Arrangements of Pseudocircles: Triangles and Drawings

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Abstract. A pseudocircle is a simple closed curve on the sphere or in the plane. The study of arrangements of pseudocircles was initiated by Grünbaum, who defined them as collections of simple closed curves that pairwise intersect in exactly two crossings. Grünbaum conjectured that the number of triangular cells p_3 in digon-free arrangements of n pairwise intersecting pseudocircles is at least $2n-4$. We present examples to disprove this conjecture. With a recursive construction based on an example with 12 pseudocircles and 16 triangles we obtain a family with $p_3(\mathscr{A})/n \to 16/11 = 1.\overline{45}$. We expect that the lower bound $p_3(\mathscr{A}) \geq 4n/3$ is tight for infinitely many simple arrangements. It may however be that digon-free arrangements of n pairwise intersecting circles indeed have at least $2n-4$ triangles.

For pairwise intersecting arrangements with digons we have a lower bound of $p_3 \geq 2n/3$, and conjecture that $p_3 \geq n-1$.

Concerning the maximum number of triangles in pairwise intersecting arrangements of pseudocircles, we show that $p_3 \leq 2n^2/3 + O(n)$. This is essentially best possible because families of pairwise intersecting arrangements of n pseudocircles with $p_3/n^2 \to 2/3$ as $n \to \infty$ are known.

The paper contains many drawings of arrangements of pseudocircles and a good fraction of these drawings was produced automatically from the combinatorial data produced by the generation algorithm. In the final section we describe some aspects of the drawing algorithm.

1 Introduction

Arrangements of pseudocircles generalize arrangements of circles in the same vein as arrangements of pseudolines generalize arrangements of lines. The study of arrangements of pseudolines was initiated 1918 with an article of Levi [7] where he studied triangles in arrangements. Since then arrangements of pseudolines were intensively studied and the handbook article on the topic [3] lists more than 100 references.

* Partially supported by DFG Grant FE 340/11-1
** Partially supported by ERC Advanced Research Grant no 267165 (DISCONV)

Grünbaum [6] initiated the study of arrangements of pseudocircles. By stating a large number of conjectures he was hoping to attract the attention of researchers for the topic. The success of this program was limited and several of Grünbaum's 45 year old conjectures remain unsettled. In this paper we report on some progress regarding conjectures involving numbers of triangles and digons in arrangements of pseudocircles.

Some of our results and new conjectures are based on a program written by the second author that enumerates all arrangements of up to 7 pairwise intersecting pseudocircles. Before formally stating our main results we introduce some terminology:

An *arrangement of pseudocircles* is a collection of closed curves in the plane or on the sphere, called *pseudocircles*, with the property that the intersection of any two of the pseudocircles is either empty or consists of two points where the curves cross. An arrangement $\mathscr{A}$ of pseudocircles is

simple, if no three pseudocircles of $\mathscr{A}$ intersect in a common point.

pairwise intersecting, if any two pseudocircles of $\mathscr{A}$ have non-empty intersection. We will frequently abbreviate and just write "*intersecting*" instead of "pairwise intersecting".

cylindrical, if there are two cells of the arrangement which are separated by each of the pseudocircles.

digon-free, if there is no cell of the arrangement which is incident to only two pseudocircles.

We consider the sphere to be the most natural ambient space for arrangements of pseudocircles. Consequently, we call two arrangements isomorphic if they induce homeomorphic cell decompositions of the sphere. In many cases, in particular in all our figures, arrangements of pseudocircles are embedded in the Euclidean plane, i.e., there is a distinguished outer/unbounded cell. An advantage of such a representation is that we can refer to the inner and outer side of a pseudocircle. Note that for every cylindrical arrangement of pseudocircles it is possible to choose the unbounded cell such that there is a point in the intersection of the interior pseudodiscs of all pseudocircles.

In an arrangement $\mathscr{A}$ of pseudocircles, we denote a cell with k crossings on its boundary as a *k-cell* and let $p_k(\mathscr{A})$ be the number of k-cells of $\mathscr{A}$. Following Grünbaum we call 2-cells *digons* and remark that some other authors call them *lenses*. 3-cells are *triangles*, 4-cells are *quadrangles*, and 5-cells are *pentagons*.

Conjecture 3.7 from Grünbaum's monograph [6] is: *Every (not necessarily simple) digon-free arrangement of n pairwise intersecting pseudocircles has at least* $2n-4$ *triangles*. Grünbaum also provides examples of arrangements with $n \geq 6$ pseudocircles and $2n-4$ triangles.

Snoeyink and Hershberger [10] showed that the sweeping technique, which serves as an important tool for the study of arrangements of lines and pseudolines, can be adapted to work also in the case of arrangements of pseudocircles. They used sweeps to show that, in an intersecting arrangement of pseudocircles, every pseudocircle is incident to two cells which are digons or triangles on either side.

Therefore, $2p_2 + 3p_3 \geq 4n$, and whence, every intersecting digon-free arrangement of n pseudocircles has at least $4n/3$ triangles.

Felsner and Kriegel [4] observed that the bound from [10] also applies to non-simple intersecting digon-free arrangements and gave examples of arrangements showing that the bound is tight on this class for infinitely many values of n. These examples disprove the conjecture in the non-simple case.

In Section 2, we give counterexamples to Grünbaum's conjecture which are simple. With a recursive construction based on an example with 12 pseudocircles and 16 triangles we obtain a family with $p_3/n \xrightarrow{n\to\infty} 16/11 = 1.\overline{45}$. We then replace Grünbaum's conjecture by Conjecture 2: *The lower bound $p_3(\mathscr{A}) \geq 4n/3$ is tight for infinitely many non-isomorphic simple arrangements.*

A specific arrangement $\mathscr{N}_6$ of 6 pseudocircles with 8 triangles appears as a subarrangement in all known simple, intersecting, digon-free arrangements with $p_3 < 2n - 4$. From [5] it is known that $\mathscr{N}_6$ is not circularizable, i.e., not representable by circles. This motivates the question, whether indeed Grünbaum's conjecture is true when restricted to intersecting arrangements of circles, see Conjecture 1. In Subsection 2.1 we discuss arrangements with digons. We give an easy extension of the argument of Snoeyink and Hershberger [10] to show that these arrangements contain at least $2n/3$ triangles. All arrangements known to us have at least $n - 1$ triangles and therefore our Conjecture 3 is that $n - 1$ is a tight lower bound for intersecting arrangements with digons.

In Section 3 we study the maximum number of triangles in arrangements of n pseudocircles. We show an upper bound of order $2n^2/3 + O(n)$. For the lower bound construction we glue two arrangements of n pseudolines into an arrangement of n pseudocircles. Since respective arrangements of pseudolines are known, we obtain arrangements of pseudocircles with $2n(n-1)/3$ triangles for $n \equiv 0, 4 \pmod 6$.

The paper contains many drawings of arrangements of pseudocircles and a good fraction of these drawings was produced automatically from the combinatorial data produced by the generation algorithm. In Section 4 we describe some aspects of the drawing algorithm which is based on iterative calls to a Tutte embedding a.k.a. spring embedding with adapting weights on the edges.

From now on (unless explicitly stated otherwise) the term *arrangement* is used as equivalent to *simple arrangement of pairwise intersecting pseudocircles.*

2 Arrangements with few Triangles

The main result of this section is the following theorem, which disproves Grünbaum's conjecture.

Theorem 1. *The minimum number of triangles in digon-free arrangements of n pseudocircles is*

(i) 8 *for* $3 \leq n \leq 6$.
(ii) $\lceil \frac{4}{3}n \rceil$ *for* $6 \leq n \leq 14$.
(iii) $< \frac{16}{11}n$ *for all* $n = 11k + 1$ *with* $k \in \mathbb{N}$.

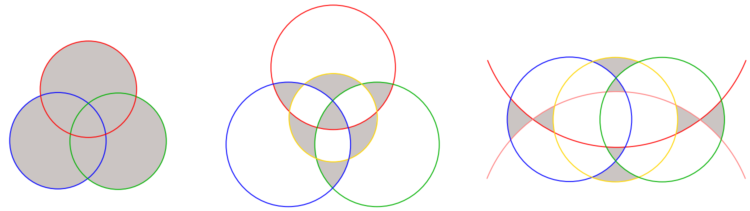


Fig. 1: Arrangements of $n = 3, 4, 5$ circles and $p_3 = 8$ triangles each. Triangles (except the outer face) are colored gray.

Figures 1 and 2 show arrangements with the minimum number of triangles for up to 8 pseudocircles. We remark that, in total, there are three non-isomorphic arrangements of $n = 8$ pseudocircles with $p_3 = 11$ triangles, these are the smallest counterexamples to Grünbaum's conjecture (cf. Lemma 1). We refer to our website [8] for further examples.

The basis for Theorem 1 was laid by exhaustive computations, which generated all simple arrangements of up to $n = 7$ pseudocircles. Starting with the unique arrangement of two intersecting pseudocircles, our program recursively inserted pseudocircles in all possible ways. Since counting arrangements is also interesting, we state the numbers in Table 1. The table shows the number of simple intersecting pseudocircle arrangements on the sphere. The first row shows the numbers when digons are allowed and the second row shows the numbers of digon-free arrangements. The arrangements and more information can be found on the companion website [8].

n	2	3	4	5	6	7
general	1	2	8	278	145 058	447 905 202
digon-free	0	1	2	14	2 131	3 012 972

Table 1: Number of combinatorially different arrangements of n pseudocircles.

From the complete enumeration we know the minimum number of triangles for $n \leq 7$. In the range from 8 to 14, we iteratively used arrangements with n pseudocircles and a small number of triangles and digons to generate arrangements with $n + 1$ pseudocircles and the same property. Using this strategy, we found arrangements with $\lceil 4n/3 \rceil$ triangles for all n in this range. The corresponding lower bound $p_3(\mathscr{A}) \geq 4n/3$ is known from [10].

A result of the computations was that the triangle-minimizing example for $n = 6$ is unique, i.e., there is a unique simple arrangement $\mathscr{N}_6$ with 6 pseudocircles and only 8 triangles. In [5] we have shown that $\mathscr{N}_6$ is not circularizable. The arrangement $\mathscr{N}_6$ is a subarrangement of all known arrangements with less than $2n - 4$ triangles. Therefore, the following weakening of Grünbaum's conjecture may be true.

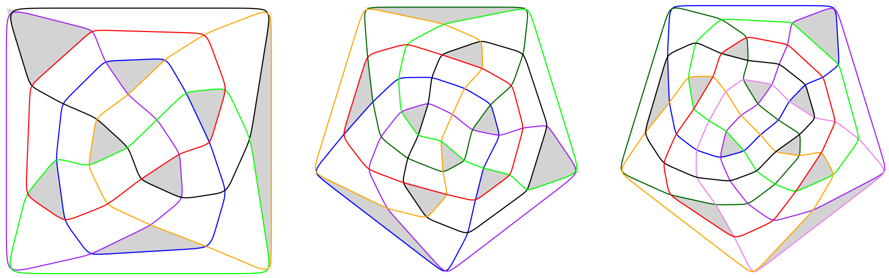


Fig. 2: Arrangements with $n = 6, 7, 8$ and 8, 10, 11 triangles respectively.

Conjecture 1 (Weak Grünbaum Conjecture). Every digon-free arrangement of n circles has at least $2n - 4$ triangles.

We know that this conjecture is true for all $n \leq 9$. The claim, that we have checked all arrangements with $p_3(\mathscr{A}) < 2n - 4$ in this range, is justified by the following lemma, which restricts the pairs (p_2, p_3) for which there exist arrangements of n pseudocircles whose extensions have $p_3(\mathscr{A}) < 2n - 4$. In particular, to get all digon-free arrangements with $n = 9$ and 12 triangles we only had to extend arrangements with $n = 7$ and $n = 8$, where $p_3 + 2p_2 \leq 12$. It turned out, that all arrangements on $n = 9$ pseudocircles with 12 triangles are non-circularizable since all of them contain $\mathscr{N}_6$ as a subarrangement.

Lemma 1. *Let $\mathscr{A}$ be an arrangement of pseudocircles. Then for every subarrangement $\mathscr{A}'$ of $\mathscr{A}$ we have*

$$p_3(\mathscr{A}') + 2p_2(\mathscr{A}') \leq p_3(\mathscr{A}) + 2p_2(\mathscr{A}).$$

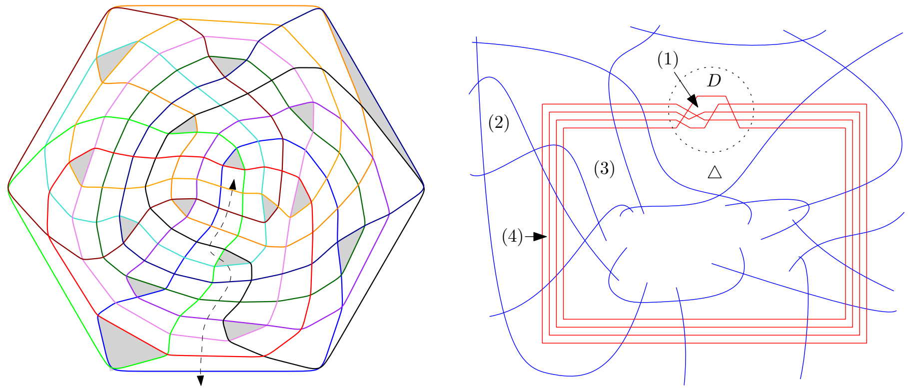
Proof. We show the statement for a subarrangement $\mathscr{A}'$ in which one pseudocircle C is removed from $\mathscr{A}$. The inequality then follows by iterating the argument. The arrangement $\mathscr{A}'$ partitions the pseudocircle C into arcs. Reinsert these arcs one by one.

Consider a triangle of $\mathscr{A}'$. After adding an arc, one of the following cases occurs: (1) the triangle remains untouched, or (2) the triangle is split into a triangle and a quadrangle, or (3) a digon is created in the region of the triangle.

Now consider a digon of $\mathscr{A}'$. After adding an arc, either (1) there is a new digon inside this digon, or (2) the digon has been split into two triangles. □

We now prepare for the proof of Theorem 1(iii). The basis of the construction is the arrangement $\mathscr{A}_{12}$ with 12 pseudocircles and 16 triangles shown in Figure 3a. This arrangement will be used iteratively for a 'merge' as described by the following lemma.

Lemma 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be digon-free arrangements of $n_{\mathscr{A}} \geq 3$ and $n_{\mathscr{B}} \geq 3$ pseudocircles, respectively. If there is a simple curve $P_{\mathscr{A}}$ that (1) intersects every pseudocircle of $\mathscr{A}$ exactly once (2) contains no vertex of $\mathscr{A}$, (3) traverses $\tau \geq 1$*



(a) (b)

Fig. 3: (a) A digon-free, intersecting arrangement $\mathscr{A}_{12}$ of $n = 12$ pseudocircles with exactly 16 triangles. The dashed curve intersects every pseudocircle exactly once. (b) An illustration of the construction in Lemma 2. Pseudocircles of $\mathscr{A}$ ($\mathscr{B}$) are drawn red (blue).

triangles of $\mathscr{A}$*, and (4) forms* δ *triangles with pairs of pseudocircles from* $\mathscr{A}$*, then there is a digon-free arrangement* $\mathscr{C}$ *of* $n_{\mathscr{A}} + n_{\mathscr{B}} - 1$ *pseudocircles with* $p_3(\mathscr{C}) = p_3(\mathscr{A}) + p_3(\mathscr{B}) + \delta - \tau - 1$ *triangles.*

Proof. Take a drawing of $\mathscr{A}$ and make a hole in the two cells, which contain the ends of $P_{\mathscr{A}}$. This yields a drawing of $\mathscr{A}$ on a cylinder such that none of the pseudocircles is contractible. The path $P_{\mathscr{A}}$ connects the two boundaries of the cylinder. In fact, the existence of a path with the properties of $P_{\mathscr{A}}$ is characterizing cylindrical arrangements.

Stretch the cylindrical drawing such that it becomes a narrow belt, where all intersections of pseudocircles take place in a small disk, which we call *belt-buckle*. This drawing of $\mathscr{A}$ is called a *belt drawing*. The drawing of the red subarrangement in Figure 3b shows a belt drawing.

Choose a triangle $\triangle$ in $\mathscr{B}$ and a pseudocircle B which is incident to $\triangle$. Let b be the *edge* of B on the boundary of $\triangle$. Specify a disk D, which is traversed by b and disjoint from all other edges of $\mathscr{B}$. Now replace B by a belt drawing of $\mathscr{A}$ in a small neighborhood of B such that the belt-buckle is drawn within D; see Figure 3b.

The arrangement $\mathscr{C}$ obtained from *merging* $\mathscr{A}$ and $\mathscr{B}$, as we just described, has $n_{\mathscr{A}} + n_{\mathscr{B}} - 1$ pseudocircles. Moreover if $\mathscr{A}$ and $\mathscr{B}$ are digon-free/intersecting, then $\mathscr{C}$ has the same property. Most of the cells c of $\mathscr{C}$ are of one of the following four types:

(a) All boundary edges of c belong to pseudocircles of $\mathscr{A}$.
(b) All boundary edges of c belong to pseudocircles of $\mathscr{B}$.

(c) All but one of the boundary edges of c belong to pseudocircles of $\mathscr{B}$ and the remaining edge belongs to $\mathscr{A}$. (These cells correspond to cells of $\mathscr{B}$ with a boundary edge on B.)
(d) Quadrangular cells, whose boundary edges alternatingly belong to $\mathscr{A}$ and $\mathscr{B}$.

From the cells of $\mathscr{B}$, only $\triangle$ and the other cell containing b (which is not a triangle since $\mathscr{B}$ is digon-free) have not been taken into account. In $\mathscr{C}$, the corresponding two cells have at least two boundary edges from $\mathscr{B}$ and at least two from $\mathscr{A}$. Consequently, neither of the two cells are triangles. The remaining cells of $\mathscr{C}$ are bounded by pseudocircles from $\mathscr{A}$ together with one of the two bounding pseudocircles of $\triangle$ other than B. These two pseudocircles cross through $\mathscr{A}$ following the path prescribed by $P_{\mathscr{A}}$. There are δ triangles among these cells, but τ of these are obtained because $P_{\mathscr{A}}$ traverses a triangle of $\mathscr{A}$. Among cells of $\mathscr{C}$ of types (1) to (4) all the triangles have a corresponding triangle in $\mathscr{A}$ or $\mathscr{B}$. But $\triangle$ is a triangle of $\mathscr{B}$ which does not occur in this correspondence. Hence, there are $p_3(\mathscr{A}) + p_3(\mathscr{B}) + \delta - \tau - 1$ triangles in $\mathscr{C}$. □

Proof of Theorem 1(iii). We use $\mathscr{A}_{12}$, the arrangement shown in Figure 3a, in the role of $\mathscr{A}$ for our recursive construction. The dashed path in the figure is used as $P_{\mathscr{A}}$ with $\delta = 2$ and $\tau = 1$. Starting with $\mathscr{C}_1 = \mathscr{A}_{12}$ and defining $\mathscr{C}_{k+1}$ as the merge of $\mathscr{C}_k$ and $\mathscr{A}_{12}$, we construct a sequence $\{\mathscr{C}_k\}_{k\in\mathbb{N}}$ of digon-free arrangements with $n(\mathscr{C}_k) = 11k + 1$ pseudocircles and $p_3(\mathscr{C}_k) = 16k$ triangles. The fraction $16k/(11k + 1)$ is increasing with k and converges to $16/11 = 1.\overline{45}$ as n goes to ∞. □

We remark that using other arrangements from Theorem 1(ii) (which also admit a path with $\delta = 2$ and $\tau = 1$) in the recursion, we obtain arrangements with $p_3 = \lceil \frac{16}{11} n \rceil$ triangles for all $n \geq 6$.

Since the lower bound $\lceil \frac{4}{3} n \rceil$ is tight for $6 \leq n \leq 14$, we believe that the following is true:

Conjecture 2. There are digon-free arrangements $\mathscr{A}$ with $p_3(\mathscr{A}) = \lceil 4n/3 \rceil$ for infinitely many values of n.

2.1 Arrangements with Digons

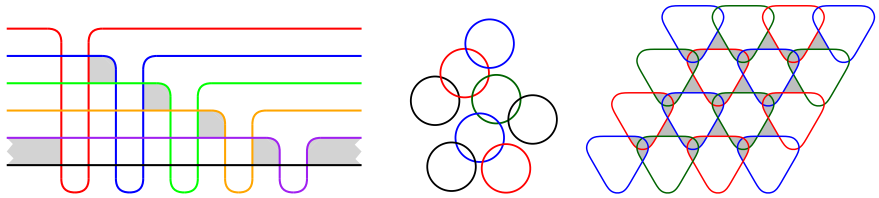
We know arrangements of n pseudocircles with digons and only $n - 1$ triangles. The example shown in Figure 4a is part of an infinite family of such arrangements.

Using ideas based on sweeps (cf. [10]), we can show that every pseudocircle is incident to at least two triangles. This implies the following theorem:

Theorem 2. *Every arrangement of $n \geq 3$ pseudocircles has at least $2n/3$ triangles.*

The proof of the theorem is based on the following lemma:

Lemma 3. *Let C be a pseudocircle in an arrangement of $n \geq 3$ pseudocircles. Then all digons incident to C lie on the same side of C.*



(a) (b) (c)

Fig. 4: Example arrangements (a) n pseudocircles with n digons and $n-1$ triangles (b) "trees of circles" with no triangles (c) connected arrangements of n pseudocircles with triangle-cell-ratio of $\frac{5}{6} - O(\frac{1}{\sqrt{n}})$.

Proof. Consider a pseudocircle C' that forms a digon D' with C that lies, say, "inside" C. If C'' also forms a digon D'', then C'' has to cross C' in the exterior of C. Hence D'' also has to lie "inside" C. Consequently, all digons incident to C lie on the same side of C. □

Proof of Theorem 2. Let $\mathscr{A}$ be an arrangement and consider a drawing of $\mathscr{A}$ in the plane. Snoeyink and Hershberger [10] have shown that starting with any circle C from $\mathscr{A}$ the outside of C can be swept with a closed curve γ until all of the arrangement is inside of γ. During the sweep γ is intersecting every pseudocircle from $\mathscr{A}$ at most twice. The sweep uses two types[3] of move to make progress:

(1) *take a crossing*, in [10] this is called 'pass a triangle';
(2) *leave a pseudocircle*, this is possible when γ and some pseudocircle form a digon which is on the outside of γ, in [10] this is called 'pass a hump'.

Let C be a pseudocircle of $\mathscr{A}$. By the previous lemma, all digons incident to C lie on the same side of C. Redraw $\mathscr{A}$ so that all digons incident to C are inside C. The first move of a sweep starting at C has to take a crossing, and hence, there is a triangle $\triangle$ incident to C. Redraw $\mathscr{A}$ such that $\triangle$ becomes the unbounded face. Again consider a sweep starting at C. The first move of this sweep reveals a triangle $\triangle'$ incident to C. Since $\triangle$ is not a bounded triangle of the new drawing we have $\triangle \neq \triangle'$, and hence, C is incident to at least two triangles. The proof is completed by double counting the number of incidences of triangles and pseudocircles. □

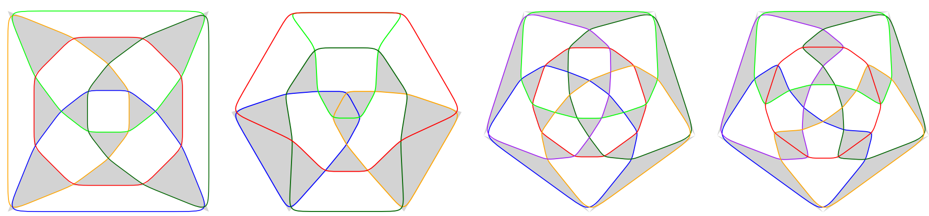
Since for $3 \leq n \leq 7$ every arrangement has at least $n-1$ triangles, we believe that the following is true:

Conjecture 3. Every intersecting arrangement of $n \geq 3$ pseudocircles has at least $n-1$ triangles.

[3] There is a third type of move for sweeps of arrangements of pseudocircles, it is called *take a hump* and does not occur in our case, as each two pseudocircles already intersect.

If the arrangement is not required to be intersecting, then the proof of Lemma 3 fails and indeed there are examples of non-intersecting arrangements without triangles, e.g., a "tree of circles", see Figure 4b.

3 Maximum Number of Triangles



(a) (b) (c) (d)

Fig. 5: (a) and (b) show arrangements with $n = 5$ pseudocircles. The first one is digon-free and has 12 triangles and the second one has 13 triangles and one digon. (c) and (d) show arrangements with $n = 6$ and 20 triangles. The arrangement in (c) is the skeleton of the Icosidodecahedron.

Regarding the maximal number of triangles the complete enumeration provides precise data for $n \leq 7$. We used heuristics to generate examples with many triangles for larger n. Table 2 shows the results. For $n \geq 4$ there is only one instance where we know an arrangement with more than $\frac{4}{3}\binom{n}{2}$ triangles. This number is 1/3 times the number of edges of the arrangement, i.e., it is an upper bound for the number of triangles in arrangements where each edge is incident to at most one triangle. In the next subsection we show that asymptotically the contribution of edges that are incident to two triangles is neglectable. The last subsection gives a construction of arrangements which show that $\lfloor\frac{4}{3}\binom{n}{2}\rfloor$ is attained for infinitely many values of n.

	2	3	4	5	6	7	8	9	10
simple	0	8	8	13	20	29	≥ 37	≥ 48	≥ 60
digon-free	-	8	8	12	20	29	≥ 37	≥ 48	≥ 60
$\lfloor\frac{4}{3}\binom{n}{2}\rfloor$	1	4	8	13	20	28	37	48	60

Table 2: Upper bound on the number of triangles.

Recall that we only study simple arrangements. Grünbaum [6] also looked at non-simple arrangements. His Figures 3.30, 3.31, and 3.32 show drawings of simplicial arrangements that have $n = 7$ with $p_3 = 32$, $n = 8$ with $p_3 = 50$, and

$n = 9$ with $p_3 = 62$, respectively. Hence, non-simple arrangements can have more triangles.

Theorem 3. $p_3(\mathscr{A}) \leq \frac{2}{3}n^2 + O(n)$.

The proof of this theorem can be found in Appendix A.

Remarks.

– Since intersecting arrangements have $2\binom{n}{2} + 2 = n^2 - O(n)$ faces we can also state the bound as: at most $\frac{2}{3} + O(\frac{1}{n})$ of all cells of an arrangement are triangles. However, this is not true if we consider non-intersecting arrangements. Figure 4c shows a construction where this ratio converges to $\frac{5}{6}$ as $n \to \infty$. It can be shown with a counting argument that $\frac{5}{6}$ is an upper bound for the triangle-cell-ratio of simple arrangements.
– It would be interesting to get more precise results. In particular, we would like to know whether $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$ is true for all n.

3.1 Constructions using Arrangements of Pseudolines

Great circles on the sphere are a well known model for projective arrangements of lines. Antipodal pairs of points on the sphere correspond to points of the projective plane. Hence, the great circle arrangement corresponding to a projective arrangement $\mathscr{A}$ of lines has twice as many vertices, edges, and faces of every type as $\mathscr{A}$. The same idea can be applied to projective arrangement of pseudolines. If $\mathscr{A}$ is a projective arrangement of pseudolines take a drawing of $\mathscr{A}$ in the unit disk D such that every line ℓ of $\mathscr{A}$ connects two antipodal points of D. Project D to the upper hemisphere of a sphere S, such that the boundary of D becomes the equator of S. Use a projection through the center of ℓ to copy the drawing from the upper hemisphere to the lower hemisphere of S. By construction the two copies of a pseudoline ℓ from $\mathscr{A}$ join together to form a pseudocircle. The collection of these pseudocircles yields an arrangement of pseudocircles on the sphere with twice as many vertices, edges, and faces of every type as $\mathscr{A}$. Arrangements of pseudocircles obtained by this construction have a special property:

– If three pseudocircles C, C', and C'' have no common crossing, then C'' separates the two crossings of C and C'.

Grünbaum calls arrangements with this property 'symmetric'. In the context of oriented matroids the property is part of the definition of arrangements of pseudocircles.

Arrangements of pseudolines which maximize the number of triangles have been studied intensively. The end of this line of research is marked by Blanc [2]. This paper gives precise bounds for the maximum both in the Euclidean and in the projective case. In particular, Blanc constructs examples of projective arrangements of pseudolines with $\frac{2}{3}\binom{n}{2}$ triangles for an infinite number of values of n. This directly yields arrangements of pseudocircles with $\frac{4}{3}\binom{n}{2}$ triangles.

The ‘doubling method’ that has been used for constructions of arrangements of pseudolines with many triangles, see [2], can also be applied for pseudocircles. In fact, in the case of pseudocircles there is more flexibility for applying the method. Therefore, it is possible that $\lfloor \frac{4}{3}\binom{n}{2} \rfloor$ triangles can be achieved for all n.

4 Visualization

Most of the figures in this paper have been automatically generated by our framework, which was written in the mathematical software SageMath [11] and is available on demand. We encode an arrangement of pseudocircles by its dual graph. Each face in the arrangement is represented by a vertex and two vertices share an edge if and only if the two corresponding faces share a common pseudosegment. As our arrangements are intersecting, it is easy to see that the dual graph is 3-connected and thus its embedding is unique on the sphere (up to isomorphism).

To visualize an arrangement of pseudocircles, we draw the primal (multi)graph using straight-line segments, in which vertices represent crossings of pseudocircles and edges connect two vertices if they are connected by a pseudocircle segment. Note that in the presence of digons we obtain double-edges.

In our drawings, pseudocircles are colored by distinct colors, and triangles (except the outer face) are filled gray. In straight-line drawings, edges corresponding to digons are drawn dashed in the two respective colors alternatingly, while in the curved drawings digons are represented by a point where the two respective pseudocircles touch.

4.1 Iterated Tutte Embeddings

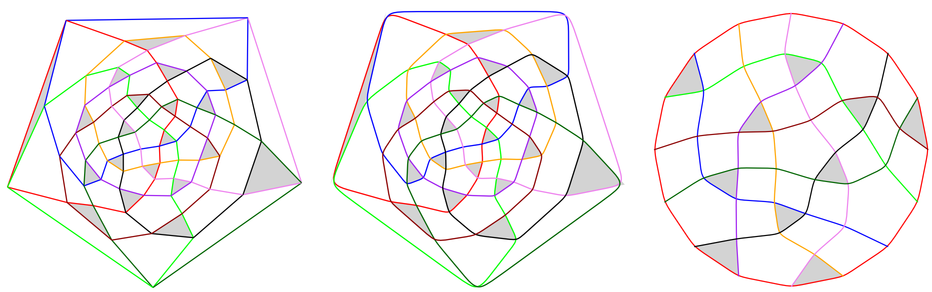
To generate nice aesthetic drawings automatically, we iteratively use weighted Tutte embeddings. We fix a non-digon cell as the outer cell and arrange the vertices of the outer cell as the corners of a regular polygon. Starting with edge-weights all equal to 1, we obtain an ordinary plane Tutte embedding.

For iteration j, we set the weights (force of attraction) of an edge $e = \{u, v\}$ proportional to $p(A(f_1)) + p(A(f_2)) + q(\|u - v\|/j)$ where f_1, f_2 are the faces incident to e, $A(.)$ is the area function, $\| \cdot \|$ is the Euclidean norm, and p, q are suitable monotonically increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ (we use $p(x) = x^4$ and $q(x) = x^2/10$).

Intuitively, if the area of a face becomes too large, the weights of its incident edges are increased and will rather be shorter so that the area of the face will also get smaller in the next iteration. It turned out that in some cases the areas of the faces became well balanced but some edges were very short and others long. Therefore we added the dependence on the edge length which is strong at the beginning and decreases with the iterations. The particular choice of the functions was the result of interactive tuning. The iteration is terminated when the change of the weights is small.

4.2 Visualization using Curves

On the basis of the straight-line embedding obtained with the Tutte iteration we use splines to smoothen the curves. The details are as follows. First we take a 2-subdivision of the graph, where all subdivision-vertices adjacent to a given vertex v are placed at the same distance $d(v)$ from v. We choose $d(v)$ so that it is at most $1/3$ of the length of an edge incident to v. We then use B-splines to visualize the curves. Even though one can draw Bézier curves directly with Sage, we mostly generated ipe files (xml-format) so that we can further process the arrangements. Figures 6a and 6b show the straight-line and curved drawing of the same arrangement.



(a) (b) (c)

Fig. 6: (a) Straight-line and (b) curved drawings of the arrangement of pseudo(great)circles, which consists of two copies of (c) the (non-stretchable) non-Pappus pseudoline arrangement of pseudolines.

4.3 Visualization of Arrangements of Pseudolines

We also adopted the code to visualize arrangements of pseudolines nicely. One of the lines is considered as the "line at infinity" which is then drawn as a regular polygon. Figure 6c gives an illustration.

For arrangements of pseudolines we used the framework pyotlib, which originated from the Bachelor's thesis of Scheucher [9].

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A Proof of Theorem 3

Let $\mathscr{A}$ be an arrangement of $n \geq 4$ pseudocircles. We view $\mathscr{A}$ as a 4-regular plane graph, i.e., the set X of crossings is the vertex set and edges are the segments which connect consecutive crossings on a pseudocircle.

Claim. A. No crossing is incident to 4 triangular cells.

Assume that a crossing u of C_i and C_j is incident to four triangular cells. Then there is a pseudocircle C_k which bounds those 4 triangles, see Figure 7(a). Now C_k only intersects C_i and C_j. This, however, is impossible because $n \geq 4$ and $\mathscr{A}$ is intersecting. $\triangle$

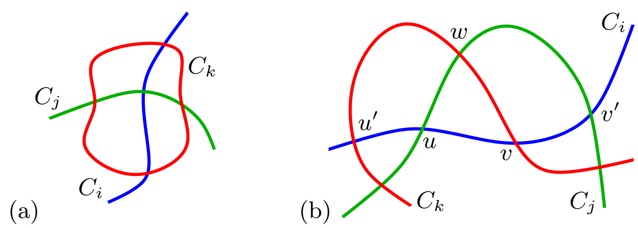


Fig. 7: Illustrations of the proof of Claim A and Claim B.

Let $X' \subseteq X$ be the set of crossings of $\mathscr{A}$ that are incident to 3 triangular cells. Our aim is to show that $|X'|$ is small, in fact $|X'| \in O(n)$. When this is shown we can bound the number of triangles in $\mathscr{A}$ as follows. The number of triangles incident to a crossing in X' clearly is in $O(n)$. Now let $Y = X \setminus X'$. Each of the remaining triangles is incident to three crossings from Y and each crossing of Y is incident to at most 2 triangles. Hence, there are at most $2|Y|/3 + O(n)$ triangles. Since $|Y| \leq |X| = n(n-1)$ we obtain the bound claimed in the statement of the theorem.

To show that $|X'|$ is small we need some preparation.

Claim. B. Two adjacent crossings u, v in X' share two triangles.

Since u and v are both incident to 3 triangles, there is at least one triangle $\triangle$ incident to both of them. Assume for a contradiction that the other cell which is incident to the segment uv is not a triangle. Let C_i, C_j, C_k be the three pseudocircles such that u is a crossing of C_i and C_j, v is a crossing of C_i and C_k, and $\triangle$ is bounded by C_i, C_j, C_k; see Figures 7(b). We denote the third vertex of $\triangle$ by w and note that w is a crossing of C_j and C_k.

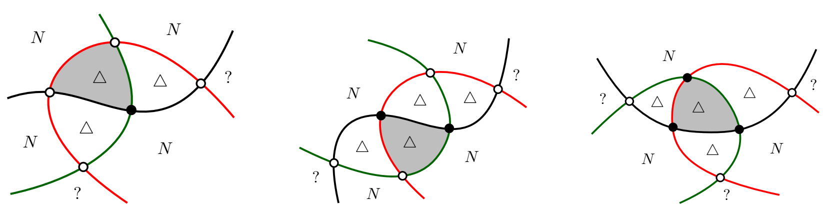
Since u is incident to three triangles, the segment uw bounds another triangle, which is again defined by C_i, C_j, C_k. Let u' be the third vertex incident to that triangle. Similarly, the segment vw is incident to another triangle which is also defined by C_i, C_j, C_k, and has a third vertex v'.

Again, by the same argument, the segments uu' and vv', respectively, are both incident to another triangle. However, this is impossible as the two circles C_j and C_k intersect three times. Thus both faces incident to segment uv are triangles. $\triangle$

Claim. C. Let u, v, w be three distinct crossings in X'. If u is adjacent to both v and w, then v is adjacent to w.

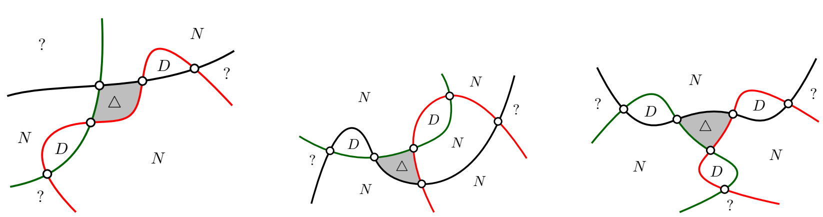
Since u is incident to three triangles and the segments uv and uw are both incident to two triangles, there is a triangle $\triangle$ with corners u, v, w. This triangle shows that u, v, and w are adjacent to each other. $\triangle$

Claim C implies that each connected component of the graph induced by X' is a complete graph. It is easy to see that a K_4 induced by X' is impossible, and therefore, all components induced by X' are either singletons, edges, or triangles. Figure 8 shows the local structure of the arrangement around components of these three types.



(a) (b) (c)

Fig. 8: An illustration. In this figure $\triangle$ marks a triangle, "N" marks a k-cell with $k \geq 4$ ("neither a triangle, nor a digon"), "?" marks an arbitrary cell. Crossings with 3 incident triangles are shown as black vertices (these are the crossings in X').



(a) (b) (c)

Fig. 9: The configurations in (a), (b), and (c) are obtained by flipping the gray triangle in the configuration from Figure 8(a), 8(b), and 8(c), respectively. The digons created by the flip are marked "D".

To show that $|X'|$ is small, we are going to trade crossings of X' with digons and then refer to a result of Agarwal et al. [1]. They have shown that the number of digons in intersecting arrangements of pseudocircles is at most linear in n.

To convert crossings of X' into digons we use *triangle flips*. Each of the configurations shown in Figure 8 has a gray triangle. By flipping these triangles we obtain the configurations shown in Figure 9. These so-obtained configurations have at least as many new digons as the original configurations contain crossings in X'. It may be that with the flip we create new triangles and even new vertices which are incident to 3 triangles. However, the flips that we apply never remove a digon.

Therefore, thanks to the result from [1] we can make no more than $O(n)$ flips before all the crossings are incident to at most 2 triangles. □

Remarks.

- Using flips we can also trade segments which are incident to two triangles against digons.
- It can be shown that at most one component of the graph induced by X' is a K_3.