The Number of Convex Polyominoes with Given Height and Width

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March 12, 2019

Abstract

We give a new combinatorial proof for the number of convex polyominoes whose minimum enclosing rectangle has given dimensions. We also count the subclass of these polyominoes that contain the lower left corner of the enclosing rectangle (*directed* polyominoes). We indicate how to sample random polyominoes in these classes. As a side result, we calculate the first and second moments of the number of common points of two monotone lattice paths between two given points.

Contents

1   Convex Polyominoes: Overview of Results

A *polyomino* is a connected set of lattice squares, where two squares are considered as adjacent if they share an edge. A polyomino is *convex* if the intersection with every horizontal or vertical line is a connected interval, see Figure 1 for an example. The *enclosing rectangle R* is the smallest axis-parallel rectangle containing the polyomino. The *height h* and *width w* of a polyomino are defined as the corresponding dimensions
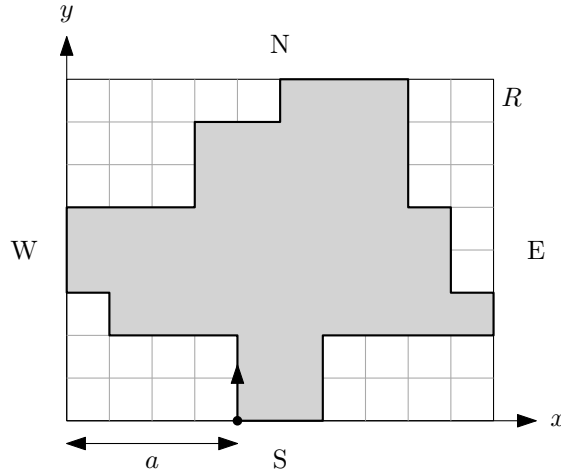
Figure 1: A convex polyomino of width $w = 10$ and height $h = 8$ in its enclosing rectangle R. We traverse the boundary clockwise, starting at the leftmost point $(a, 0)$ on the bottom side. The distance a of this point from the left edge will be used as a parameter.

of R. The perimeter (length of the boundary) of a convex polyomino is $2(h + w)$. Throughout, we will denote the semi-perimeter by $s := h + w$.

Theorem 1. *The number of convex polyominoes of width $w \geq 1$ and height $h \geq 1$ is*

$$P_{wh} = \binom{2s - 4}{2w - 2} + \frac{2s - 5}{2}\binom{2s - 6}{2w - 3} - 2(s - 3)\binom{s - 2}{w - 1}\binom{s - 4}{w - 2}, \qquad (1)$$

where $s = w + h$, see Table 1.

This formula was first proved by Gessel [Ges], who derived it from the generating function

$$\sum_{h=1}^{\infty}\sum_{w=1}^{\infty} P_{wh}x^{2w}y^{2h} = \frac{x^2y^2\, B(x^2, y^2)}{\Delta(x, y)^2} - \frac{4x^4y^4}{\Delta(x, y)^{3/2}} \qquad (2)$$

with

$$B(x, y) = 1 - 3x - 3y + 3x^2 + 3y^2 + 5xy - x^3 - y^3 - x^2y - xy^2 - xy(x - y)^2,$$

$$\Delta(x, y) = 1 - 2x^2 - 2y^2 + (x^2 - y^2)^2 = (1 + x + y)(1 + x - y)(1 - x + y)(1 - x - y),$$

which is due to Lin and Chang [LC, Eq. (27)]. The two terms in (2) correspond to the positive and the negative terms in (1). Another proof for the formula (1) was given by Guo and Zeng [GZ]. They exploit a correspondence between convex polyominoes and noncrossing lattice paths, similar as in our proof. They use generating functions of Jacobi polynomials in order to evaluate the expressions that arise. We will review this proof in Section 4.4.

We present a new combinatorial proof of formula (1) in Section 4, and we assign some meaning to its terms. Based on this interpretation, in Section 6, we discuss an acceptance-rejection method for sampling from this set of convex polyominoes, with an acceptance probability tending to one for large parameters w and h.

The summation of P_{wh} over all pairs w, h with fixed sum s gives the number of convex

s									P_{wh}											P_s
2									1											1
3								1		1										2
4							1		5		1									7
5						1		13		13		1								28
6					1		25		68		25		1							120
7				1		41		222		222		41		1						528
8			1		61		555		1110		555		61		1					2344
9		1		85		1171		3951		3951		1171		85		1				10416
10	1		113		2198		11263		19010		11263		2198		113		1			46160
11	1		145	3788		27468		70438		70438		27468		3788		145		1		203680

Table 1: The number P_{wh} of convex polyominoes of width w and height h for $2 \le s = w + h \le 11$ [OEIS, A093118]. The width w increases from 1 to $s - 1$ in row s, while h decreases and $s = w + h$ remains constant. The row sum P_s is given in the rightmost column. It is the number of convex polyominoes of perimeter $2s$ [OEIS, A005436].

polyominoes with perimeter $2s$, which we denote by P_s. The resulting formula,

$$P_s = \sum_{w+h=s} P_{wh} = 4^{s-4}(2s+3) - 4(2s-7)\binom{2s-8}{s-4}$$

$$= 4^{s-4}(2s+3) - (2s-6)\binom{2s-6}{s-3}, \quad \text{for } s \ge 4 \qquad (3)$$

has been known before [DV, Kim]. (The reader should beware that the literature contains formulas in terms of various different quantities instead of the parameter s that we have chosen.)

A *directed convex polyomino* is a convex polyomino that contains the lower left corner of its enclosing rectangle R.

Theorem 2. *The directed convex polyominoes of width $w \ge 1$ and height $h \ge 1$ are counted by the squared binomial coefficients:*

$$D_{wh} = \binom{s-2}{w-1}^2, \qquad (4)$$

where $s = w + h$, see Table 2.

This formula was proved by Barcucci, Frosini, and Rinaldi [BFR]. It is implicit also in Guo and Zeng [GZ], see Section 4.4. We will give a different proof in Section 3. The corresponding generating function

$$\sum_{h=1}^{\infty} \sum_{w=1}^{\infty} D_{wh} x^{2w} y^{2h} = x^2 y^2 / \sqrt{\Delta(x,y)}$$

has been known as well, see Lin and Chang [LC, Eq. (25)].

The formula (4) suggests that there should be a bijection between these polyominoes and pairs of objects that are counted by the binomial coefficients. Indeed, from our proof of Theorem 2, we can derive a bijection to pairs of monotone lattice paths. This is described in Section 3.1. Such a bijection can be used to generate a random sample from the polyominoes from this class, since it is straightforward to generate random monotone lattice paths. The proof of Barcucci et al. [BFR] mentioned above is also bijective: They used a bijection between directed polyominoes and so-called *two-colored*

s					D_{wh}						D_s
2						1					1
3					1		1				2
4				1		4		1			6
5			1		9		9		1		20
6		1		16		36		16		1	70
7		1	25		100		100		25	1	252
8	1	36		225		400		225	36	1	924
9	1	49	441		1225		1225	441	49	1	3432
10	1	64	784	3136		4900	3136	784	64	1	12870
11	1	81	1296	7056	15876	15876	7056	1296	81	1	48620

Table 2: The number D_{wh} of directed convex polyominoes of width w and height h. These are the squared binomial coefficients [OEIS, A008459].

Grand-Motzkin paths. It is easy to see that two-colored Grand-Motzkin paths are in one-to-one correspondence with pairs of lattice paths, but the bijection to polyominoes is different from ours. We will say more in Section 3.2.

As above, we can sum the rows of the table and conclude that the number of directed convex polyominoes with perimeter $2s$ are the central binomial coefficients

$$D_s = \binom{2s-4}{s-2},$$

which was also known before [LC, B-M], [OEIS, A000984].

Our investigation of pairs of lattice paths has led, as a side result, to formulas for the first and second moments of the number of intersections between two lattice paths, see Section 5.

For completeness, we state the well-known formulas for *parallelogram polyominoes*, which contain both the lower left and the upper right corner of R. A proof can be easily obtained by Lemma 1 below (see Section 3).

Proposition 1. *The number of parallelogram polyominoes of width $w \geq 1$ and height $h \geq 1$ is given by a Narayana number*

$$B_{wh} = \frac{1}{s-1}\binom{s-1}{w}\binom{s-1}{h}. \qquad [\text{OEIS, A001263}]$$

The number of convex parallelogram polyominoes of perimeter $2s$ is equal to the $(s-1)$th Catalan number

$$B_s = \frac{1}{s}\binom{2s-2}{s-1}. \qquad [\text{OEIS, A000108}] \qquad \square$$

For a bijective proof of the second formula and some references to its history see Stanley [Sta, p. 66].

2   Conventions

Before going into the proofs, we clarify some notations that we will use. We label the four sides of R by N, S, E, W. Accordingly, we call the corners of R NE corner, SE corner, and so on. We place R in the first quadrant with the SW corner at the origin.

By an *edge*, we mean a unit-length boundary segment of a grid square. On the boundary of a polomino, several edges may lie straight in succession, but they are considered as separate edges.
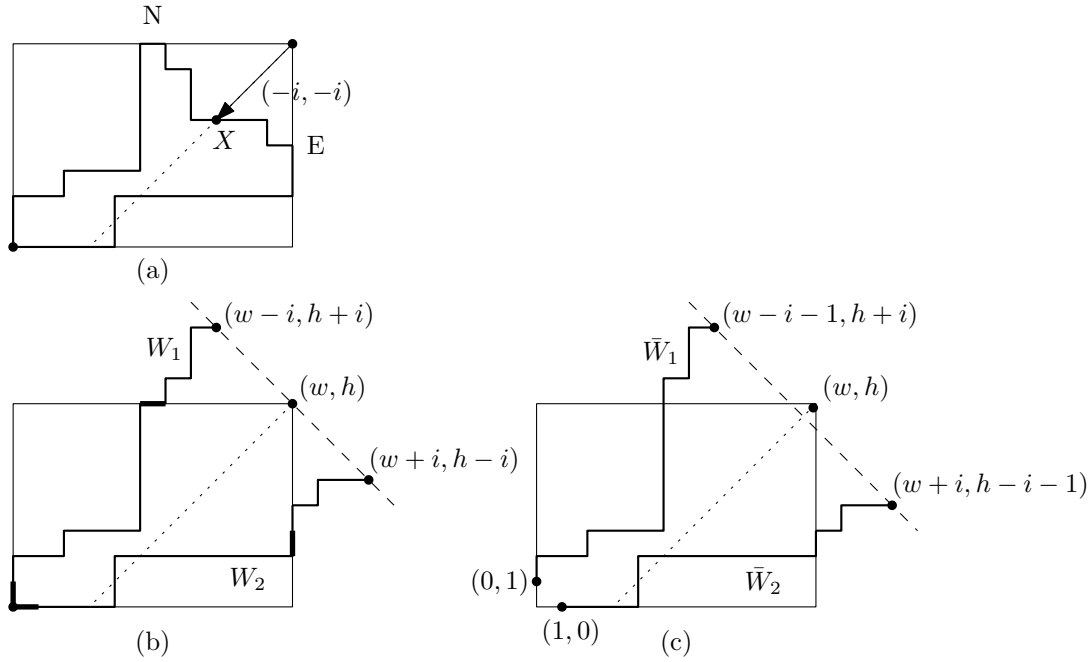
Figure 2: Splitting the boundary at the diagonal (a) and reflecting at the N-side and E-side. (b) An implicit edge on each side of R is highlighted. (c) The implicit edges from the N-side and the E-side are removed.

If P is a polyomino with enclosing rectangle R and $K \in \{N, S, E, W\}$, then the K-*side* of P is the intersection of P with the respective side of R.

3   Directed Convex Polyominoes

We start with the easier and more restricted case of directed polyominoes, i.e., polyominoes that contain the SW corner $(0,0)$ (Theorem 2). This serves also as a preparation for our treatment of general polyominoes in Section 4, where some of the arguments will recur.

We will relate directed polyominoes to certain monotone paths in the square lattice. A *monotone path* is a lattice path whose steps go in only two orthogonal directions.

Traverse the boundary clockwise, starting from the origin. Between the N-side and the E-side, the boundary intersects the diagonal $y - x = h - w$ through the NE corner at a unique point $X = (w, h) - (i, i)$ for some $i \geq 0$. see Figure 2.

We cut the boundary path at X. We reflect the path between the N-side and X across the N-side of R. Similarly, we reflect the path between X and the E-side across the E-side of R. We get two monotone paths W_1 and W_2, ending in $(w, h) + (-i, i)$, and $(w, h) + (i, -i)$. It would be easy to count pairs of these paths but the transformation would not be bijective. We have to ensure two properties.

(a) The boundary must not cross itself.

(b) The boundary must have at least one edge on each of the four sides of R.

We take care of (b) by declaring the first edge on the E- and the N-side to be an *implicit edge*. If we remove these edges from the two paths, we obtain paths $\bar{W}_1$ and $\bar{W}_2$ that end in $(w, h) + (-i - 1, i)$, and $(w, h) + (i, -i - 1)$. The implicit edges on the W-side and the S-side are easier to take care of: We know that the boundary starts with an up-step and ends with a left-step. Thus, we simply let $\bar{W}_1$ start in the point $(0, 1)$, and

we let $\bar{W}_2$ start $(1,0)$. Now we have a bijection: There is a one-to-one correspondence between directed polyominoes and pairs $(\bar{W}_1, \bar{W}_2)$ of *non-intersecting* monotone lattice paths from $(0,1)$ to $(w,h) + (-i-1, i)$, and from $(1,0)$ to $(w,h) + (i,-i-1)$, for some $0 \le i < \min\{w, h\}$. Indeed, given such a pair of non-intersecting paths, we re-insert the implicit edge into each path, reflect the paths outside R back, and join the two pieces together. This process is reversible, and it does not introduce or destroy intersections.

We still have to take care of self-intersections. It is known how to count pairs of non-intersecting monotone lattice paths. We will use the basic Lemma 1 below, which is the first nontrivial case of the Gessel–Viennot Lemma. We include its proof because we will extensively use the underlying uncrossing idea. The Gessel–Viennot Lemma is more general: it counts families of non-intersecting paths between more than two pairs of terminal nodes in acyclic directed graphs.

For integer points p, q, u, v in the plane, we denote by $\bar{X}_{p,q}^{u,v}$ the number of *non-intersecting* pairs of monotone grid paths in the NE direction from p to u and from q to v, and by $X_{p,q}^{u,v}$ the number of *intersecting* pairs of such paths. By N_p^u, we denote the number of monotone paths in the NE direction from p to u. This is equal to a binomial coefficient, for which we temporarily introduce the abbreviation

$$Z_x^y = \binom{x+y}{x}.$$

Thus,

$$N_{(a,b)}^{(c,d)} = N_{(0,0)}^{(c-a,d-b)} = Z_{c-a}^{d-b}.$$

Lemma 1. *Assume that the number of non-intersecting pairs of paths from p to v and from q to u is zero. Then the number of intersecting pairs of paths from p to u and from q to v is*

$$X_{p,q}^{u,v} = N_p^v N_q^u. \tag{5}$$

Consequently, the number of non-intersecting pairs of paths is

$$\bar{X}_{p,q}^{u,v} = N_p^u N_q^v - N_p^v N_q^u. \tag{6}$$

Proof. For each intersecting pair of paths we can perform a crossover switch at the first intersection point, leading to a pair of paths where the end terminals are swapped. Conversely, every pair of paths with swapped endpoints must intersect, by assumption, and hence we can perform a crossover at the first intersection point. These two operations are inverses, thus establishing (5) by bijection. $\square$

The lemma gives

$$
\begin{aligned}
D_{wh} &= \sum_{i \ge 0} \bar{X}_{(0,1),(1,0)}^{(w-i-1,h+i),(w+i,h-i-1)} \\
&= \sum_{i \ge 0} (Z_{h+i-1}^{w-i-1} \cdot Z_{h-i-1}^{w+i-1} - Z_{h+i}^{w-i-2} \cdot Z_{h-i-2}^{w+i}) \\
&= Z_{h-1}^{w-1} Z_{h-1}^{w-1} - Z_h^{w-2} Z_{h-2}^w + Z_h^{w-2} Z_{h-2}^w - Z_{h+1}^{w-3} Z_{h-3}^{w+1} + Z_{h+1}^{w-3} Z_{h-3}^{w+1} - \cdots . \tag{7}
\end{aligned}
$$

This is a telescoping sum. Its terms become eventually zero, because the horizontal distances $w - i - 1$ and the vertical distances $h - i - 1$ become negative for large i. Thus, only the very first term remains, and therefore

$$D_{wh} = Z_{h-1}^{w-1} Z_{h-1}^{w-1} = \binom{s-2}{w-1}^2. \qquad \square$$
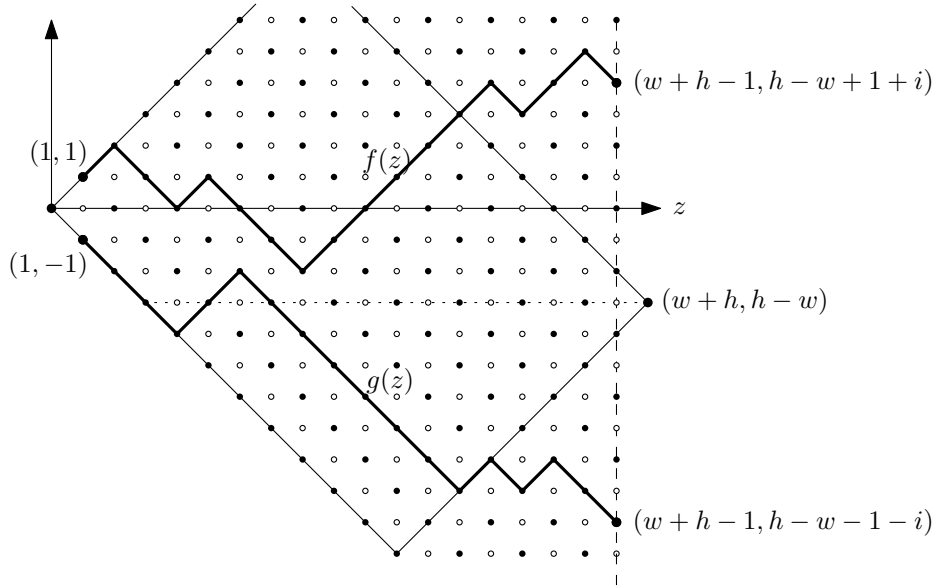
Figure 3: Two functions f and g.

3.1 A bijection for directed convex polyominoes

The formula $\binom{s-2}{w-1}^2$ has an obvious combinatorial interpretation: it counts ordered *pairs* of monotone paths from $(0,0)$ to $(w-1, h-1)$. By analyzing the cancellation that happens in the telescoping sum (7), we can derive a bijection to directed convex polyominoes of height h and width w.

We start from a pair of paths from $(0,0)$ to $(w-1, h-1)$. It is more convenient to turn the grid by $45°$ and scale it by $\sqrt{2}$, using the transformation $\binom{x}{y} \mapsto \binom{x+y}{y-x}$. We let the first path start at $(1,1)$ and the second path at $(1,-1)$, like the desired paths of Figure 2c, see Figure 3. This leads to two paths with endpoints $(w+h-1, h-w+1)$ and $(w+h-1, h-w-1)$. The paths can be viewed as graphs of two functions $f, g \colon \{1, 2, \ldots, s-1\} \to \mathbb{Z}$ that make steps ± 1: $|f(z+1) - f(z)| = 1$ and $|g(z+1) - g(z)| = 1$. Now we will modify these two functions to ensure that they don't intersect. In general, we will have two functions on $\{1, 2, \ldots, s-1\}$. The starting heights are $+1$ and -1, and the ending heights are given by $h - w + 1 + 2i$ and $h - w - 1 - 2i$, for some $i \geq 0$. Initially $i = 0$. If the paths intersect we make a crossover at the rightmost intersection point, swapping the initial parts to the left of this point. Then we shift the first path by $+2$ and the second path by -2. Schematically, this is shown as follows (see also Figure 4 for an example):

$$\begin{bmatrix} +1 \to h-w+1 + 2i \\ -1 \to h-w-1 - 2i \end{bmatrix} \stackrel{\text{crossover}}{\Longrightarrow} \begin{bmatrix} -1 \to h-w+1 + 2i \\ +1 \to h-w-1 - 2i \end{bmatrix} \stackrel{\text{vertical shift}}{\Longrightarrow} \begin{bmatrix} +1 \to h-w+1 + 2i + 2 \\ -1 \to h-w-1 - 2i - 2 \end{bmatrix}$$

We repeat this procedure as long as the functions intersect. The same procedure would also work if we systematically take the *leftmost* intersection. Both versions, if implemented naively, take quadratic time. The current version is better because the parts to the right of the intersection point are always moved further apart. This means that no new intersection points appear there, and we can find the sequence of rightmost intersection points in a single right-to-left sweep.

The changes left of the intersection points are best understood in terms of their effect on the *difference* function $\delta(z) := f(z) - g(z)$. Swapping f and g amounts to negating δ, and the vertical shift will add 4 to δ. Together, these two operations will perform the mapping $\delta \mapsto 2 - \delta$.
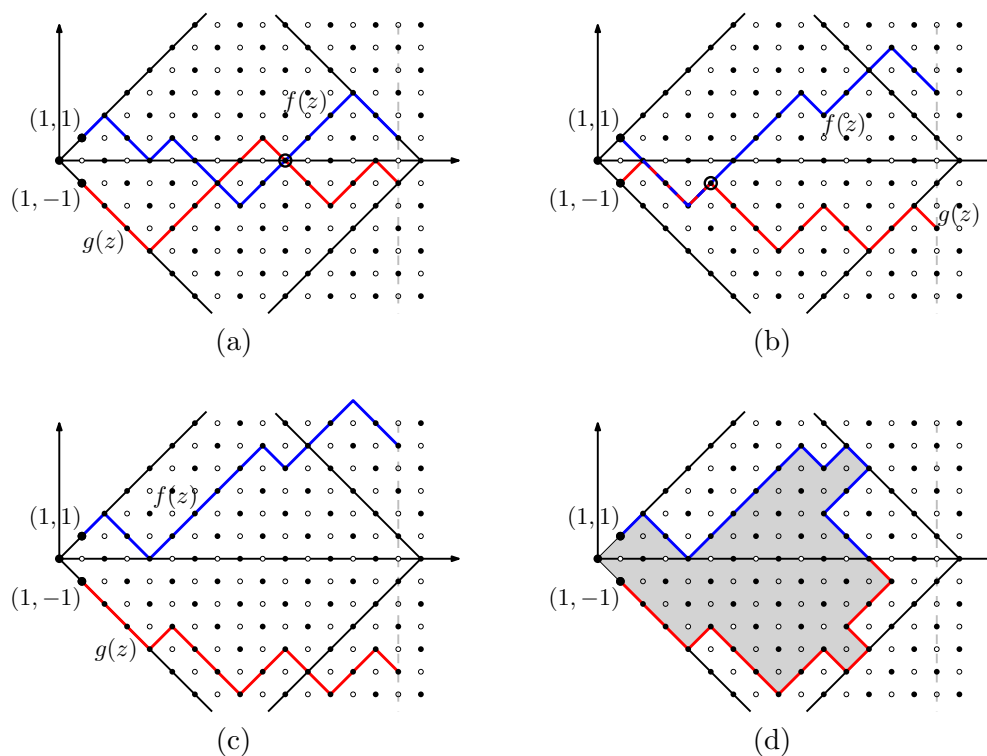
Figure 4: (a)–(c): A sequence of untangling operations for an intersecting path pair. The rightmost intersection pairs are marked. (d): The final mapping from the non-intersecting paths to the polyomino.

Performing this operation twice will restore the original function. We can therefore leave the functions intact and, instead of searching for zeros of δ, alternately search for the values 0 and 4. The following is a streamlined version of this algorithm, where δ does not explicitly appear. After the sequence of intersection points is determined in the first phase, we compute the resulting noncrossing paths $F, G: \{1, 2, \ldots, s-1\} \to \mathbb{Z}$ of the bijection in a left-to-right sweep.

```
−− Phase 1. Right-to-left sweep:
−− identify all intersection points
swapped := False
target := 0
s := 0
for z := s − 1, s − 2, . . . , 2, 1:
    if f[z] − g[z] = target :
        intersection[z] := True
        swapped := not swapped
        target := 4 − target
    else:
        intersection[z] := False
```

```
−− Phase 2. Left-to-right sweep:
if swapped:
    shift := 2
else:
    shift := 0
for z := 1, 2, . . . , s − 1:
    if swapped :
        F[z] = g[z] + shift
        G[z] = f[z] − shift
    else:
        F[z] = f[z] + shift
        G[z] = g[z] − shift
    if intersection[z] :
        swapped := not swapped
        shift := shift + 2
```

3.2 Two-colored Grand-Motzkin paths

When we found our proof, we were not aware of the bijective proof of Theorem 2 by Barcucci, Frosini, and Rinaldi [BFR]. There are differences as well as commonalities, which we will now discuss, glossing over superficial details. In particular, the idea of cutting at the point of intersection X with the diagonal through the upper right corner is also used. This leads, in our case, to a pair of functions f, g such that f always remains above g. Barcucci et al. [BFR, Section 3] represent the same information in terms of the difference function $\delta = f - g$. This function has only even values. When δ makes a step of size 2, i.e., $\delta(z + 1) = \delta(z) \pm 2$, it is possible to infer the change of f and g. However, if δ is stationary, this means that f and g either both go up or both go down. One therefore has to distinguish these possibilities by marking each horizontal step of δ with one of two colors. After dividing δ by 2, this leads to the notion of a "two-colored Grand-Motzkin path" as an alternative way to represent the pair (f, g): A Grand-Motzkin path is a sequence of steps from $\{0, \pm 1\}$ starting and ending at the same height, and a Motzin path has the additional requirement of never going below the initial height.

The main part of the proof concerns the bijective mapping from a pair (f, g) of functions with difference $\delta(s - 1) = 2$ at the endpoint to a pair (F, G) of *noncrossing* functions with endpoints at difference $F(s-1) - G(s-1) = 2+4i$, for some i. In contrast to our bijection, which scans the sequence from right to left, Barcucci et al. [BFR, Section 3.1.3] process the sequence from left to right. Their bijection has a very elegant pictorial description in terms of Grand-Motzkin paths [BFR, Figs. 13–14]. Translated to our notation, it can be described as follows: Look for the leftmost point z where $\delta(z) = 0$, $\delta(z) = -2$, $\delta(z) = -4$, etc. Arriving at such a point, f must have taken a down-step: $f(z) = f(z-1) - 1$, and g must have taken an up-step: $g(z) = g(z-1) + 1$. Modify f by adding 2 to $f(z), f(z+1), \ldots$, changing the down-step into an up-step, and modify g similarly by subtracting 2. If the minimum value of δ is initially $2-2i$, then this procedure is carried out cumulatively for i different positions z, and the final difference at the endpoints is $F(s-1) - G(s-1) = 2+4i$. We have achieved that $F(z) \geq G(z)+2$ throughout. This bijection is clearly different from our algorithm, which was derived from the cancellation in (7) and from Lemma 1. It is simpler as it does not need to swap parts of f and g.

The inverse mapping, from (F, G) to (f, g) is just as easy: If $F(s-1) - G(s-1) = 2+4i$, perform a right-to-left scan: look for the rightmost points z where $F(z) - G(z) = 2i$, $2i - 2, \ldots, 4, 2$. These are the points where up-shifts must be inverted to down-shifts and vice versa.

4 General Convex Polyominoes

4.1 S-walks

We will first introduce a definition that includes all convex polyominoes but also allows some self-intersection of the boundaries.

An *S-walk* is a shortest closed walk on the grid that spans its enclosing rectangle R. By *spanning*, we mean that it contains at least one edge on each of the four sides of R, while *shortest* says that the length of the walk equals the perimeter of R; in particular, the S-walk is contained in R. (The letter S in S-walks has nothing to do with the South direction.) We orient the S-walk by the convention that the edges on the S-side are traversed from right to left, i.e., in the W direction. (If the S-walk is non-intersecting, this means that it is oriented clockwise.) If R is an $h \times w$ rectangle, then an S-walk consists of exactly h steps in each of the E and W directions, and exactly w steps in each
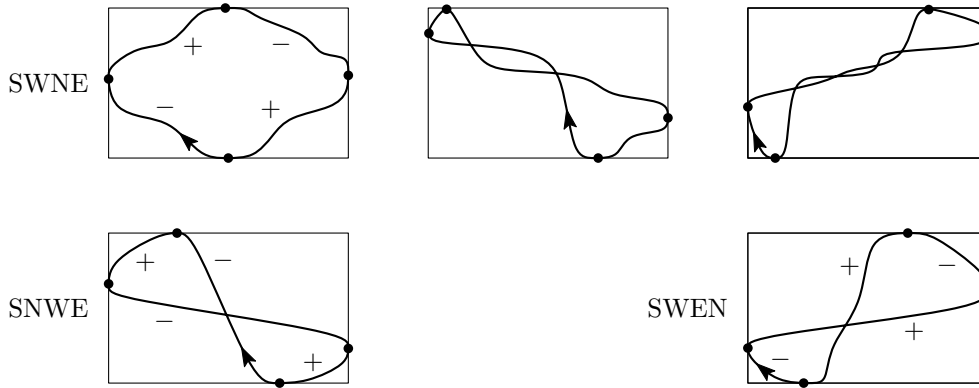
Figure 5: The possible orders in which an S-walk can touch the four sides of the enclosing rectangle, and the possibilities of self-intersections. Some rising and falling parts are marked by + and −.

of the N and S directions, so its total length is $2(h + w)$. An S-walk forms the boundary of a convex polyomino if and only if it does not intersect itself.

The S-walk will not necessarily visit the four sides in the correct order. We choose the leftmost point on the S-side as the starting point, as shown in Figure 1. Since we are moving in the W direction, the S-walk will meet the W-side before the E-side. This leaves three possible orders in which the walk can hit the sides: SWNE, SNWE, and SWEN. They are shown schematically in Figure 5. The "normal" order, which corresponds to the non-intersecting case, is SWNE. In the two other orders, an intersection is forced. Removing the parts where the S-walk touches the rectangle boundary decomposes it into four pieces, each of which may be empty. Each piece is a monotone grid path, and we classify these paths either as *rising* or *falling*, by looking at them as the graph of a function and reading them from left to right (independent of the traversal direction). Thus, for example the path between the W-side and the N-side is always rising. Whenever the walk hits a boundary edge, it alternates between rising and falling. Thus, there will always be two rising paths and two falling paths. Looking at Figure 5, it is easy to convince oneself that a self-intersection can occur either between the two rising paths, or between the two falling paths, but not both simultaneously. For example, in the SWNE case, the two falling paths can cross only if the rightmost point of the walk on the N-side is to the left of the leftmost point of the walk on the S-side, and the bottommost point of the walk on the W-side is higher than the topmost point of the walk on the E-side. This precludes the two rising paths from intersecting. Of course, there may be multiple intersections between two rising/falling paths.

Thus, given an $h \times w$ enclosing rectangle, denoting by $\widetilde{P}_{wh}$ the number of all S-walks, and by P^*_{wh} the number of S-walks in which the *rising* paths intersect, we can compute the number P_{wh} of convex polyominoes by the formula

$$P_{wh} = \widetilde{P}_{wh} - 2P^*_{wh}. \tag{8}$$

The factor 2 reflects the fact that the intersection of the falling paths is symmetric to the rising case. In the following two subsections, we will determine $\widetilde{P}_{wh}$ and P^*_{wh}.

4.2 The number of all S-walks

We start the walk at the leftmost point $(a, 0)$ on the S-side with an up-step. Given the starting point in the range $0 \le a \le w - 1$, we code the walk by recording a sequence over the alphabet $\{V, H\}$, representing the walking steps sequence whether we make a
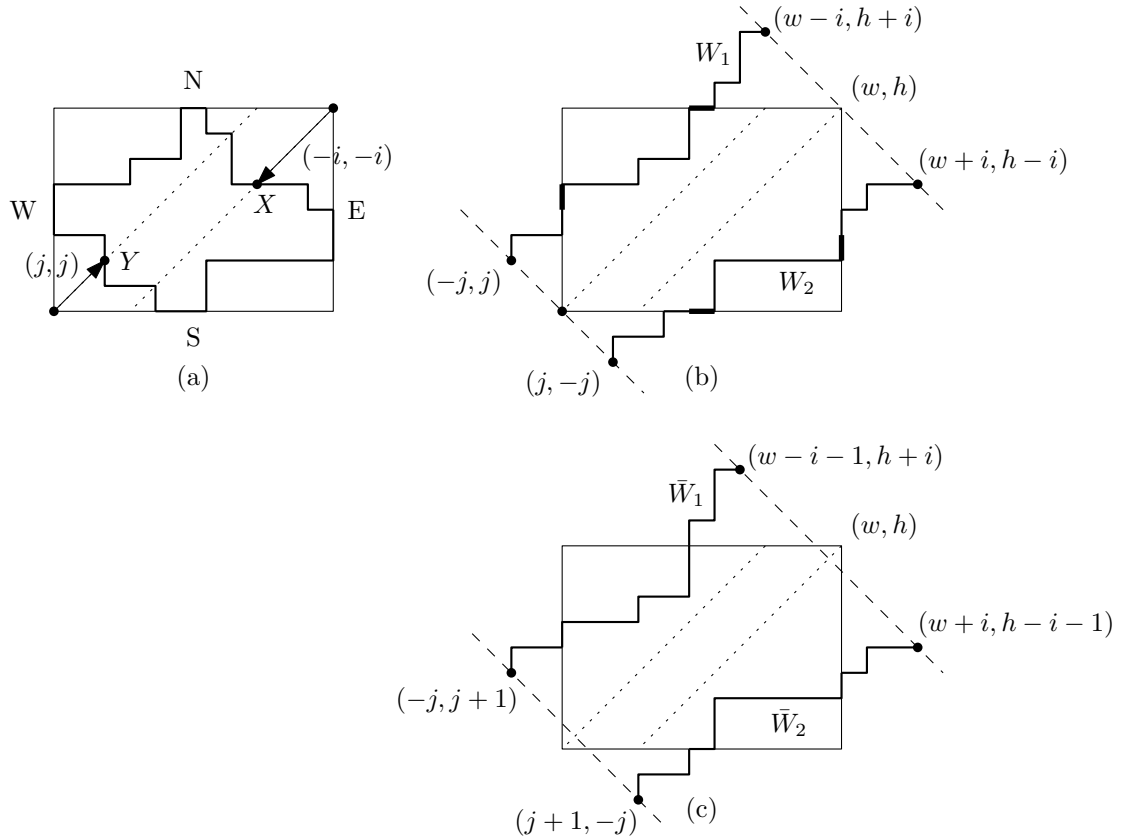
Figure 6: Splitting the boundary at two diagonals (a) and reflecting at each side. (b) An implicit edge on each side of R is highlighted. (c) The four implicit edges are removed.

vertical or horizontal step. It is not necessary to record the orientation in which the steps go (left versus right, or up versus down), because this information can be recovered by tracing the path when it hits the boundary of the rectangle. Furthermore, the walk must contain at least one edge on each of the four sides of the rectangle. These four steps are *implicit steps* and are not coded. A code is translated into an S-walk as follows: We start at $(a, 0)$ and follow the $\{V, H\}$-code to trace the walk. Whenever we touch one of the four sides W,N,S,E of R for the first time, an implicit step along that side is inserted. In addition, the first up-step at $(a, 0)$ is also not coded. If $a = 0$, the first up-step coincides with the implicit W-step. (This corresponds to directed polyominoes.) Then the code consists of $2h - 2$ V's and $2w - 2$ H's. If $1 \le a \le w - 1$, we need only $2h - 3$ V's. In total, the number of S-walks is therefore

$$\widetilde{P}_{wh} = \binom{2s-4}{2w-2} + (w-1)\binom{2s-5}{2w-2} = \binom{2s-4}{2w-2} + \binom{2s-6}{2w-3}\frac{2s-5}{2}. \tag{9}$$

The factor $w - 1$ counts the number of nonzero possibilities for a.

This coding is very similar to the code used by Hochstättler, Loebl and Moll [HLM] for generating convex polyominoes of given perimeter, see also Section 6.1.

4.3   Self-intersecting S-walks

To count S-walks where the two rising paths intersect, we adapt the approach of Section 3, splitting the walk at diagonals and moving falling paths outside by reflection. The two relevant cases are SWNE (the "normal case") and SWEN (the "flipped case"). This time, we split the curve not only at the upper right part but also at the lower left

part, see Figure 6. More precisely, in addition to splitting the S-walk at the intersection point X of the falling path between the N-side and E-side with the diagonal through the NE corner, we also split it at the intersection point Y of the falling path between the S-side and W-side with the diagonal through the SW corner. Both in the normal case SWNE, which is illustrated in Figure 6, and in the flipped case SWEN, there is a falling path between S and W and another falling path between N and E, so that the intersection points X and Y are well-defined.

The two resulting paths intersect if and only if the two rising paths of the S-walk intersect. Thus we are interested in the number of intersecting pairs of paths. In the normal case SWNE, this is

$$\sum_{i\geq 0}\sum_{j\geq 0} X_{(j+1,-j),(-j,j+1)}^{(w+i,h-i-1),(w-i-1,h+i)} = \sum_{i\geq 0}\sum_{j\geq 0} N_{(-j,j+1)}^{(w+i,h-i-1)} N_{(j+1,-j)}^{(w-i-1,h+i)}. \tag{10}$$

For the flipped case SWEN, we get a pair of paths from $(-j, j+1)$ to $(w+i, h-i-1)$ and from $(j+1, -j)$ to $(w-i-1, h+i)$. All these path pairs are forced to intersect by the positions of their endpoints, and their number equals the same expression (10). Consequently,

$$P_{wh}^* = 2\sum_{i\geq 0}\sum_{j\geq 0} N_{(-j,j+1)}^{(w+i,h-i-1)} N_{(j+1,-j)}^{(w-i-1,h+i)} = 2\sum_{i\geq 0}\sum_{j\geq 0}\binom{s-2}{w+i+j}\binom{s-2}{w-2-i-j}.$$

We note for later use that the number of S-walks of the normal type SWNE in which the rising paths intersect is $P_{wh}^*/2$.

Since i and j always occur together in the above summation for P_{wh}^*, we group terms with the same sum $i+j$:

$$P_{wh}^* = 2\sum_{i\geq 0}\sum_{j\geq 0}\binom{s-2}{w+i+j}\binom{s-2}{w-2-i-j}$$
$$= 2\sum_{k\geq 0}\sum_{\substack{i+j=k\\i,j\geq 0}}\binom{s-2}{w+i+j}\binom{s-2}{w-2-i-j} = 2\sum_{k\geq 0}(k+1)\binom{s-2}{w+k}\binom{s-2}{w-2-k}.$$

In order to get rid of the factor $(k+1)$ we split it into two terms $(w+k) - (w-1)$, which can be more readily incorporated into the binomial coefficients:

$$P_{wh}^* = 2\sum_{k\geq 0}(w+k)\binom{s-2}{w+k}\binom{s-2}{w-2-k} - 2\sum_{k\geq 0}(w-1)\binom{s-2}{w+k}\binom{s-2}{w-2-k}$$
$$= 2\sum_{k\geq 0}(s-2)\binom{s-3}{w+k-1}\binom{s-2}{w-2-k} - 2\sum_{k\geq 0}(w-1)\binom{s-2}{w+k}\binom{s-2}{w-2-k}$$
$$= 2(s-2)T_1 - 2(w-1)T_2.$$

We will analyze the two sums T_1 and T_2 in the last expression separately. For the first sum

$$T_1 = \sum_{k\geq 0}\binom{s-3}{w+k-1}\binom{s-2}{w-2-k},$$

we have the following combinatorial interpretation, see Figure 7. Let $A = s - 3$. From the numbers $1, 2, \dots$ up to $(s-3) + (s-2) = 2s - 5 = 2A + 1$, we have to select a subset S of $(w+k-1) + (w-2-k) = 2w - 3$ elements, under the additional constraint that
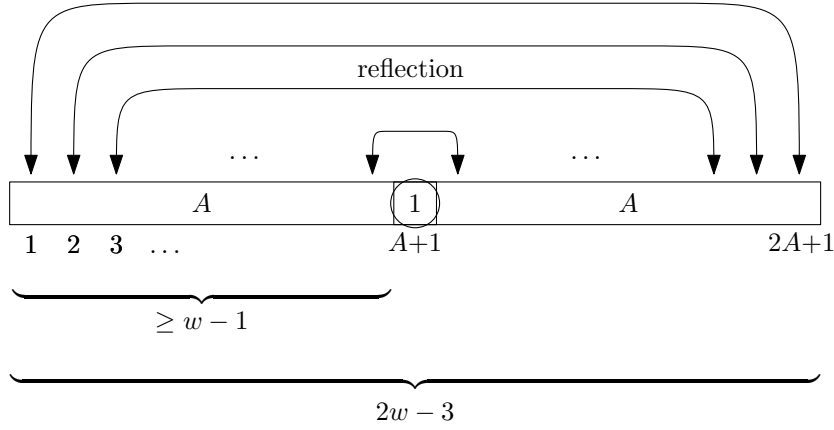
Figure 7: The combinatorial model for the expression T_1.

the number of elements selected out of the first $A = s - 3$ integers is at least $w - 1$. (More precisely, it is $w - 1 + k$.) T_1 counts the number of these subsets.

We consider the subsets S under the reflection operation $S \mapsto \bar{S}$ which maps each integer x to its "mirror image" $2A + 2 - x$. Obviously, for any pair $(S, \bar{S})$, at most one of the two sets can satisfy the additional constraint. If S contains at least $w - 1$ elements in the first A integers, then S contains at most $2w - 3 - (w - 1) = w - 2$ in the last A integers, so $\bar{S}$ contains at most $w - 2$ elements in the first A integers. If both S and $\bar{S}$ contain at most $w - 2$ elements in the first A integers, we have the *exceptional pairs* $(S, \bar{S})$, which are characterized as those pairs where S contains exactly $w - 2$ elements from the first A integers, exactly $w - 2$ elements from the last A integers, and also the central element $A + 1$.

Thus, to count the subsets that fulfill the additional constraint, we take all subsets of size $2w - 3$, subtract the exceptional pairs, and divide by 2.

$$T_1 = \frac{1}{2} \left[\binom{2s - 5}{2w - 3} - \binom{s - 3}{w - 2}^2 \right]$$

For the second sum

$$T_2 = \sum_{k \geq 0} \binom{s - 2}{w + k} \binom{s - 2}{w - 2 - k}, \tag{11}$$

an argument similar to T_1 is possible, but it is easier to argue algebraically: Substituting $k = 2 - k'$ and renaming k' to k again gives a summation of the same expression over a nearly complementary range of k.

$$T_2 = \sum_{k' \leq -2} \binom{s - 2}{w - 2 - k'} \binom{s - 2}{w + k'} = \sum_{k \leq -2} \binom{s - 2}{w + k} \binom{s - 2}{w - 2 - k} \tag{12}$$

Summing up (11) and (12) gives almost the full range of integers k. We add the term for $k = -1$ and get

$$\sum_{k \leq -2} + \sum_{k = -1} + \sum_{k \geq 0} = \sum_{k - \infty}^{\infty} \binom{s - 2}{w + k} \binom{s - 2}{w - 2 - k} = \binom{2s - 4}{2w - 2}.$$

On the other hand, from (11–12),

$$\sum_{k \leq -2} + \sum_{k = -1} + \sum_{k \geq 0} = T_2 + \binom{s - 2}{w - 1}^2 + T_2,$$

and thus

$$T_2 = \frac{1}{2}\left[\binom{2s-4}{2w-2} - \binom{s-2}{w-1}^2\right].$$

On the whole we get

$$P_{wh}^* = (s-2)\left[\binom{2s-5}{2w-3} - \binom{s-3}{w-2}^2\right] - (w-1)\left[\binom{2s-4}{2w-2} - \binom{s-2}{w-1}^2\right]$$

$$= -(s-2)\times\binom{s-3}{w-2}^2 + (w-1)\times\binom{s-2}{w-1}^2,$$

because the two leading terms cancel. Taking out common factors, we get

$$P_{wh}^* = \left(\frac{(s-3)!}{(w-1)!(h-1)!}\right)^2\left[-(s-2)\times(w-1)^2 + (w-1)\times(s-2)^2\right]$$

$$= \left(\frac{(s-3)!}{(w-1)!(h-1)!}\right)^2(s-2)(w-1)\left[-(w-1) + (s-2)\right]$$

$$= \left(\frac{(s-3)!}{(w-1)!(h-1)!}\right)^2(s-2)(w-1)(h-1) = (s-3)\binom{s-2}{w-1}\binom{s-4}{w-2}. \quad (13)$$

Substituting (13) and (9) into (8) proves Theorem 1. $\qquad\square$

4.4   Our proof and its relation to the proof of Guo and Zeng

Some of the ideas that we use are implicit in the proof of Guo and Zeng [GZ], of which we were not aware when we developed this work. Their derivation makes heavy use of Lemma 1, but the interpretation of the terms as crossing paths is soon lost in the algebraic manipulations. We will review their approach, using our notation and terminology.

The number of convex polyominoes are represented as a sum of five expressions:

$$P_{wh} = S_0 + S_1 + S_2 - S_3 - S_4$$

The first term, $S_0 = B_{wh}$, represents the number of parallelogram polyominoes, which contain both the SW and the NE corners (Proposition 1). S_1 represents the number of directed polyominoes and the "opposite directed polyominoes" that contain the NE corner, but exclude the parallelogram polyominoes. Thus, $S_1 = 2(D_{wh} - B_{wh})$, and it is possible to recover the formula for D_{wh} (Theorem 2) from the expressions for S_0 and S_1 in [GZ, Eqs. (4) and (5)]. Like in our proof, the calculations leading to S_1 involve a telescoping sum.

The number of the remaining polyominoes are represented by $S_2 - S_3 - S_4$. Here S_2 counts S-walks that visit the sides in the normal SWNE order, with the exclusion of the SW and NE corners. From this, one has to subtract the number S_3 of those walks where the falling pieces intersect, and the number S_4 of walks where the rising pieces intersect. In our notation, we can express this as $S_3 = P_{wh}^*/2$, and indeed, formula (13) is in accordance with the expression for S_3 in [GZ, Eq. (9)]. S_4 has to be computed separately because the special treatment of the SW and NE corner has destroyed the symmetry between ascending and descending intersections. As mentioned in the introduction, the computation of S_3 and S_4 involves the use of generating functions.

By contrast, in our proof, we have tried to reduce the amount of computation and to give bijective proofs whenever possible. Some ideas, like cutting at diagonals and reflecting along the boundary, were helpful in this respect. Another distinguishing feature

is that we include S-walks that touch the boundary in an arbitrary order. This has allowed us to keep the proof simple and elementary. We did not have to resort to generating functions, and we needed only moderate algebraic manipulations.

S-walks that touch the boundary in an arbitrary order also show up in the rejection-sampling procedure of Hochstättler, Loebl and Moll [HLM] for convex polyominoes of given perimeter, see Section 6.1, but their connection to the positive terms in (3) was not noted.

5 Moments of Intersection Numbers Between Two Paths

In an early attempt to establish Theorem 1 we tried a different approach to treat intersecting paths: In contrast to Lemma 1, which deals with intersections by swapping the endpoints, we split the walk into an initial part up to the last intersection point, plus a final part. The final part is a directed convex polyomino, for which we know the formula. This approach leads to a sum that can be interpreted as the total number of intersections of all pairs of monotone paths between two points. We failed to evaluate this sum directly. However, as shown in Theorem 1, we found a different way to calculate the result. Turning the argument around, we can therefore say something about the total number of intersections of all path pairs (the unnormalized first moment). We can even extend the method and compute the sum of squares of these intersection numbers (the unnormalized second moment).

Theorem 3. *Let $w, h \geq 0$, and set $s = w + h$. Let $\mathcal{P}_{wh}$ denote the set of the $\binom{s}{w}$ monotone grid paths from $(0,0)$ to (w, h).*

$$\sum_{U \in \mathcal{P}_{wh}} \sum_{V \in \mathcal{P}_{wh}} |U \cap V| = \binom{2s+2}{2w+1}/2 \qquad \text{[OEIS, A091044]} \tag{14}$$

$$\sum_{U \in \mathcal{P}_{wh}} \sum_{V \in \mathcal{P}_{wh}} |U \cap V|^2 = (s+1)\binom{s+2}{w+1}\binom{s}{w} - \binom{2s+2}{2w+1}/2 \qquad \text{[OEIS, A324010]} \tag{15}$$

Dividing these formulas by the number $\binom{s}{w}^2$ of path pairs yields the (normalized) first moment, i.e., the mean, and the second moment, from which the variance can be calculated. The intersection points of U and V include the common endpoints, and therefore, $|U \cap V|$ is always at least 2 (unless $w = h = 0$).

Proof. We will establish a bijection between the left side of (14) and *weak directed S-walks* in a $(w+1) \times (h+1)$ rectangle, see Figure 8. Such a walk starts and ends at $(0,0)$ but it need not contain an edge of the S- or W-side. The edges on the N- and E-side are, however, required. Using the code in Section 4.2, such a walk can be coded with $2w + 1$ V's and $2h + 1$ H's. We orient the walk by requiring that the N-side is touched before the E-side.

The number D_{wh}^- of these walks is

$$D_{wh}^- = \binom{2s+2}{2w+1}/2.$$

The factor $1/2$ accounts for the orientation.

In a weak directed S-walk, a self-intersection can only occur between the two rising pieces. Now we split the walk at the highest and rightmost intersection point (i, j). If there is no intersection point, we set $(i, j) = (0, 0)$. Now the part P_2 that was split off is
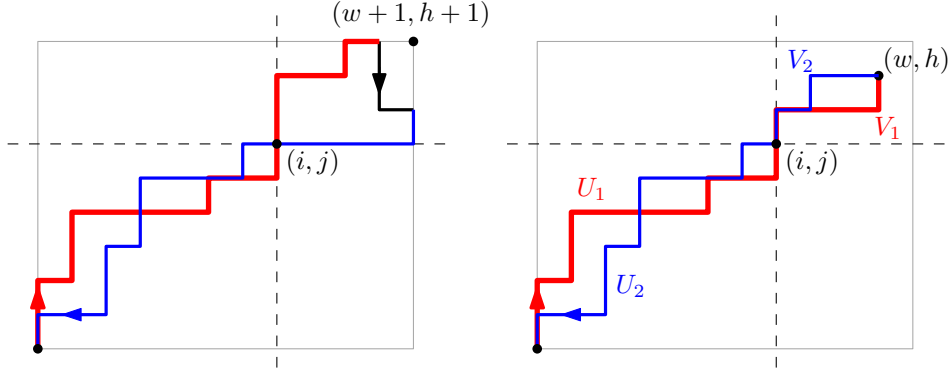
Figure 8: A weak directed S-walk is split at the highest intersection point and converted into a pair of paths.

a directed convex polyomino, spanning the rectangle $[i, w+1] \times [j, h+1]$, and containing the SW corner (i, j) of this rectangle. We have established through Theorem 2 that such directed convex polyominoes are in bijection with ordered pairs (V_1, V_2) of monotone paths from (i, j) to (w, h). After splitting off P_2 from the S-walk, an initial path U_1 from $(0, 0)$ to (i, j) and a final path U_2 from (i, j) to $(0, 0)$ remain. We connect U_1 with V_1 and the reverse of U_2 with V_2, obtaining a pair of paths from $(0, 0)$ to (w, h). We mark the intersection point (i, j).

Conversely, consider a pair of paths from $(0, 0)$ to (w, h), in which one of the intersection points (i, j) is marked. We split off the parts after (i, j) and turn them into a directed polyomino, using the bijection of Theorem 2. We obtain a directed S-walk.

Thus we have established a bijection between weak directed S-walks and pairs of paths $W_1, W_2 \in \mathcal{P}_{wh}$ in which an intersection in $W_1 \cap W_2$ is marked. The number of objects of the latter type is the number of path pairs, weighted by the number of intersections; this is the quantity that is counted in (14).

We note that both the first point $(0, 0)$ and the last point (w, h) are valid intersection points to be marked. A mark on the first point corresponds to directed S-walks without self-intersections. If the last point is marked, the bijection will expand it into a path around a unit square.

To get the second moment (15), we perform the replacement on both ends. We start with a pair of paths (U, V) in which *two* indistinguishable marks are placed on intersection points. The two marks can be placed on the same point. The number of these objects is

$$\sum_{U \in \mathcal{P}_{wh}} \sum_{V \in \mathcal{P}_{wh}} \binom{|U \cap V| + 1}{2}.$$

The pair of paths after the higher mark is replaced by a directed polyomino as above. The pair of paths up to the lower mark is replaced by a another directed polyomino, which is rotated by 180°: instead of the lower left corner, it contains the upper right corner of its enclosing rectangle. We obtain an S-walk of the normal type SWNE in a $(w + 2) \times (h + 2)$ rectangle, whose rising parts intersect. As argued in Section 4.3 this number is $P^*_{w+2,h+2}/2$, the other half being the flipped type SWEN. By (13), this gives

$$\sum_{U \in \mathcal{P}_{wh}} \sum_{V \in \mathcal{P}_{wh}} \binom{|U \cap V| + 1}{2} = (s+1)\binom{s+2}{w+1}\binom{s}{w}/2. \tag{16}$$

Since $\binom{k+1}{2} = (k^2 + k)/2$, with $k = |U \cap V|$, the claimed formula (15) for the second moment is obtained by multiplying (16) by 2 and subtracting (14). □

6 Rejection Sampling for Convex Polyominoes

Since the formula (1) contains a subtraction, it does not suggest a way to bijectively sample convex polyominoes with given height and width. However, it is straightforward to randomly generate one of the $\tilde{P}_{wh} = \binom{2s-4}{2w-2} + (w-1)\binom{2s-5}{2w-2}$ S-walks. Note that they correspond to the positive terms in formula (1). If the S-walk intersects itself, it is rejected, and another random S-walk must be tried. The *efficiency* of the method is the success probability of generating a valid polyomino.

For the square case $w = h$ the probability of self-intersection goes to 0 like $4\sqrt{2/\pi w}$, and hence the efficiency goes to 1. In general, the efficiency is $1 - O(1/\sqrt{\min\{w,h\}})$. For small values of h or w, the efficiency is around $1/2$. The worst observed efficiency is around 48%, for $h = w = 4$.

6.1 Sampling for given perimeter $2s$

For sampling from the set of polyominoes with a given perimeter $2s$, without regarding the specific width and height, we can give a more explicit sampling procedure with an efficiency that approaches 1 as s increases. A very similar procedure is proposed in Hochstättler, Loebl and Moll [HLM]: its efficiency approaches $1/2$ for large s. By adding a small twist to their procedure, we can avoid throwing away 50 % of the samples.

The procedure is the same as above: We generate all S-walks. The sum

$$\tilde{P}_s = \sum_{w+h=s} \tilde{P}_{wh} = 4^{s-2}/2 + 4^{s-3}/2 \cdot (2s-5)/2 = 4^{s-4}(2s+3)$$

has a term $4^{s-2}/2$ for the directed polyominoes ($a = 0$) and another term $4^{s-3}/2 \cdot (2s-5)/2$ for the remaining polyominoes ($a \geq 1$).

The S-walks for the directed polyominoes are obtained from strings of length $2s - 4$ over the alphabet $\{V, H\}$ with an even number of H's. Their number is $(1/2)2^{2s-4} = 4^{s-2}/2$.

Let us now concentrate on the case $a \geq 1$. We assume $s \geq 4$. We start by generating a random string of length $2s - 6$ over the alphabet $\{V, H\}$ with an odd number of H's (2^{2s-7} possibilities). We generate a random position where to insert an additional H ($2s - 5$ possibilities). The result is a random string of length $2s - 5$ with some even number $\#H \geq 2$ of H's and an odd number $\#V \geq 1$ of V's. We choose w and h such that $\#H = 2w - 2$ and $\#V = 2h - 3$.

Suppose that the additional, inserted H is the $\bar{a}$-th H from the left. This gives a random number $\bar{a}$ in the range $1 \leq \bar{a} \leq 2w - 2$. We reduce this number modulo $w - 1$ so that the resulting value a lies in the interval $1 \leq a \leq w - 1$. This gives all the data we need for an S-walk of height h and width w with $1 \leq a \leq w - 1$.

We have started with $2^{2s-7}(2s - 5)$ possibilities, and we have used them to generate one of $4^{s-3}/2 \cdot (2s - 5)/2 = 2^{2s-8} \cdot (2s - 5)$ random objects. Thus we have wasted one bit. This happened at the reduction from $\bar{a}$ to a modulo $w - 1$. If this loss is an issue, the lost bit can be recovered from the reduction (by comparing $\bar{a}$ against $w - 1$) and recycled for the next round of random generation. We leave it as a challenge to avoid the generation of the extra bit from the start.

The overall procedure, including the case $a = 0$, is as follows. We generate a random string x of length $2s - 7$ over the alphabet $\{V, H\}$ and a random number Q in the range $1 \leq Q \leq 2s + 3$. If $Q > 2s - 5$, we translate the eight possibilities into 3 bits and use them to extend x by 2 more symbols. One bit is wasted. We add a parity symbol so that we get a string of length $2s - 4$ with an even number of H's and V's. This is used as an S-walk with $a = 0$.

If $Q \leq 2s - 5$, we are in the case $a \geq 1$. We extend x by a parity symbol to make the number of H's odd, and we use Q as the insertion point for the additional H, as described above.

The efficiency of the method is easy to analyze. By comparing the negative term of formula (3) against the positive term, one can see the probability of self-intersection is goes to 0 like $4/\sqrt{\pi s}$ as $s \to \infty$. (If we don't recycle the wasted bit, we additionally loose 1 bit out of $2s - 7 + \log_2(2s + 3)$ bits.) Hence the efficiency approaches 1 as s increases.

The main difference in the procedure of Hochstättler, Loebl and Moll [HLM] is that they generate the code string and the parameter $a \in \{0, 1, 2, \ldots, s - 1\}$ independently. If the number of H's is bigger than $2a$, they certainly have to reject the sample. This causes them to loose $50\,\%$ of the samples right away. Our improvement is essentially due to a combinatorial interpretation of the equality in (9), followed by a summation over all pairs w, h with $w + h = s$.

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