POSETS AND PLANAR GRAPHS

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Abstract. Usually *dimension* should be an integer valued parameter, we introduce a refined version of dimension for graphs which can assume a value $[t-1 \updownarrow t]$ which is thought to be between $t-1$ and t. We have the following two results:

- A graph is outerplanar if and only if its dimension is at most $[2 \updownarrow 3]$.

This characterization of outerplanar graphs is closely related to the celebrated result of W. Schnyder [16] who proved that a graph is planar if and only if its dimension is at most 3.

- The largest n for which the dimension of the complete graph $\mathbf{K}_n$ is at most $[t-1 \updownarrow t]$ is the number of antichains in the lattice of all subsets of a set of size $t-2$.

Accordingly, the refined dimension problem for complete graphs is equivalent to the classical combinatorial problem known as Dedekind's problem. This result extends work of Hoşten and Morris [14].

The main results are enriched by background material which links to a line of reserch in extremal graph theory which was stimulated by a problem posed by G. Agnarsson: Find the maximum number of edges in a graph on n nodes with dimension at most t.

1. Introduction

Let $\mathbf{G} = (V, E)$ be a finite simple graph.

Definition 1.1. A nonempty family $\mathcal{R}$ of linear orders on the vertex set V of a graph $\mathbf{G} = (V, E)$ is called a *realizer* of $\mathbf{G}$ provided

(∗) For every edge $S \in E$ and every vertex $x \in X - S$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in S$.

The *dimension* of $\mathbf{G}$, denoted $\dim(\mathbf{G})$, is then defined as the least positive integer t for which $\mathbf{G}$ has a realizer of cardinality t.

In order to avoid trivial complications when the condition (∗) is vacuous, throughout the remainder of the paper, we restrict our attention to connected graphs with three or more vertices.

For those readers who are new to the concept of dimension for graphs, we present the following elementary example.

Example 1.2. *The dimension of the complete graph* $\mathbf{K}_5$ *is* 4*, but the removal of any edge reduces the dimension to* 3*.*

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Proof. Consider the complete graph with vertex set $\{1,2,3,4,5\}$. Any family of 4 linear orders $\{L_1, L_2, L_3, L_4\}$ with i the highest element and 5 the second highest element in L_i for all i is a realizer. So $\dim(\mathbf{K}_5) \leq 4$. On the other hand, suppose $\dim(\mathbf{K}_5) \leq 3$, and let $\mathcal{R} = \{M_1, M_2, M_3\}$ be a realizer. Without loss of generality, 4 and 5 are not the highest element of any linear order in $\mathcal{R}$. Also, without loss of generality $4 > 5$ in both M_1 and M_2. Now let j be the largest element of M_3. Then there is no element $i \in \{1,2,3\}$ for which 5 is over both 4 and j in M_i. The contradiction shows that $\dim(\mathbf{K}_5) = 4$, as claimed.

Now let $e = \{3,4\}$. The following three linear orders form a realizer of $\mathbf{K}_5 - e$:

$L_1 = [2 < 3 < 5 < 4 < 1]$
$L_2 = [1 < 3 < 5 < 4 < 2]$
$L_3 = [1 < 2 < 4 < 5 < 3]$ □

Here is a second example. We leave its elementary proof as an exercise.

Example 1.3. *The dimension of the complete bipartite graph* $\mathbf{K}_{3,3}$ *is* 4*, but the removal of any edge reduces the dimension to* 3.

The preceding two examples help to motivate the following, now classic, theorem of W. Schnyder [16].

Theorem 1.4. *A graph* $\mathbf{G}$ *is planar if and only if its dimension is at most* 3. □

Schnyder's original version of Theorem 1.4 used a slightly different concept of dimension. With a finite graph $\mathbf{G} = (V, E)$, we associate a height two poset $\mathbf{P} = \mathbf{P_G}$ whose ground set is $V \cup E$. The order relation is defined by setting $x < S$ in $\mathbf{P_G}$ if $x \in V$, $S \in E$ and $x \in S$. $\mathbf{P_G}$ is called the *incidence poset* of $\mathbf{G}$. Schnyder proved: A graph $\mathbf{G}$ is planar if and only if the dimension of its incidence poset is at most 3.

The close relationship between the dimension of a graph and the dimension of its incidence poset can be described as follows:

Proposition 1.5. *Let* $\mathbf{G}$ *be a graph and let* $\mathbf{P_G}$ *be its incidence poset. Then*

1. $\dim(\mathbf{G}) \leq \dim(\mathbf{P_G}) \leq 1 + \dim(\mathbf{G})$.
2. $\dim(\mathbf{G}) = \dim(\mathbf{P_G})$ *if* $\mathbf{G}$ *has no vertices of degree* 1. □

Although the preceding proposition admits an elementary proof, it can be stated in a somewhat more general form:

Proposition 1.6. *The dimension of a graph equals the interval dimension of its incidence poset.* □

Graphs and incidence orders of dimension at most two are easy to characterize:

Proposition 1.7. *Let* $\mathbf{G}$ *be a graph and let* $\mathbf{P_G}$ *be its incidence poset. Then*

1. $\dim(\mathbf{G}) \leq 2$ *if and only if* $\mathbf{G}$ *is a caterpillar.*
2. $\dim(\mathbf{P_G}) \leq 2$ *if and only* $\mathbf{G}$ *is a path.* □

We will not use the concepts of dimension and interval dimension for posets extensively in this article, but for those readers who would like additional information on how this parameter relates to graph theory problems, we suggest looking at Trotter's monograph [20] or survey articles [21], [22] and [23].

Although we do not include a proof of Schnyder's theorem here, we pause for some comments related to it.

The fact that graphs with $\dim(\mathbf{G}) \leq 3$ are planar is relatively easy to prove. This was shown by Babai and Duffus [3]. The difficult part is to show that $\dim(\mathbf{G}) \leq 3$ when $\mathbf{G}$ is planar. This proof required Schnyder to develop several elegant structural results for planar graphs, and these results have interest independent from their application to Theorem 1.4. Schnyder's theorem has been generalized by Brightwell and Trotter [5], [6] with the following two results.

Theorem 1.8. *Let D be a plane drawing without edge crossings of a 3-connected planar graph $\mathbf{G}$ and let $\mathbf{P}$ be the poset of vertices, edges and faces of this drawing, partially ordered by inclusion. Then $\dim(\mathbf{P}) = 4$. Furthermore, the subposet of $\mathbf{P}$ generated by the vertices and faces is 4-irreducible.* □

Theorem 1.9. *Let D be a plane drawing without edge crossings of a planar multigraph $\mathbf{G}$ and let $\mathbf{P}$ be the poset of vertices, edges and faces of this drawing, partially ordered by inclusion. Then $\dim(\mathbf{P}) \leq 4$.* □

Simplified proofs of Theorem 1.8 have been given by Felsner [10], [11].

2. Other Combinatorial Connections

In order to provide further motivation for the results which follow, we pause to discuss two other recent research directions. One such theme is to determine (or estimate) the dimension of the complete graph $\mathbf{K}_n$. Note that the dimension of $\mathbf{K}_n$ and the dimension of its incidence poset are the same when $n \geq 3$.

For a positive integer t, let $\mathcal{B}(t)$ denote the set of all subsets of $\{1, 2, \dots, t\}$. A subset $\mathcal{A} \subset \mathcal{B}(t)$ is called an *antichain* if no two sets in $\mathcal{A}$ are ordered by inclusion. We then let $D(t)$ count the number of antichains[1] Starting with $D(1) = 3$ the next values are: 6, 20, 168, 7781. Exact values are known for $t \leq 8$ The evaluation (or estimation) of the function $D(t)$ is popularly known as *Dedekind's Problem* [18].

We then let $\mathrm{HM}(t)$ count the number of antichains $\mathcal{A}$ in $\mathcal{B}(t)$ which satisfy the following additional property:

$$(**) \quad S_1 \cup S_2 \neq \{1, 2, \dots, t\} \text{ for every } S_1, S_2 \in \mathcal{A}.$$

Starting with $\mathrm{HM}(1) = 2$ the next values are: 4, 12, 81. These numbers arise in several combinatorial problems [18], but here is one particularly surprising one recently discovered by Hoşten and Morris [14].

Theorem 2.1. *Let $t \geq 2$. Then $\mathrm{HM}(t-1)$ is the largest n so that $\dim(\mathbf{K}_n) \leq t$.* □

So it is natural to ask whether there is a connection between dimension and Dedekind's problem which avoids the technical restriction $(**)$ described above.

But perhaps there is even a more significant motivation involving minor-monotone graph parameters—a subject which has attracted considerable attention in the last few years. For example, let $\mu(\mathbf{G})$ denote the Colin de Verdière graph invariant introduced in [8]. The parameter $\mu(\mathbf{G})$ is minor-monotone. Furthermore:

(1) $\mu(\mathbf{G}) \leq 1$ if and only if $\mathbf{G}$ is a path.
(2) $\mu(\mathbf{G}) \leq 2$ if and only if $\mathbf{G}$ is outerplanar.
(3) $\mu(\mathbf{G}) \leq 3$ if and only if $\mathbf{G}$ is planar.
(4) $\mu(\mathbf{G}) \leq 4$ if and only if $\mathbf{G}$ is linklessly embeddable in 3-space.

[1] In this count we include the empty antichain.

We refer the reader to Schrijver's survey article [17] for an extensive discussion of the Colin de Verdière invariant. However, in view of our previous remarks, it is striking that in the list of results for this invariant, we see both a characterization of paths and of planar graphs. So it is natural to explore the concept of dimension of graphs to see if one can find a characterization of outerplanar graphs, a characterization of linklessly embeddable graphs and a natural extension to a minor-monotone parameter. We have solved the first of these three challenges.

3. A New Characterization of Outerplanar Graphs

Let L and M be linear orders on a finite set X. We say that L and M are *dual* and write $L = M^d$ if $x < y$ in L if and only if $x > y$ in M for all $x, y \in X$. Reflecting on the problem of characterizing outerplanar graphs in terms of dimension, one is also faced with the problem of finding a number between 2 and 3, this object[2] will be denoted $[2 \updownarrow 3]$.

Definition 3.1. For an integer $t \geq 2$, we say that the dimension of a graph is $[t-1 \updownarrow t]$ if it has dimension greater than $t-1$ yet has a realizer of the form $\{L_1, L_2, \ldots, L_t\}$ with $L_{t-1} = L_t^d$.

As the reader will see, the following theorem is not difficult to prove. It is the statement which is a bit surprising.

Theorem 3.2. *A graph* $\mathbf{G}$ *is outerplanar iff it has dimension at most* $[2 \updownarrow 3]$.

Proof. Let $\mathbf{G}$ be a graph and suppose that $\dim(\mathbf{G}) \leq [2 \updownarrow 3]$. We show that $\mathbf{G}$ is outerplanar. Choose a realizer $\{L_1, L_2, L_3\}$ for $\mathbf{G}$ with $L_2 = L_3^d$. Then let $\mathbf{H}$ be the graph formed by adding a new vertex x adjacent to all vertices of $\mathbf{G}$. We show that $\mathbf{H}$ is planar. To accomplish this, consider the family $\mathcal{R} = \{M_1, M_2, M_3\}$ of three linear orders on the vertex set of $\mathbf{H}$ formed by adding x at the top of L_1, the bottom of L_2 and the bottom of L_3. We claim that $\mathcal{R}$ is a realizer of $\mathbf{H}$. To see this, let u be a vertex in $\mathbf{H}$ and let f be an edge not containing u as one of its endpoints. If $u = x$, then x is over both points of f in M_1. So we may assume $u \neq x$. If $f = \{x, v\}$, with v a vertex from $\mathbf{G}$ and $u \neq v$, then u is over both x and v in exactly one of M_2 and M_3. Finally, if $f = \{v, w\}$, where both v and x are vertices in $\mathbf{G}$, then there is some $i \in \{1, 2, 3\}$ for which u is over both v and w in L_i. It follows that u is over v and w in M_i. Thus by Schnyder's theorem, $\mathbf{H}$ is planar. In turn, $\mathbf{G}$ is outerplanar.

Now suppose that $\mathbf{G}$ is outerplanar. We show that the dimension of $\mathbf{G}$ is at most $[2 \updownarrow 3]$. Without loss of generality, we may assume that $\mathbf{G}$ has $n \geq 4$ vertices and is maximal outerplanar, i.e., adding any missing edge to $\mathbf{G}$ produces a graph which is no longer outerplanar.

As before, let $\mathbf{H}$ be formed from $\mathbf{G}$ by adding a new vertex x adjacent to all vertices of $\mathbf{G}$. Then $\mathbf{H}$ is maximal planar. Choose a plane drawing without edge crossings of $\mathbf{H}$ so that the vertex x appears on the exterior triangle. Let u_1 and u_n be the other two vertices on this triangle. Then there is a natural labelling of the vertices of $\mathbf{G}$ as $u_1, u_2, \ldots, u_n$ so that $\{u_i, u_{i+1}\}$ is an edge and $\{x, u_i, u_{i+1}\}$ is a triangular face in the drawing for all $i = 1, 2, \ldots, n-1$. Let L_2 be the subscript order $u_1 < u_2 < \cdots < u_n$ and let L_3 be the dual of L_2.

[2]In the original manuscript we have used the fraction $\frac{5}{2}$ for this purpose. This, however, could be confused with with the independent notion of *fractional dimension* (see [4], [12]).

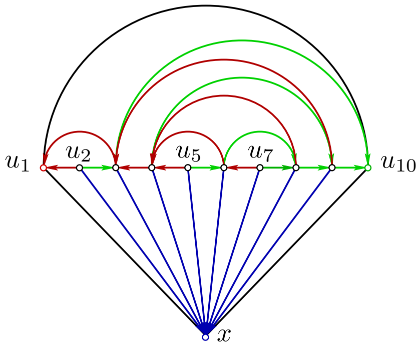

FIGURE 1. An example for the construction. Shortest path trees for u_1 and u_{10} are color coded. A corresponding permutation L_1 is $u_1, u_{10}, u_3, u_2, u_9, u_4, u_8, u_6, u_5, u_7, x$.

Call a path $u_{i_1}, u_{i_2}, \ldots, u_{i_r}$ in $\mathbf{G}$ *monotonic* if $i_1 < i_2 < \cdots < i_r$. For each integer i with $1 < i < n$, note that there is a unique shortest monotonic path $P(u_1, u_i)$ from u_1 to u_i. Likewise, there is a unique shortest monotonic path $P(u_i, u_n)$ in $\mathbf{G}$ from u_i to u_n. Then let S_i be the region consisting of all points in the plane belonging to the closed region bounded by the edges in these two paths together with the edge $\{u_1, u_n\}$. By convention, we take S_1 and S_n as the degenerate region consisting of those points in the plane which are on the edge $\{u_1, u_n\}$. Define a strict partial order Q on the set $\{u_1, u_2, \ldots, u_n\}$ by setting $u_i < u_j$ in Q if and only if S_i is a proper subset of S_j. Then let L_1 be any linear extension of Q.

We claim that $\{L_1, L_2, L_3\}$ is a realizer of $\mathbf{G}$. To see this, let u be a vertex of $\mathbf{G}$ and let $e = \{y, z\}$ be an edge not containing u. We show that there is some $i \in \{1, 2, 3\}$ for which u is over both y and z in L_i. This conclusion is straightforward except possibly when there exist integers i, j, k with $1 \leq i < j < k \leq n$ so that $\{y, z\} = \{u_i, u_k\}$ and $u = u_j$. However, in this case, it is easy to see that u is over y and z in L_1. □

4. The Connection with Dedekind's Problem

In this section, we show that our refined dimension concept for complete graphs yields a full equivalence with the classical problem of Dedekind. Again, the proof is not difficult, and we find the statement the real surprise.

Theorem 4.1. *For $t \geq 3$ the largest n so that $\dim(\mathbf{K}_n) \leq [t-1 \updownarrow t]$ is $D(t-2)$.*

Proof. We first show that if $\dim(\mathbf{K}_n) \leq [t-1 \updownarrow t]$, then $D(t-2) \geq n$. Let $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ be a realizer which shows that $\dim(\mathbf{K}_n) \leq [t-1 \updownarrow t]$. By relabelling, we may assume that:

(1) The vertex set of $\mathbf{K}_n$ is $\{1, 2, \ldots, n\}$,
(2) $1 < 2 < \cdots < n$ in L_t, and
(3) $1 > 2 > \cdots > n$ in L_{t-1}.

Now for each $i, j \in \{1, 2, \ldots, n\}$ with $1 \leq i < j \leq n$, let $S(i < j) = \{\alpha \in \{1, 2, \ldots, t-2\} : i < j \text{ in } L_\alpha\}$. Then for each $i = 1, 2, \ldots, n-1$, let $\mathcal{C}_i = \{S(i < j) : i < j \leq n\}$. Order the sets in each $\mathcal{C}_i$ by inclusion and let $\mathcal{A}_i$ denote the set of maximal elements of $\mathcal{C}_i$. By construction, each $\mathcal{A}_i$ is an antichain in $\mathcal{B}(t-2)$, in fact a non-empty antichain. Finally, set $\mathcal{A}_n = \emptyset$.

We claim that $\mathcal{A}_i \neq \mathcal{A}_j$ for all $1 \leq i < j \leq n$. In fact, we claim that there exists a set $S \in \mathcal{A}_i$ so that $S \not\subseteq T$ for every $T \in \mathcal{A}_j$. This is clearly true if $j = n$. But suppose that this claim fails for some pair i, j with $1 \le i < j < n$. Consider the set $S(i < j)$. Then there is a set $S \in \mathcal{A}_i$ with $S(i < j) \subseteq S$. Suppose that there is a set $T \in \mathcal{A}_j$ so that $S \subseteq T$. Choose k with $j < k \le n$ so that $T = S(j < k)$. It follows that whenever $\alpha \in \{1, 2, \ldots, t-2\}$ and $i < j$ in L_α, then $j < k$ in L_α. So there is no α in $\{1, 2, \ldots, t-2\}$ for which j is over both i and k. Since j is between i and k in both L_{t-1} and L_t, it follows that $\mathcal{R}$ is not a realizer. The contradiction completes the first part of the proof.

Now suppose that $D(t-2) \ge n$. We want to show that $\dim(\mathbf{K}_n) \le [t-1 \updownarrow t]$. Here we only provide a sketch of the argument since it follows immediately from the next lemma, a result due to Hoşten and Morris. It is also presented in somewhat more compact form in Kierstead's survey paper [15] and has its roots in Spencer's paper [19], where the asymptotic behavior of the dimension of the complete graph is first discussed.

First, let $s \ge 1$ and let $L = (S_1, S_2, \ldots, S_{2^s})$ be a listing of all the subsets of $\{1, 2, \ldots, s\}$ so that $i < j$ whenever $S_i \subset S_j$, i.e., this listing is a linear extension of the inclusion ordering. Then suppose that $D(s) = n$ and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be the unique listing of the antichains in $\mathcal{B}(s)$ so that

> For all $i < j$ with $1 \le i < j \le n$, if k is the largest integer in $\{1, 2, \ldots, 2^s\}$ so that S_k belongs to one of $\mathcal{A}_i$ and $\mathcal{A}_j$ but not the other, then S_k belongs to $\mathcal{A}_i$.

In other words, the listing of antichains is in reverse lexicographic order as determined by the listing L. The proof of the following lemma is given in [14].

Lemma 4.2. *Let $s \ge 1$, let L be a linear extension of the inclusion order on the subsets of $\{1, 2, \ldots, s\}$ and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be the antichains of $\mathcal{B}(s)$ listed in reverse lexicographic order as determined by L. For each i and j with $1 \le i < j \le n$, let k be the largest integer in $\{1, 2, \ldots, 2^s\}$ so that S_k belongs to one of $\mathcal{A}_i$ and $\mathcal{A}_j$ but not the other, and set $S(i < j) = S_k$. Then for each $\alpha \in \{1, 2, \ldots, s\}$, the binary relation*

$$L_\alpha = \{(i, j) : \alpha \in S(i < j)\| \cup \{(j, i) : \alpha \notin S(i < j)\}$$

is a total order on the antichains of $\mathcal{B}(s)$. □

It is easy to see that the orders $\{L_1, L_2, \ldots, L_s\}$ together with the subscript order and its dual form a realizer of the complete graph of size n with the vertices being the antichains in $\mathcal{B}(s)$. With this observation, the proof is complete. □

5. A New Extremal Graph Theory Problem

G. Agnarsson [1] first proposed to investigate the following extremal graph theory problem. For integers n and t, find the maximum number $\mathrm{ME}(n, t)$ of edges in a graph on n vertices having dimension at most t. Agnarsson was motivated by ring theoretic problems which are discussed in [1] and [2].

Based on the results presented thus far, we can also attempt to find the maximum number of edges $\mathrm{ME}(n, [t-1 \updownarrow t])$ in a graph on n vertices having dimension at most $[t-1 \updownarrow t]$. For small values, we know everything, since we are just counting respectively the maximum number of edges in a caterpillar, an outerplanar graph and a planar graph.

Proposition 5.1. *For $n \ge 3$, $\mathrm{ME}(n, 2) = n-1$, $\mathrm{ME}(n, [2 \updownarrow 3]) = 2n-3$ and $\mathrm{ME}(n, 3) = 3n-6$.* □

In [2], Agnarsson, Felsner and Trotter investigated the asymptotic behavior of $\mathrm{ME}(n, 4)$ and used Turán's theorem [24], the product Ramsey theorem (see [13], for example) and the Erdős/Stone theorem [9] to obtain the following result.

Theorem 5.2.

$$\lim_{n\to\infty} \frac{\mathrm{ME}(n,4)}{n^2} = \frac{3}{8}.$$

□

The lower bound in this formula comes from the fact that any graph with chromatic number at most 4 has dimension at most 4. So the Turán graph, a balanced complete 4-part graph has dimension at most 4. This is enough to show that $\lim_{n\to\infty} \mathrm{ME}(n,4)/n^2 \geq 3/8$.

Theorem 5.3.

$$\lim_{n\to\infty} \frac{\mathrm{ME}(n,[3 \updownarrow 4])}{n^2} = \frac{1}{4}.$$

Proof. As the argument is a straightforward modification of the proof of Theorem 5.2, we provide only a sketch. First, note that the balanced complete bipartite graph has dimension at most $[3 \updownarrow 4]$ and has $\lceil n^2/4 \rceil$ edges. This shows that 1/4 is a lower bound for the limes.

Now suppose that $\epsilon > 0$ and $\mathbf{G}$ is any graph on n vertices with more than $(1/4+\epsilon)n^2$ edges. We show that $\dim(\mathbf{G}) > [3 \updownarrow 4]$ provided n is sufficiently large. Suppose that $\dim(\mathbf{G}) \leq [3 \updownarrow 4]$ and choose a realizer $\mathcal{R} = \{L_1, L_2, L_3, L_4\}$ with $L_3 = L_4^d$. From the Erdös/Stone theorem, we know that for every $p \geq 1$, $\mathbf{G}$ contains a complete 3-partite graph with p vertices in each part—provided n is sufficiently large in terms of p. Choose such a subgraph and label the three parts as V_1, V_2 and V_3. Using the product ramsey theorem, it follows that if p is sufficiently large, there exists $W_1 \subset V_1$, $W_2 \subset V_2$ and $W_3 \subset V_3$, with $|W_1| = |W_2| = |W_3| = 2$, so that for each $i, j, k = 1, 2, 3$ with $i \neq j$, either all points of W_i are under all points of W_j in L_k or all points of W_i are over all points of W_j in L_k.

Label the points so that $W_1 = \{x_1, x_2\}$, $W_2 = \{y_1, y_2\}$ and $W_3 = \{z_1, z_2\}$. Without loss of generality, we may assume that $x_1 < x_2 < y_1 < y_2 < z_1 < z_2$ in L_3, so that $z_2 < z_1 < y_2 < y_1 < x_2 < x_1$ in L_4.

Consider the vertex y_1 and the edge $\{x_1, y_2\}$. Since $y_1 < y_2$ in L_3 and $y_1 < x_1$ in L_4, we may assume without loss of generality that y_1 is over both x_1 and y_2 in L_1. Thus y_1 and y_2 are over x_1 and x_2 in L_1. Similarly, considering the vertex y_2 and the edge $\{z_1, y_1\}$, we may conclude that y_2 is over both z_1 and y_1 in L_2. Thus y_1 and y_2 are over z_1 and z_2 in L_2.

Following this pattern, we may then conclude that z_1 is over both z_2 and y_1 in L_1, while x_2 is over both x_1 and y_1 in L_2. It follows that the middle two points of $W_1 \cup W_2 \cup W_3$ in each of the four linear orders are y_1 and y_2. This is a contradiction, since it implies that y_1 is never higher than both x_1 and z_1. The contradiction completes the proof. □

Remark. A previous version of this paper contained two conjectures regarding the structure of the extremal graphs of dimension at most $[3 \updownarrow 4]$ and 4. The conjectures where that these graphs can be obtained from complete four-partite and bipartite graphs by adding an maximal outerplaner graph on each of the color classes.

Both of these conjectures have been disproved recently by de Mendez and Rosenstiehl [7]. An independent example disproving the second of the conjectures was brought to our attention by an anonymous referee.

6. Minor-monotone Issues

It follows from Schnyder's theorem that the property of having dimension at most 3 is minor closed, i.e., if $\mathbf{G}$ has dimension at most 3, then any minor of $\mathbf{G}$ has dimension at most 3. However, we no of no direct proof of this assertion—other than to appeal to the full power of Schnyder's theorem. Ideally, one would like to find an alternative proof of Schnyder's theorem by combining the following three assertions:

(1) For every $n \geq 1$, the $n \times n$ grid has dimension at most 3.
(2) If $\mathbf{G}$ is a planar graph, there is some $n \geq 1$ for which $\mathbf{G}$ is a minor of an $n \times n$ grid.
(3) Every minor of a graph of dimension at most 3 has dimension at most 3.

Of course, each of these three statements is true, and simple proofs are known for the first two. So we just want to find a direct proof of the third.

We also know that the property of having dimension at most $[2 \updownarrow 3]$ is minor closed. However, we do not know of a simple proof of this statement either.

For $t \geq [3 \updownarrow 4]$, it is easy to see that the property $\dim(\mathbf{G}) \leq t$ is no longer minor closed. For example, $\dim(\mathbf{K}_n) \to \infty$ but if we subdivide each edge, then we obtain a bipartite graph which has dimension at most $[3 \updownarrow 4]$. We may then ask whether there is an appropriate generalization of the concept of dimension which coincides with the original definition when $t < [3 \updownarrow 4]$ and is minor closed when $t \geq [3 \updownarrow 4]$. We could also ask whether there is any way to characterize linklessly embeddable graphs in this framework.

7. Complexity Issues

Yannakakis [25] showed that testing for $\dim(\mathbf{P}) \leq t$ is NP-complete for every fixed $t \geq 3$. Yannakakis also proved that testing for $\dim(\mathbf{P}) \leq t$ is NP-complete even for height 2 posets when $t \geq 4$. However, he was not able to settle whether testing for $\dim(\mathbf{P}) \leq 3$ is NP-complete for height 2 posets. This problem remains open.

Our original definition for dimension was formulated for a graph. However, it applies equally as well to hypergraphs. In a similar manner, we can speak of the incidence poset $\mathbf{P_H}$ of a hypergraph $\mathbf{H}$. When $\mathbf{G}$ is a graph, testing for $\dim(\mathbf{G}) \leq 3$ is linear, since this is just a test for planarity. A similar remark holds when testing for $\dim(\mathbf{G}) \leq [2 \updownarrow 3]$. When $\mathbf{H}$ is a hypergraph, we do not know if testing for $\dim(\mathbf{H}) \leq 3$ is NP-complete. Also, we do not know whether testing for $\dim(\mathbf{H}) \leq [2 \updownarrow 3]$ is NP-complete. We suspect that testing for $\dim(\mathbf{G}) \leq [3 \updownarrow 4]$ is NP-complete, but have not been able to settle the question.

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