Klasse*, 88:141–164, 1936.

[26] K. Koźmiński and E. Kinnen. Rectangular duals of planar graphs. *Networks*, 15:145–157, 1985.

[27] A. Lee and I. Streinu. Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 308(8):1425–1437, 2008.

[28] M. S. Rahman, K. Miura, and T. Nishizeki. Octagonal drawings of plane graphs with prescribed face areas. *Computational Geometry*, 42(3):214–230, 2009.

[29] W. Schnyder. Embedding planar graphs on the grid. In *Symposium on Discrete Algorithms (SODA'90)*, pages 138–148, 1990.

[30] P. Ungar. On diagrams representing graphs. *Journal of London Mathematical Society*, 28:336–342, 1953.

[31] M. J. van Kreveld and B. Speckmann. On rectangular cartograms. *Computational Geometry*, 37(3):175–187, 2007.

[32] K.-H. Yeap and M. Sarrafzadeh. Floor-planning by graph dualization: 2-concave rectilinear modules. *SIAM Journal on Computing*, 22:500–526, 1993.