Convex drawings of Planar Graphs and the Order Dimension of 3-Polytopes

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Abstract. We define an analogue of Schnyder's tree decompositions for 3-connected planar graphs. Based on this structure we obtain:

- Let G be a 3-connected planar graph with f faces, then G has a convex drawing with its vertices embedded on the $(f-1) \times (f-1)$ grid.
- Let G be a 3-connected planar graph. The dimension of the incidence order of vertices, edges and bounded faces of G is at most 3.

The second result is originally due to Brightwell and Trotter. Here we give a substantially simpler proof.

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1 Introduction

Schnyder discovered that every plane triangulation admits a special decomposition of its interior edges into three trees. Based on these Schnyder 3-tree decompositions he proved two beautiful theorems about planar graphs. In [9] Schnyder characterized planar graphs in terms of the order dimension of their incidence order:

- The dimension of the incidence order of vertices and edges of a graph G is at most 3 $\iff$ G is planar.

This striking result recently found applications even in algebra [6]. Schnyder's Theorem has found several extensions: Brightwell and Trotter proved in [2] that the incidence order of vertices, edges and faces of a planar map has dimension at most 4. The proof of this result is inductive and required them to first establish the following theorem.

Theorem 1. *Let G be a 3-connected planar graph. The dimension of the incidence order of vertices, edges and bounded faces of G is at most 3.*

Felsner and Trotter [4] used Schnyder's Theorem to give a characterization of outerplanar graphs in terms of order dimension. That paper should also give a good overview of other results on the dimension of graphs.

The second application Schnyder had for his tree decompositions concerns straight line drawings of planar graphs. The existence of straight line embeddings for planar graphs was independently proven by Wagner [16], Fáry [3] and Stein [11]. The question whether every planar graph has a straight line embedding of reasonable resolution, i.e., on a grid of polynomial size, was raised by Rosenstiehl and Tarjan [8]. Schnyder [9] shows how to construct a barycentric representation which yields an embedding on the $(2n-5) \times (2n-5)$ grid. In [10] Schnyder improved on his first result and shows

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- Let G be a planar graph with n vertices, then G has a straight line drawing with its vertices embedded on the $(n-2) \times (n-2)$ grid.

The proof is constructive and the embedding be computed in $O(n)$ time (see e.g. [1]). The method of Schnyder's still gives the strongest result regarding the size of a grid embedding arbitrary planar graphs. Xin He [5] gives a comprehensive history of the problem.

Tutte [14, 15] shows that every 3-connected planar graph G admits a strictly convex drawing, i.e. a drawing such that the boundary of every face is a strictly convex polygon. Actually this result can be obtained as an easy consequence of much older theorems of Steinitz and of the Koebe circle packing theorem, a reviews of these connections is given by Ziegler [17]. Again the early approaches to convex drawings give no reasonable guarantee on resolution.

We define *triorientations* of the edges of 3-connected planar graphs so that they resemble the 3-tree decompositions of Schnyder. Based on this structure we prove:

Theorem 2. *Let G be a 3-connected planar graph with f faces, then G has a convex drawing with its vertices embedded on the $(f-1) \times (f-1)$ grid.*

Straight line grid drawings with area $O(n^2)$ are not strictly convex, in fact, every strictly convex embedding of the n-cycle requires an area of $\Omega(n^3)$. The smallest grid size known to admit convex drawings of 3-connected planar graphs is $(n-2) \times (n-2)$. This result has been announced by two groups of authors Schnyder and Trotter as well as Chrobak and Kant, however, it has so far not appeared in print.

Based on the drawing result we give a new proof of the result of Brightwell and Trotter (Theorem 1). This proof is much shorter than the original one. Theorem 1 immediately implies that face lattices of 3-polytopes are critical of order dimension 4.

2 Convex Embeddings of 3-Connected Plane Graphs

2.1 Schnyder labeling the angles of a plane graph

A *plane graph* is a planar graph with a particular planar embedding. Let $G = (V, E)$ be a 3-connected plane graph, specify three vertices a_1, a_2, a_3 in clockwise order on the boundary cycle of the outer face. A *Schnyder labeling* with respect to a_1, a_2, a_3 is a labeling of the angles of G with the labels $1, 2, 3$ satisfying three rules. By convention that there is a cyclic structure on the labels so that $i+1$ and $i-1$ is always defined.

- The outer angle at the special vertex a_i has labels $i+1$ and $i-1$ in clockwise order. All the other angles of the graph have exactly one label.
- *Rule of vertices:* The labels of the angles at each vertex form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's.
- *Rule of faces:* The labels of the angles at each interior face form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's; at the outer face the same is true in counterclockwise order.

Figure 2 shows a plane graph with a Schnyder labeling. We give the proof that every 3-connected plane graph admits a Schnyder labeling in Section 3 and continue with a discussion of some consequences.

Lemma 1. *Let G be a plane graph with a Schnyder labeling, then the four angles of each edge contain all three labels 1,2,3. Thus every edge has one of the two forms shown in Figure 3.*

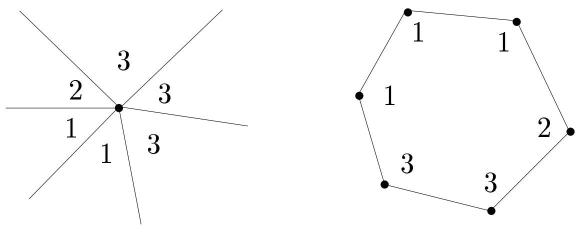

Figure 1: Rule of vertices and rule of faces

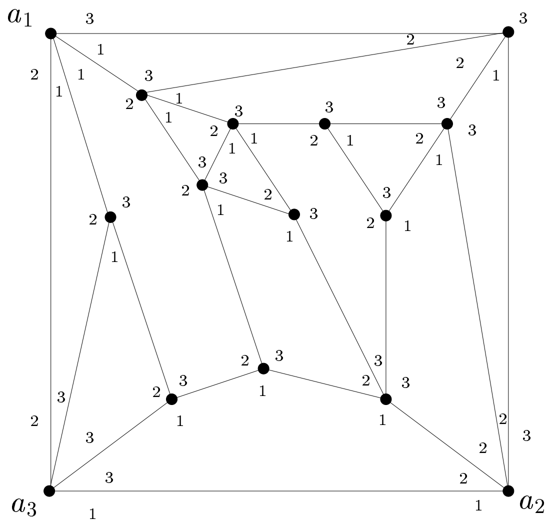

Figure 2: A plane graph with a Schnyder labeling

Proof. To make the statement true for all edges we distribute the two labels at the outer angles of special vertices to the two edges of this angle.

The proof is based on double counting and Euler's formula. Define the degree $d(v)$ of a vertex v as the number of edges incident with v whose angles at v have distinct labels. By the rule of vertices $d(v) = 3$ for every non-special vertex v and $d(v) = 2$ for each of the three special vertices. Similarly, the degree $d(F)$ of a face F as the number of boundary edges of F whose angles in F have distinct labels. By the rules of faces $d(F) = 3$ for every interior face, while the degree of the outer face is zero. Therefore,

$$S = \sum_v d(v) + \sum_F d(F) = 3n - 3 + 3(f - 1) = 3|E|.$$

Now consider the four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of an edge in counterclockwise order as shown in Figure 4.

Define $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ so that $\alpha_2 = \alpha_1 + \epsilon_1$, $\alpha_3 = \alpha_2 + \epsilon_2$, $\alpha_4 = \alpha_3 + \epsilon_3$ and $\alpha_1 = \alpha_4 + \epsilon_4$. Form the rules of vertices and faces $\epsilon_j \in \{0, 1\}$, for all j. The cyclic nature of the linear system implies that $\sum_{j=1}^{4} \epsilon_j = 0 \mod 3$, hence, either $\sum_j \epsilon_j = 0$ or $\sum_j \epsilon_j = 3$. The contribution of an edge e to the degree sum S is $\sum_j \epsilon_j(e)$. Since $S = 3|E|$ we conclude $\sum_j \epsilon_j(e) = 3$ for every edge e. Up to rotational symmetry this only leaves the two cases shown in Figure 3. □

Note that from the labels of the exterior angles at special vertices and the rule for the outer face we know all the edge labels at outer angles. Every outer angle on the clockwise outer path from a_i to a_{i+1} has label $i - 1$. With the lemma we know all four labels of the two outer edges at a_i and conclude:

Figure 3:

Figure 4:

Corollary 1. *In a Schnyder labeling all interior angles at the special vertex a_i are labeled i.*

2.2 Three oriented trees

A *triorientation* of a set of edges is an assignment of one ore two opposite directions to each of these edges, such that each edge is labeled with one of the labels 1,2,3 and two opposite directions have distinct labels.

A Schnyder labeling of the angles of a plane graph induces a triorientation of the edges. If an edge has different angular labels i and j at one of its ends, we orient the edge from this end towards the other and give this orientation the third label k, see Figure 5.

Figure 5: Orienting edges.

Henceforth, we write *labeled graph* to denote a 3-connected plane graph with a Schnyder labeling and the induced labeling and orientation of the edges. With T_i we denote the digraph induced by the edges having a direction labeled i and oriented in this direction. The three digraphs T_1, T_2, T_3 have notable properties following from the rule of vertices and Corollary 1.

(1) Each non-special vertex v has outdegree one in each T_i and the edges e_1, e_2, e_3 leaving v in labels 1,2,3 occur in clockwise order. Each edge entering v in label i enters v in the clockwise sector from e_{i+1} to e_{i-1}. See Figure 7

(2) Every edge incident with the special vertex a_i enters in label i and the two extremal edges at a_i are bidirectional, one leaves a_i in label $i-1$ the other in label $i+1$.

With the next lemma we show that each of the digraphs T_1, T_2, T_3 is acyclic, actually we show a little more.

Lemma 2. *No cycle of a labeled plane graph can be traversed by following each of its edges in a direction labeled i or in the reversal of a direction labeled $i-1$ or $i+1$, i.e., no cycle of G is directed in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$.*

Proof. Suppose there is a cycle in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ and choose such a cycle Z enclosing the minimum number of faces. We first show that the interior region F of Z is a face.

Suppose F contains a vertex x. Starting at x and following edges in T_i we construct a path P_i connecting x to Z. By minimality of Z path P_i has no repeated vertex. Similarly, there is a path P_{i-1} directed in label $i-1$ from x to Z. By the minimality of Z the paths P_i and P_{i-1} have no common vertex other than x. Together with one of the

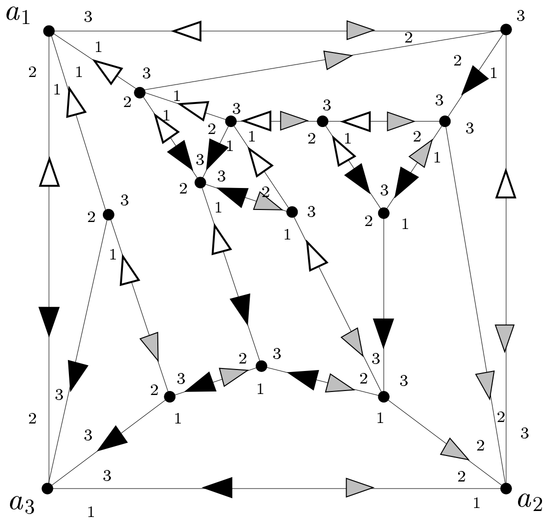

Figure 6: Edge orientations for the graph of Figure 2

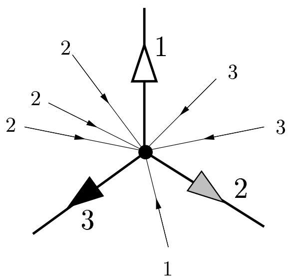

Figure 7: Edge orientations at a vertex.

segments they determine on Z these two paths form a directed cycle in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ which encloses a smaller region than Z. This contradiction shows that Z contains no vertex. An edge lying in F and joining two nonconsecutive vertices of Z would similarly determine a cycle enclosing a smaller region than Z.

Therefore, F is a face and Z its boundary cycle. If the traversal of Z is clockwise no angle of F has label $i+1$ and if this traversal is counterclockwise no angle has label $i-1$. Both cases are excluded by the rule of faces. $\square$

The digraph T_i includes all vertices and all vertices except a_i have outdegree one in T_i, while a_i has outdegree 0. Since T_i is acyclic we obtain:

Corollary 2. *T_i is a directed tree rooted at a_i, for $i = 1, 2, 3$.*

Let G be a labeled graph, for a vertex v we define the i-path $P_i(v)$ as the path in T_i from v to the root a_i of T_i. Lemma 2 implies that for $i \neq j$ the paths $P_i(v)$ and $P_j(v)$ have v as the only common vertex. Therefore, $P_1(v), P_2(v), P_3(v)$ divide G into three regions $R_1(v)$, $R_2(v)$ and $R_3(v)$, where $R_i(v)$ denotes the region bounded by and including the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$, see Fig. 8. The open interior of region $R_i(v)$, denoted $R_i^o(v)$, is $R_i(v) \setminus (P_{i-1}(v) \cup P_{i+1}(v))$.

Lemma 3. *If u and v are vertices of a labeled graph with $u \in R_i(v)$, then $R_i(u) \subseteq R_i(v)$. If $u \in R_i^o(v)$, then the inclusion is proper: $R_i(u) \subset R_i(v)$.*

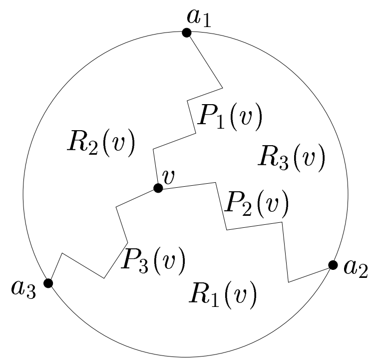

Figure 8: The three regions of a vertex

Proof. By symmetry it suffices to consider the case $i = 1$. Suppose $u \in R_1^o(v)$ and let x be the first vertex of $P_2(u)$ that belongs to $P_2(v) \cup P_3(v)$. From the edge orientations at x (Figure 7) it follows that $x \notin P_3(v)$. By the same reason $x \neq v$, hence $x \in P_2(v)$. Similarly the first vertex y of $P_3(u)$ that belongs to $P_2(v) \cup P_3(v)$ is on $P_3(v)$ and $y \neq v$. Hence, $R_1(u) \subseteq R_1(v)$, see Figure 9, the inclusion is proper as $v \notin R_1(u)$.

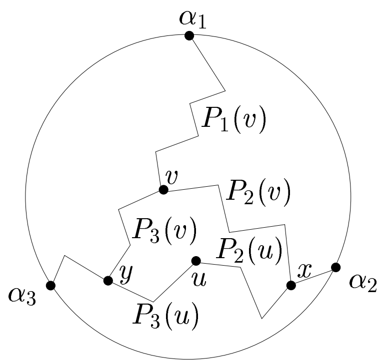

Figure 9: If $u \in R_1^o(v)$ then $R_1(u)$ is a proper subset of $R_1(v)$.

Now let $u \in R_1(v) \setminus R_1^o(v)$, by symmetry we only consider the case $u \in P_3(v)$. If at u the outgoing edge in label 2 is different from the incoming edge on $P_3(v)$ then a reasoning as in the previous case shows that the inclusion is proper, $R_1(u) \subset R_1(v)$. Otherwise, if u' is the other endvertex of the bidirected edge leaving u in label 2 and entering in label 3, then $R_1(u') = R_1(u)$. However, $R_1(u') \subseteq R_1(v)$ by induction on the number of vertices between u and v on $P_3(v)$. $\square$

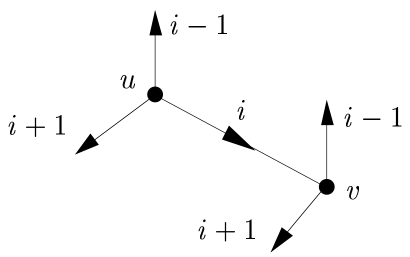

Figure 10:

Let vertices u, v be neighbors such that the edge $e = (u, v)$ is directed from u to v in label i, see Figure 10. Since $v \in P_i(u)$ vertex v is contained in $R_{i-1}(u)$ and $R_{i+1}(u)$. The orientations of edges at v imply $u \in R_i(v)$. Therefore the following inclusions of regions hold:

- If $e = (u, v)$ is an unidirectional edge in label i then $R_i(u) \subset R_i(v)$ and $R_{i-1}(u) \supset R_{i-1}(v)$ and $R_{i+1}(u) \supset R_{i+1}(v)$.
- If $e = (u, v)$ is bidirectional with label i from u to v and label $i - 1$ from v to u, then $R_{i+1}(u) = R_{i+1}(v)$ and $R_i(u) \subset R_i(v)$ and $R_{i-1}(u) \supset R_{i-1}(v)$.

2.3 Coordinates and embeddings

Let G be a labeled graph a *coordinate-mapping* associates a triple (v_1, v_2, v_3) of real numbers with every vertex v such that:

(1) $v_1 + v_2 + v_3 = 1$ for all vertices v of G.

(2) If $R_i(u) \subset R_i(v)$ then $u_i < v_i$ and if $R_i(u) = R_i(v)$ then $u_i = v_i$.

From the previous discussion we already know some further properties of coordinate mappings

(3) If $u \in R_i(v)$ then $u_i \leq v_i$ and if $u \in R_i^o(v)$ then $u_i < v_i$.

(4) If an edge of G is directed from u to v in label i then $u_i < v_i$, $u_{i+1} \geq v_{i+1}$ and $u_{i-1} \geq v_{i-1}$.

(5) For every edge (u, v) of a labeled graph then there are indices i, j such that $u_i < v_i$ and $u_j > v_j$.

Given three non-collinear points α_1, α_2 and α_3 in the plane. These points and a coordinate-mapping of G can be used to define an embedding of G in the plane. A vertex of G is mapped to the point

$$\mu : v \to v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3,$$

an edge (u, v) is represented by the line segment $(\mu(u), \mu(v))$. Note that any two drawings based on the same coordinate mapping but on different points α_1, α_2 and α_3 and β_1, β_2 and β_3 can be mapped onto each other by an affine map, therefore, we ignore the base points and denote the resulting drawing with $\mu(G)$.

The main result of this section is that coordinate mappings yield convex drawings of plane graphs.

Theorem 3. *If $v \to (v_1, v_2, v_3)$ is a coordinate-mapping of a labeled graph G, then the drawing $\mu(G)$ is a convex drawing of G.*

Let v be a vertex with coordinates (v_1, v_2, v_3). The three lines $x_1 = v_1$, $x_2 = v_2$ and $x_3 = v_3$ cross in $\mu(v)$ and partition the triangle with vertices α_1, α_2 and α_3 into six regions, see Figure 11. Each of the three closed shaded parallelograms contains exactly one neighbor of v. This is because if (u, v) is directed towards v then by property (4) vertex u is contained in one of the three white triangles. Iterating this observation we obtain:

Lemma 4. *In the embedding $\mu(G)$ every vertex v of G has a path from v to each exterior vertex α_i which is completely contained in the shaded parallelogram with corners v and α_i.*

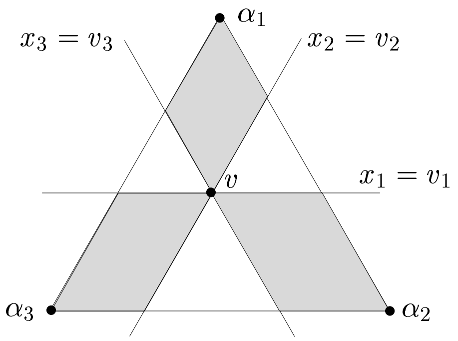

Figure 11: Each of the three shaded parallelograms contains exactly one neighbor of v, these are the outgoing edges at v of the three Schnyder trees.

A first step towards a proof of the theorem is to show that $\mu(G)$ is a plane embedding of G, i.e., if two segments representing edges intersect, then they have a common endpoint. The next lemma deals with a special case of intersection.

Lemma 5. *In the drawing $\mu(G)$ of a labeled graph G there is no vertex w which is placed on an edge (u, v).*

Proof. Assume that $\mu(w)$ is contained in the segment $(\mu(u), \mu(v))$. The edge (u, v) is contained in one of the regions of w, say in $R_1(w)$. Therefore, $u_1 \leq w_1$ and $v_1 \leq w_1$ and due to the assumption $w_1 = u_1 = v_1$. Since $u, v \in R_1(w)$ this is only possible if $R_1(u) = R_1(v) = R_1(w)$, by Lemma 3.

If u and v are both on $P_2(w)$ then $u_2 > w_2$ and $v_2 > w_2$, by property (3). This contradicts the assumption $\mu(w) \in (\mu(u), \mu(v))$. Similarly not both of u and v are on $P_3(w)$.

For the remaining case suppose $u \in P_3(w)$ and $v \in P_2(w)$. Since $R_1(u) = R_1(v) = R_1(w)$ we find u, w, v in this order on $P_2(u)$. Label 2 and orientation $u \to v$ for edge (u, v) would lead to two outgoing edges in label 2 at u. Together with $P_2(u)$ every other label and orientation for (u, v) would generate a cycle in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ for $i = 1$ or 3 in contradiction to Lemma 2. □

Corollary 3. *The mapping μ is injective from V to the plane.*

Lemma 6. *In the drawing $\mu(G)$ of a labeled graph G there is no pair of crossing edges.*

Proof. Consider two edges (u, v) and (x, y) sharing no vertex. Let i, j, k, l be such that

$$(u, v) \in R_i(x), \quad (u, v) \in R_j(y), \quad (x, y) \in R_k(u), \quad (x, y) \in R_l(v).$$

If $i = j$ or $k = l$ then the two edges are separated by a coordinate in $\mu(G)$, hence, disjoint.

By symmetry we may assume that $i = k = 1$. From $x \in R_1(u)$ and $u \in R_1(x)$ we infer with Lemma 3 that $R_1(u) = R_1(x)$. Let P^* be the bidirectional path in labels 2 and 3 between u and x, w.l.o.g. P^* is directed from u to x in label 2, see Figure 12. In terms of coordinates we have

$$x_1 = u_1, \quad u_2 > x_2, \quad \text{and} \quad u_3 < x_3.$$

Lemma 3 implies $y_1 \leq u_1$ and $v_1 \leq x_1$, if one of the two inequalities is an equality we are done by Lemma 5. Therefore let $y_1 < u_1$ and $v_1 < x_1$.

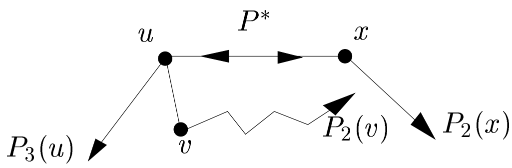

Figure 12:

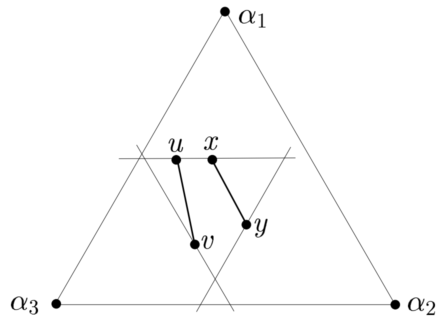

Figure 13:

We claim that with these assumptions v is the vertex with the maximal third coordinate and y is the vertex with the maximal second coordinate. This implies that the position of the two edges in $\mu(G)$ is as shown Figure 13. Clearly the edges do not intersect.

To prove the claim we first show that $x \in R_3^o(v)$. Recall that $v \in R_1(x) = R_1(u)$ and (u, v) is an edge. If v is the neighbor of u on $P_3(u)$ then the first edge of $P_2(v)$ is in the interior of region $R_1(x)$ (otherwise we would have the contradiction $R_1(v) = R_1(x) = R_1(u)$). If $v \in R_1^o(u)$ then edge (v, u) is unidirectional in label 1. In either case let s be the first vertex of $P_2(u) \cap P_2(v)$. If $s \in P^*$ then there is a cycle in $T_1 \cup T_2^{-1} \cup T_3^{-1}$, hence $s \in P_2(x)$ and $s \neq x$. This proves $x \in R_3^o(v)$ and since (x, y) is an edge also $y \in R_3(v)$. Expressed in terms of coordinates this yields $v_3 > x_3$ and $v_3 \geq y_3$, i.e., the claim. A symmetric argument shows that $y_2 \geq v_2$ and $y_2 > x_2$. □

To complete the proof of Theorem 3 it remains to shown that the embedding of every face in the drawing $\mu(G)$ is convex.

Lemma 7. *In the drawing $\mu(G)$ of a labeled graph G every interiour face is a convex polygon.*

Proof. Let F be an interiour face and let m_1, m_2 and m_3 denote the maximum first, second and third coordinate of the boundary vertices of F. The three lines with equations $x_1 = m_1$, $x_2 = m_2$ and $x_3 = m_3$ define a triangle ∇. The claim is that every vertex of F is placed on an edge of ∇, i.e. every vertex is maximal in one of the three coordinates, see Figure 14. From this claim it is immediate that the embedding of F is convex.

Let v be a boundary vertex of F with coordinates (v_1, v_2, v_3). If $F \subseteq R_i(v)$ then $u_i \leq v_i$ for every vertex $u \in F$. So F and also ∇ is completely contained in the halfspace $x_i \leq v_i$. By definition ∇ intersects the line $x_i = v_i$ hence this line supports an edge of ∇. □

This completes the proof of Theorem 3.

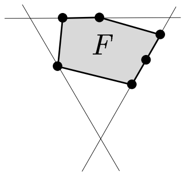

Figure 14:

2.4 Counting faces

Let G be a labeled graph with f faces. With vertex v associate the triple (v_1, v_2, v_3):

$$v_i = \frac{\text{number of faces of } G \text{ in the region } R_i(v)}{f-1}$$

Since every bounded face of G belongs to exactly one of the regions of v the three coordinates of v sum up to 1. The second defining property of coordinate-mappings is just as obvious. Hence we can apply Theorem 3 to obtain convex drawings of G. With the choice of the special vertices $\alpha_1 = (0, f-1)$, $\alpha_2 = (f-1, 0)$ and $\alpha_3 = (0,0)$ every vertex v of G is mapped by μ to an integral point in the $(f-1) \times (f-1)$ grid. Hence, Theorem 2 is obtained as a special instance of Theorem 3. Actually, the proof is not complete yet, it remains to prove the existence of Schnyder labelings. This will be fetched in the next section.

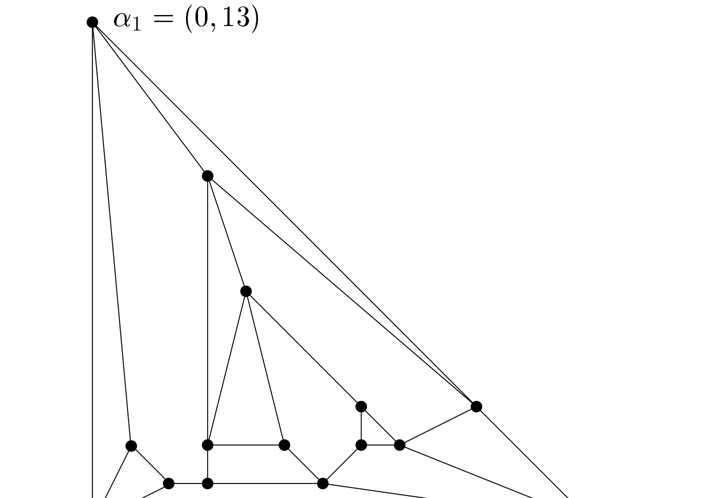

Figure 15: A convex grid drawing obtained by face-counting. Graph and labeling are taken from Figure 6

3 Existence of Schnyder Labelings

The first idea for the construction of a Schnyder labeling of a 3-connected planar graph is as follows:

(1) Choose an edge e of G and let G/e be the graph obtained by contraction of e.

(2) Recursively construct a Schnyder labeling of G/e.

(3) Expand the labeling of G/e to a Schnyder labeling of G.

The first detail that deserves some cautiousness is the choice of edge e. For the induction it is required that G/e is again 3-connected. We call an edge e of a 3-connected graph G such that G/e is again 3-connected a *contractible edge*. The existence of a contractible edge is warranted by a lemma of Thomassen. If we let e be an arbitrary contractible edge, however, the proof that the expansion of the labeling can be carried out may involve excessive case distinctions. To reduce the case analysis it would be desirable to have a contractible edge of a special form. But the existence of such an edge will likewise not come for free. Here we take a different approach.

Let G be a 3-connected planar graph with three special vertices a_1, a_2, a_3 in clockwise order on the boundary cycle C of the outer face. Let $x \notin C$ be a neighbor of a_1.

Suppose that $e = (a_1, x)$ is contractible and let $a_1, x_1, x_2, \ldots x_k$ be the neighbors of x in clockwise order. Note that only x_1 and x_k may be common neighbors of a_1 and x. Figure 16 shows a generic contraction of an edge (a_1, x) into a_1.

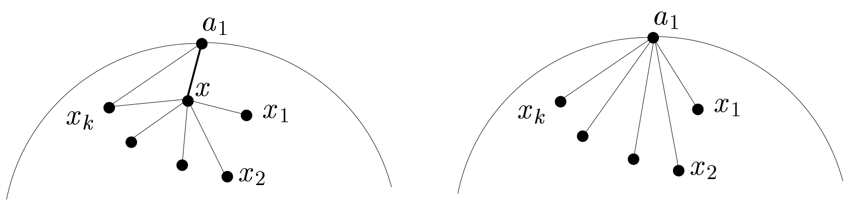

Figure 16: Contraction of edge (a_1, x) into a_1.

By the rule for the labels at special vertices all inner angles at a_1 are labeled 1. The angles of edge (a_1, x_i) at x_i have to to be labeled 2 and 3 as shown in the left part of Figure 17. The right part of Figure 17 shows that the labeling of G/e can be expanded to a Schnyder labeling of G. Note that the expansion leaves the labels in all faces that do not have (a_1, x) as boundary edge unchanged.

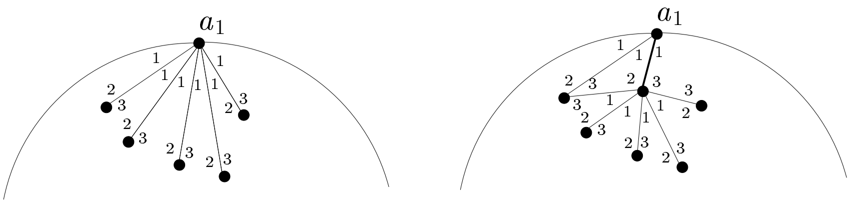

Figure 17: Expansion of edge and labels.

Next suppose that $e = (a_1, x)$ is not contractible, i.e., G/e is only 2-connected. Clearly, every cutset of size two in G/e has to contain a_1, let y be the second vertex of such a cutset. The set $S = \{a_1, x, y\}$ is a cutset of in G, denote the components of $G \setminus S$ by H and K. Let H' be $G \setminus K$ and $K' = G \setminus H$. The idea is to take Schnyder labelings of the two smaller graphs H' and K' and to show that they can be paste together.

The first problem is that H' and K' need not be 3-connected, resolve this by augmenting both graphs with the edges (a_1, y) and (x, y), provided these edges are not existent, this yields H'' and K''.

We have to consider two cases. First suppose that one of the graphs, say H'', contains all special vertices a_1, a_2 and a_3. A Schnyder labeling of H'' contains all three labels in the triangle $T = (a_1, x, y)$. The label at a_1 is 1, let the second special vertex for K'' be the vertex with label 2 in T and the third special vertex be the vertex with label 3 in T. Construct a Schnyder labeling for K'' with this assignment of special vertices. If all three edges of T have been present in G, then the pasting of the labelings makes no problem: the rules of vertices and faces can be verified in the labelings of H'' and K''.

It remains to consider a face that was cut by a new edge. We treat this case with the edge (x, y) with the assumption that the angle of x in T has label 2 in the labeling of H''. Let F be the face of G containing x and y and let F_H and F_K be the parts of this face after insertion of the edge (x, y) such that F_H belongs to H'' and F_K to K''. Figure 18 shows the situation.

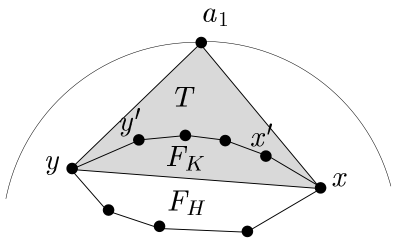

Figure 18: Separating triple, first case.

Since in labeling of H'' the angles of x and y at T are 2 and 3 the label of x at F_H is 1 or 2 and the label of y at F_H is 1 or 3. The claim is that we can use the same labels in G. Now consider the labeling of K''. Both labels of x at the edge (x, x') are 2 and both labels of y at the edge (y, y') are 3 by the rule for special vertices. Therefore, the labels of x' and y' at F_K are both 1, see Figure 3, and all vertices between x' and y' in K'' also have label 1 at F_K by the rule for the face. This proves that using the labels of H'' leads to a consistent labeling.

It remains to consider the case where the two special vertices a_2 and a_3 are separated by $\{a_1, x, y\}$. Assume $a_3 \in H''$ and $a_2 \in K''$ vertex y has to play the role of the missing special vertex in both graphs, i.e., the role of a_2 in H'' and the role of a_3 in K''. Figure 19 shows some of the labels in the Schnyder labelings of H'' and K'', we have to prove that they can be pasted together to yield a Schnyder labeling of G.

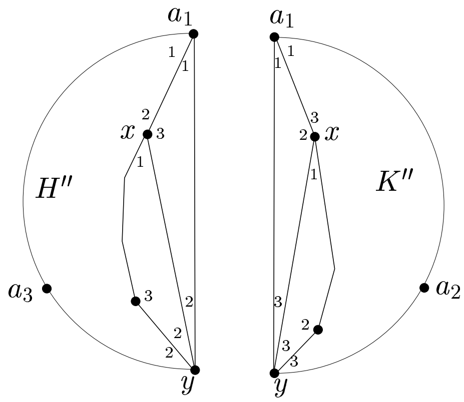

Figure 19: Separating triple, second case.

The edge a_1, y was not present in G, so we remove it from both graphs and identify the two copies of a_1, x and y. Since the edges (a_1, x) and (x, y) from the two graphs are also identified the labels in the triangles formed by a_1, x, y in H'' and K'' vanish.

However, if we assign label 1 to the outer angle at y the rule of vertices is satisfied at x and y. If the edge (x, y) is in G then the rules of all the other vertices and faces can be verified in the labelings of H'' and K''. If (x, y) has to be removed assign label 1 to the angle at x and one of the labels 2 or 3 to the angle at y. Again all the conditions for a Schnyder labeling are easily verified.

4 Order Dimension of 3-Polytopes

Let $G = (V, E)$ be a finite simple graph. A nonempty family $\mathcal{R}$ of linear orders on the vertex set V of graph G is called a *realizer* of G provided

($*$) For every edge $e \in E$ and every vertex $x \in V \setminus e$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in e$.

The *dimension* of G, denoted $\dim(G)$, is then defined as the least positive integer t for which G has a realizer of cardinality t. An intuitive formulation for condition ($*$) is as follows: For every vertex v and edge e with $v \notin e$ the vertex has to get over the edge in at least one of the orders of a realizer.

The definition we gave is not the traditional definition for the dimension of graphs. In most older paper the dimension of graphs is understood to be the dimension of the incidence order. We briefly explain the connections: With a finite graph $G = (V, E)$, associate a height two order P_G whose ground set is $V \cup E$. The order relation is defined by setting $x < e$ in P_G if $x \in V$, $e \in E$ and $x \in e$. P_G is the *incidence order* of G.

When $P = (X, <)$ is an order, and $\mathcal{R} = \{L_1, L_2, \dots, L_t\}$ is a family of linear orders on X, we call $\mathcal{R}$ a realizer of P if $P = \cap\mathcal{R}$, i.e., $x < y$ in P if and only if $x < y$ in L_i for all $i = 1, 2, \dots, t$. The *dimension* of an order is then defined as the minimum cardinality of a realizer.

G is a graph with minimum degree at least 2, then the dimension of G and the dimension of its incidence order agree. The order theoretic facts that sit in the background of the phenomenon are:

- All critical pairs of incidence orders of graphs with minimum degree at lest 2 are min-max pairs.
- If all critical pairs of an order are min-max pairs then its interval dimension equals its order dimension.
- The dimension of a graph is just the interval dimension of its incidence order.

For additional information on this background we suggest looking at Trotter's monograph [12].

If G is a graph containing a cycle then $\dim(G) \geq 3$. It is also easy to construct a realizer consisting of 3 linear orders for the cycle C_n, $n \geq 3$. The dimension of the complete graph K_5 is 4, but the removal of any edge reduces the dimension to 3. Similarly, the dimension of the complete bipartite graph $K_{3,3}$ is 4 and again the removal of any edge reduces the dimension to 3. These examples motivate the now classic, theorem of Schnyder.

Theorem 4. *A graph G is planar if and only if its dimension is at most* 3.

Proof. The easier part of the theorem is to show that $\dim(G) \leq 3$ implies that G is planar. The proof of this implication is actually due to Babai and Duffus, the argument can be found in [12] and [13].

We show that every planar graph G admits a realizer $\{L_1, L_2, L_3\}$. By monotonicity we may assume that G is a maximal planar graph, i.e., a triangulation. The Schnyder labeling of G induces three trees. Since each of the trees has $n-1$ edges and the graph has $3n-6$ edges the only bidirected edges are the three edges of the exterior triangle. Therefore, $R_i(u) \subset R_i(v)$ whenever $u \in R_i(v)$. For $i = 1, 2, 3$ let the inclusion order on the i–regions induce an order Q_i on the vertices, i.e., $u < v$ in Q_i iff $R_i(u) \subset R_i(v)$. For any edge (u, v) and vertex $w \neq u, v$ the edge is in one of the regions $R_i(w)$ of w, hence, $u < w$ and $v < w$ in Q_i. This shows that any choice of three linear extensions L_i of Q_i, $i = 1, 2, 3$, will produce a realizer for G. □

In complete analogy to the definition of the dimension of a graph the dimension of a hypergraph can be defined. A particularly interesting instance is related to polytopes and hence, by Steinitz's theorem also to planar graphs.

Let P be a polytope with vertex set $\mathcal{V}(P)$ and facets $\mathcal{F}(P)$. Given a subset $\mathcal{G}$ of $\mathcal{F}(P)$ a *realizer* for $(P, \mathcal{G})$ is a nonempty family $\mathcal{R}$ of linear orders on $\mathcal{V}(P)$ provided

(**) For every facet $F \in \mathcal{G}$ and every vertex $x \in \mathcal{V}(P) \setminus \mathcal{V}(F)$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in \mathcal{V}(F)$.

The *dimension* of $(P, \mathcal{G})$, denoted $\dim(P, \mathcal{G})$, is then defined as the least positive integer t for which $(P, \mathcal{G})$ has a realizer of cardinality t. In the case $\mathcal{G} = \mathcal{F}(P)$ we simply write $\dim(P)$ and call this the *dimension of the polytope* P. Traditionally people would be interested in the order dimension of the face lattice $\mathcal{L}(P)$ of a polytope P. However, it is not hard to see that all critical pairs of $\mathcal{L}(P)$ are min-max pairs, so that by the above remarks the two concepts of dimension coincide.

The next theorem is a lower bound which was proved by Reuter [7] in the context of Ferrer's dimension.

Theorem 5. *If P is a d-polytope with $d \geq 2$, i.e., a polytope whose affine hull is d dimensional, then* $\dim(P) \geq d + 1$.

Proof. The proof is by induction on d. If $d = 2$ then the vertices and facets of P have the structure of the cycle C_n with $n = |\mathcal{V}(P)|$. In that case $\dim(P) \geq 3$.

Let P be a d-polytope embedded in $\mathbb{R}^d$ for some $d > 2$ with realizer $L_1, L_2, \ldots, L_t$. Let v be the highest vertex in L_t and consider a hyperplane H which separates v from all the other vertices of P. The intersection $P \cap H$ is a $(d-1)$-polytope P/v, the so called *vertex figure* of P at v. The $(k-1)$-dimensional faces of P/v are in bijection with the k-dimensional faces of P that contain v. In particular an edge (u, v) of P corresponds to a vertex $u' = (u, v) \cap H$ of P/v and for every facet $\{u'_1, \ldots, u'_r\}$ of P/v there is a facet $\{v, u_1, \ldots, u_r, w_1, \ldots, w_s\}$ of P. Let $\mathcal{F}_v$ be the set of facets of P containing v, the bijection shows that $\dim(P/v) \leq \dim(P, \mathcal{F}_v)$. Since P/v is $(d-1)$-dimensional $\dim(P/v) \geq d$ by induction. Now let $F \in \mathcal{F}_v$ and $w \notin \mathcal{V}(F)$, by the choice of v the order L_t can not bring w over F, therefore, $L_1, L_2, \ldots, L_{t-1}$ is a realizer for $(P, \mathcal{F}_v)$, i.e., $\dim(P, \mathcal{F}_v) \leq t - 1$. Combining the inequalities we obtain $t \geq d + 1$. □

It is known that for $d \geq 4$ there is no bound depending only on d for the dimension of d-polytopes. For $d = 3$, however, the situation is different. By Steinitz's theorem polytopes and 3-connected plane graphs are essentially the same. Making use of the machinery of Schnyder labelings we formulate and prove a slightly stronger version of Theorem 1.

Theorem 6. *For every 3-polytope P the dimension satisfies* $\dim(P) = 4$. *Moreover, if $I \in \mathcal{F}(P)$ and $\mathcal{F}_I = \mathcal{F}(P) \setminus \{I\}$ then* $\dim(P, \mathcal{F}_I) = 3$.

Proof. Let $\mathcal{R}$ be a realizer of $(P, \mathcal{F}_I)$. To obtain a realizer for P we only have to add a single linear order with $v < w$ for all $v \in \mathcal{V}(I)$ and $w \in \mathcal{V}(P) \setminus \mathcal{V}(I)$ to $\mathcal{R}$. Combined with the lower bound from Theorem 5 this yields $4 \leq \dim(P) \leq \dim(P, \mathcal{F}_I) + 1$. To prove the theorem it remains to show $\dim(P, \mathcal{F}_I) \leq 3$.

Choose a planar embedding of the graph G of P with I as the exterior face, specify three vertices a_1, a_2, a_3 in clockwise order around I and consider a Schnyder labeling of G. As in the proof of Theorem 4 we will use linear extensions L_i of the inclusion order Q_i of regions $i = 1, 2, 3$, i.e., $u < v$ in Q_i iff $R_i(u) \subset R_i(v)$. To bring every vertex y over every face $F \in \mathcal{F}_I$ with $y \notin F$, however, more care in the choice of L_i is required.

Define Q_i^* such that $u < v$ in Q_i^* if either

(a) $u < v$ in Q_i or

(b) $u \parallel v$ in Q_i and $u < v$ in Q_{i+1}.

Lemma 8. *Q_i^* is acyclic for $i = 1, 2, 3$.*

Proof. Call (u, v) a type-a pair if $u < v$ in Q_i^* by part (a) of the definition and call it a type-b pair if $u < v$ in Q_i^* by (b). A cycle in Q_i^* has to contain both a type-a pair and a type-b pair. We claim that if $u < v$ is a type-a pair and $v < w$ is a type-b pair then $u < w$ is also in Q_i^*. Since $u < v$ and $v < u$ can not be both in Q_i^* the claim yields a contradiction to the assumption that Q_i^* contains a cycle.
Claim. If $u < v$ is a type-a pair and $v < w$ is a type-b pair then $u < w$ is also in Q_i^*.

By symmetry we may assume that $i = 1$. If $R_1(v) = R_1(w)$ then with (u, v) the pair (u, w) also is type-a. Therefore, we assume $R_1(v) \not\subseteq R_1(w)$, since (v, w) is type-b this implies $w \notin R_1(v)$ and $w \notin R_2(v)$. Therefore, $w \in R_3^o(v)$ and $R_3(w) \subset R_3(v)$. Since $u \in R_1(v)$ we either find u in $R_1(w)$ or in $R_2(w)$. If u in $R_1(w)$ then $R_1(u) \subseteq R_1(w)$ but equality is impossible since $w \notin R_1(u)$, i.e., (u, w) is type-a pair in this case. Otherwise $u \in R_2^o(w)$, i.e., $R_2(u) \subset R_2(w)$, and the i–regions of u and w are incomparable. This shows that (u, w) is a type-b pair in this case. □

Let L_i be a linear extension of Q_i^*, the claim is that L_1, L_2, L_3 is a realizer. Consider a pair (F, y), where F is a face and y is some vertex not on F. Face F is contained in one of the regions of y, by symmetry we assume $F \in R_1(y)$. Hence, $R_1(x) \subseteq R_1(y)$ for all $x \in F$. If $R_1(x) \subset R_1(y)$ for all $x \in F$ then F is below y in Q_1 and in L_1.

Assume that there is an $x \in F$ with $R_1(x) = R_1(y)$. It is impossible that F contains vertices x and x' with $R_1(x) = R_1(y) = R_1(x')$ and $x \in P_3(y)$ while $x' \in P_2(y)$. This would lead to the placement of y on some edge bounding F in the drawing $\mu(G)$, contradicting Lemma 5.

If for all $x \in F$ either $R_1(x) \subset R_1(y)$ or $R_1(x) = R_1(y)$ and $x \in P_3(y)$, then $R_2(x) \subset R_2(y)$ for all $x \in F$ with $R_1(x) = R_1(y)$. By the definition of Q_1^* this shows that F is below y in L_1.

Finally, consider the situation that for all $x \in F$ either $R_1(x) \subset R_1(y)$ or $R_1(x) = R_1(y)$ and $x \in P_2(y)$. We claim that F is below y in L_3 in this case. All x in F with $R_1(x) = R_1(y)$ have $R_3(x) \subset R_3(y)$, hence, they are below y in L_3. The other vertices x of F have $R_1(x) \subset R_1(y)$. The next lemma shows that $x || y$ in Q_3 holds for all these vertices which implies that they also also go below y in L_3.

Lemma 9. *If $R_1(x) = R_1(y)$, $x \in P_3(y)$ and F is a face in $R_1(x)$ with $x \in F$ and $y \notin F$ then $R_3(y) \not\subseteq R_3(v)$ for all $v \in F$.*

Proof. Consider the triangle ∇ enclosing F in the convex drawing $\mu(G)$ (Lemma 7). Vertex y is placed on the horizontal line ℓ_1 bounding ∇ and y is left of all vertices of F on ℓ_1, Figure 20 shows the situation.

Let x_0 be the leftmost vertex of F on ℓ_1 and u be the uppermost vertex of F on ℓ_3. Let x_1 be the other neighbor of x_0 at F. Even so x_1 need not be on ℓ_1 the edge (x_0, x_1) is the outgoing edge of x_0 in label 2, c.f. Figure 11. Also (x_0, y) is the outgoing edge of x_0 in label 3. The edge orientations at vertex x_0 imply that (u, x_0) is oriented from u to x_0 in label 1. This shows that x_0 is on $P_1(v)$ for all $v \in F \cap \ell_3$. The paths

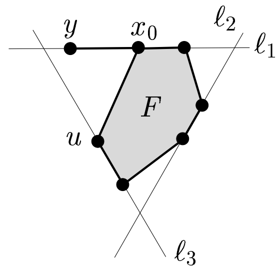

Figure 20:

$P_1(v)$ for $v \in F \cap \ell_2$ clearly cross ℓ_1 to the right of x_0. This shows that $y \notin R_3(v)$ for all $v \in F$, hence $R_3(y) \not\subseteq R_3(v)$. □

Acknowledgment.

In 1993 Tom Trotter left to me a draft of a joint manuscript with Walter Schnyder, entitled 'Convex Embeddings of 3-Connected Plane Graphs'. Though the manuscript was not completed it already contained most of the ideas of Section 2. Without the inspiring insights of Schnyder and Trotter this paper would not exist, thanks to both of them.

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