Triangles in Arrangements of Pseudocircles*

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Abstract

Grünbaum conjectured that the number of triangular cells p_3 in digon-free arrangements of n pairwise intersecting pseudocircles is at least $2n-4$. We present examples to disprove this conjecture. With a recursive construction based on an example with 12 pseudocircles and 16 triangles we obtain a family with $p_3(\mathscr{A})/n \to 16/11 = 1.\overline{45}$. We conjecture that the lower bound $p_3 \geq 4n/3$ of Hershberger and Snoeyink is tight for infinitely many arrangements. For intersecting arrangements with digons we have $p_3 \geq 2n/3$, and conjecture that $p_3 \geq n-1$.

First counterexamples to Grünbaum's conjecture were found on the basis of an exhaustive enumeration of all arrangements of n intersecting pseudocircles for $n \leq 7$. It turned out that there is a unique digon-free intersecting arrangement $\mathscr{N}_6$ with $n = 6$ and only 8 triangles. This arrangement is a subarrangement of all minimizing examples for $n = 7, 8, 9$. We show that $\mathscr{N}_6$ is not circularizable, i.e., there is no equivalent arrangement of circles. These results suggest that Grünbaum's conjecture might be true for digon-free intersecting arrangements of circles.

1 Introduction

We study *simple*, *intersecting* arrangements of pseudocircles on the sphere. Here intersecting means that any two pseudocircles cross twice, while simple means that no three pseudocircles intersect in a common point.

An arrangement of pseudocircles is called *completely intersecting* if there are two cells, which are separated by each of the pseudocircles. Note that for every completely intersecting arrangement of pseudocircles there is a stereographic projection from the sphere to the plane such that those two cells are mapped to the outer cell and a cell, which lies "inside" every pseudocircle, respectively.

In an arrangement $\mathscr{A}$ of pseudocircles, we denote a cell with k crossings on its boundary as *k-cell* and let $p_k(\mathscr{A})$ be the number of k-cells of $\mathscr{A}$. As usual we call 2-cells *digons*, 3-cells *triangles*, 4-cells *quadrangles*, and 5-cells *pentagons*.

In his monograph [3] from 1972, Grünbaum states Conjecture 3.7: *Every (not necessarily simple) digon-free arrangement of n pairwise intersecting pseudocircles has at least $2n-4$ triangles.* Grünbaum also provides examples of arrangements with $n \geq 6$ pseudocircles and $2n-4$ triangles. Snoeyink and Hershberger [7] showed that every connected digon-free arrangement of n pseudocircles has at least $4n/3$ triangles. Felsner and Kriegel [2] observed that the bound from [7] also applies to non-simple intersecting digon-free arrangements and gave examples of arrangements showing that the bound is tight on this class.

In Section 2, we give counterexamples to Grünbaum's conjecture. A specific arrangement $\mathscr{N}_6$ of 6 pseudocircles appears as subarrangement in most of the known counterexamples. In Section 3, we show that $\mathscr{N}_6$ is not circularizable, i.e., representable by circles. This motivates the question, whether indeed Grünbaum's conjecture is true when restricted to intersecting arrangements of circles. In the course of the presentation, we offer some additional conjectures, e.g., in Subsection 2.1 where we discuss arrangements with digons.

In this paper (unless explicitly stated otherwise) the term *arrangement* is used as equivalent to *simple arrangement of pairwise intersecting pseudocircles.*

2 Arrangements with few Triangles

In this section, we discuss arrangements with few triangles. The main result is the following theorem, which disproves Grünbaum's conjecture.

Theorem 1 *The minimum number of triangles in digon-free arrangements of n pseudocircles is*

(i) 8 *for* $3 \leq n \leq 6$.

(ii) $\lceil \frac{4}{3}n \rceil$ *for* $6 \leq n \leq 14$.

(iii) $< \frac{16}{11}n$ *for all* $n = 11k+1$ *with* $k \in \mathbb{N}$.

The basis for this theorem was laid by exhaustive computations, which generated all simple arrangements of up to $n = 7$ pseudocircles. We generated all possible dual graphs of such arrangements, that is, the graph on the faces, where two vertices share an edge if the correspond faces share a common segment of a pseudocircle. Since counting arrangements

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is also interesting, we digress to present the enumerative results in Table 1. The four rows of the table show the number of simple pseudocircle arrangements without fixed outer cell (sphere) and with fixed outer cell (plane). In both cases, we first state the numbers when digons are allowed and then the numbers of digon-free arrangements. The arrangements and more information can be found on our website [6].

n	2	3	4	5	6	7
sphere	1	2	8	278	145 058	447 905 202
+digon-free	0	1	2	14	2 131	3 012 972
plane	1	4	45	5 108	4 598 809	?
+digon-free	0	1	5	157	63 808	132 355 602

Table 1: Number of combinatorially different arrangements of n pseudocircles.

Starting with $n = 7$, we iteratively used arrangements with n pseudocircles and a small number of triangles and digons to generate arrangements with $n+1$ pseudocircles and the same property. Using this strategy, we found the arrangements from (i) and (ii) of Theorem 1. Details can be found at [6]. From [7] we know the lower bound: Every digon-free arrangement has at least $4n/3$ triangles.

A result of the computations was that the triangle-minimizing example for $n = 6$ is unique, i.e., there is a unique simple arrangement $\mathcal{N}_6$ with 6 pseudocircles and only 8 triangles. This arrangement is a subarrangement of each of the minimizing examples for $7 \leq n \leq 9$. The claim, that indeed we found all minimizing examples in this range, is justified by Lemma 2, which allows to quantify the range of pairs (p_2, p_3) of arrangements of n pseudocircles whose extension may yield a minimizing example for $n + 1$. In particular, to get all arrangements with $n = 9$ and 12 triangles we only had to extend arrangements with $n = 7$ and $n = 8$, where $p_3 + 2p_2 \leq 12$.

Lemma 2 *For any arrangement $\mathcal{A}$ and $C \in \mathcal{A}$, we have* $p_3(\mathcal{A}) + 2p_2(\mathcal{A}) \geq p_3(\mathcal{A} - C) + 2p_2(\mathcal{A} - C)$.

Proof. Consider a triangle of $\mathcal{A} - C$. After adding C, either the triangle remains untouched, or the triangle is split into a triangle and a quadrangle, or a digon is created in the region covered by the triangle. Now consider a digon of $\mathcal{A} - C$. After adding C, either there is a digon in this region or the digon has been split into two triangles. □

It turns out that $\mathcal{N}_6$ is a subarrangement of many arrangements, that violate Grünbaum's conjecture. In Section 3, we show that $\mathcal{N}_6$ is not circularizable, i.e., there is no equivalent arrangement of circles. This property is inherited by all arrangements, that have $\mathcal{N}_6$ as a subarrangement. For the examples with less than $2n - 4$ triangles, that do not contain a subarrangement equivalent to $\mathcal{N}_6$, we could not find realizations by circles. Therefore, the following weakening of Grünbaum's conjecture may be true.

Conjecture 1 (Weak Grünbaum Conjecture) *Every digon-free arrangement of n circles has at least $2n - 4$ triangles.*

We now come to the proof of (iii) of Theorem 1. The basis of the construction is an arrangement $\mathcal{A}_{12}$ with 12 pseudocircles and 16 triangles shown in Figure 1. This arrangement will be used iteratively in a 'merge' operation as described by the following lemma.

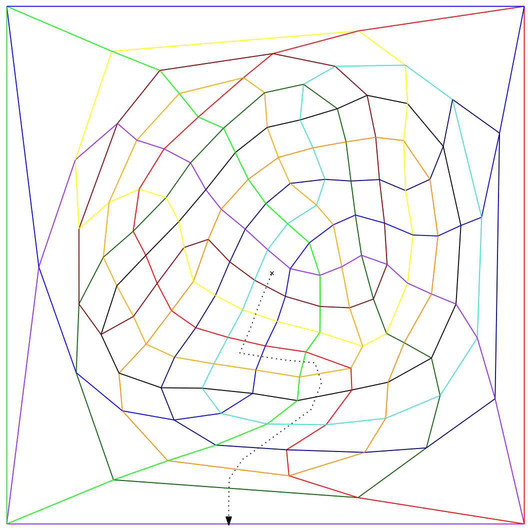

Figure 1: A digon-free, completely intersecting arrangement $\mathcal{A}_{12}$ of $n = 12$ pseudocircles with exactly 16 triangles. The dotted curve intersects every pseudocircle exactly once.

Lemma 3 *Let $\mathcal{A}$ and $\mathcal{B}$ be arrangements of $n_{\mathcal{A}}$ and $n_{\mathcal{B}}$ pseudocircles, respectively, and let $P_{\mathcal{A}}$ be a path in $\mathcal{A}$, that intersects every pseudocircle exactly once. If $P_{\mathcal{A}}$ traverses τ triangles of $\mathcal{A}$ and forms δ triangles with pairs of pseudocircles from $\mathcal{A}$, then there is an arrangement $\mathcal{C}$ of $n_{\mathcal{A}} + n_{\mathcal{B}} - 1$ pseudocircles with* $p_3(\mathcal{C}) = p_3(\mathcal{A}) + p_3(\mathcal{B}) + \delta - \tau - 1$.

Proof. Take a drawing of $\mathcal{A}$ and make a hole in the two cells, where the path $P_{\mathcal{A}}$ ends. This yields a drawing of $\mathcal{A}$ on a cylinder such that none of the pseudocircles is contractible. The path $P_{\mathcal{A}}$ connects the two boundaries of the cylinder. Now we stretch the drawing such that it becomes a narrow belt, where all intersections of pseudocircles take place in a small disk, which we call *belt-buckle*. This drawing of $\mathcal{A}$ is called a *belt drawing*. The construction is illustrated with the blue subarrangement in Figure 2.

Let B be a pseudocircle in $\mathcal{B}$ and let $\triangle$ be a triangle incident to B. Let b be the *edge* of B, which bounds $\triangle$. Specify a disk D, which is traversed by b and disjoint from all other edges of $\mathcal{B}$. Now replace B by a belt drawing of $\mathcal{A}$ in a small neighborhood of B such that the belt-buckle is drawn within D; see Figure 2.

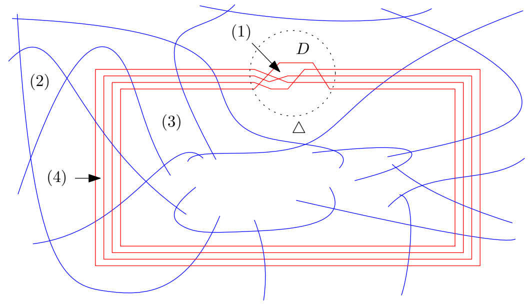

Figure 2: An illustration of the construction in Lemma 3. Pseudocircles of $\mathscr{A}$ ($\mathscr{B}$) are drawn red (blue).

The arrangement $\mathscr{C}$ obtained from the *merge* of $\mathscr{B}$ and $\mathscr{A}$ along B consists of $n_{\mathscr{A}} + n_{\mathscr{B}} - 1$ pseudocircles. Most of the cells of $\mathscr{C}$ are of one of the following four types: (1) All boundary edges belong to pseudocircles of $\mathscr{A}$. (2) All boundary edges belong to pseudocircles of $\mathscr{B}$. (3) All boundary edges but one belong to pseudocircles of $\mathscr{B}$ and the remaining edge belongs to the first or the last pseudocircle of $\mathscr{A}$ intersected by $P_{\mathscr{A}}$. These cells correspond to cells of $\mathscr{B}$ with a boundary edge on B. (4) Quadrangular cells, whose boundary edges alternatingly belong to $\mathscr{A}$ and $\mathscr{B}$.

From the cells of $\mathscr{B}$, only $\triangle$ and the other cell containing b (which is not a triangle since $\mathscr{B}$ is simple) have not been taken into account. In $\mathscr{C}$, the corresponding two cells have at least two boundary edges from $\mathscr{B}$ and at least two from $\mathscr{A}$. Consequently, neither of the two cells are triangles. The remaining cells of $\mathscr{C}$ have been created by inserting $P_{\mathscr{A}}$ into $\mathscr{A}$. To be precise, the role of $P_{\mathscr{A}}$ in these cells is taken by one of the two boundary pseudocircles of $\triangle$ other than B. There are δ triangles among these cells, but τ of these are obtained because $P_{\mathscr{A}}$ traverses a triangle of $\mathscr{A}$. All other triangles of $\mathscr{C}$ have a corresponding triangle in $\mathscr{A}$ or $\mathscr{B}$, except for $\triangle$, which does not occur in this correspondence. Altogether, there are $p_3(\mathscr{A}) + p_3(\mathscr{B}) + \delta - \tau - 1$ triangles in $\mathscr{C}$. ☐

Proof of Theorem 1(iii). We use $\mathscr{A}_{12}$, the arrangement shown in Figure 1, in the role of $\mathscr{A}$ for our recursive construction. The dotted path in the figure is used as $P_{\mathscr{A}}$ with $\delta = 2$ and $\tau = 1$. Starting with $\mathscr{C}_1 = \mathscr{A}_{12}$ and defining $\mathscr{C}_{k+1}$ as the merge of $\mathscr{C}_k$ and $\mathscr{A}_{12}$, we construct a sequence $\{\mathscr{C}_k\}_{k \in \mathbb{N}}$ of arrangements with $n(\mathscr{C}_k) = 11k + 1$ pseudocircles and $p_3(\mathscr{C}_k) = 16k$ triangles. The fraction $16k/(11k+1)$ is increasing as k increases with limit $16/11 = 1.\overline{45}$. ☐

We remark that using other arrangements from Theorem 1(ii) (which also admit a path with $\delta = 2$ and $\tau = 1$) in the recursion, we obtain arrangements with $p_3 \leq \lceil \frac{16}{11} n \rceil$ triangles for all $n \geq 6$.

Since the lower bound $\lceil \frac{4}{3} n \rceil$ is tight for $6 \leq n \leq 14$, we believe that the following is true:

Conjecture 2 *There are infinitely digon-free arrangements $\mathscr{A}$ with $p_3(\mathscr{A}) = \lceil 4n/3 \rceil$.*

2.1 Arrangements with Digons

Concerning arrangements with digons, we know of two constructions for families of arrangements with only $n-1$ triangles. An example is shown in Figure 3.

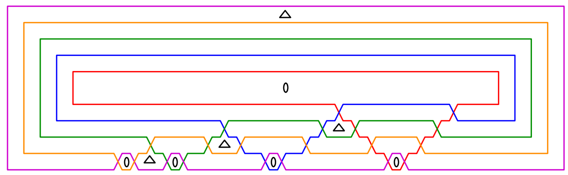

Figure 3: An illustration of an arrangement of $n = 5$ pseudocircles with n digons and $n-1$ triangles.

Using ideas based on sweeps (cf. [7]), we can show that every pseudocircle is incident to at least two triangles. This implies the following theorem:

Theorem 4 *Every arrangement of $n \geq 3$ pseudocircles has at least $2n/3$ triangles.*

Since for $3 \leq n \leq 7$ every arrangement has at least $n-1$ triangles, we believe that the following is true:

Conjecture 3 *Every arrangement of $n \geq 3$ pseudocircles has at least $n-1$ triangles.*

3 Non-circularizable Arrangements

Little is known about *circularizability*, i.e., deciding whether a given arrangement of pseudocircles is isomorphic to an arrangement of circles. Edelsbrunner and Ramos [1] proved non-circularizability of an arrangement of 6 pseudocircles with digons. Linhart and Ortner [5] found a non-intersecting arrangement of 5 pseudocircles with digons, that is non-circularizable. Kang and Müller [4] proved that all arrangements with at most 4 pseudocircles are circularizable and that deciding circularizability is NP-hard in general.

Having generated all intersecting arrangements with $n \leq 7$, we used a randomized procedure to see, which of them are realizable as circle arrangement. After realizing some remaining hard instances with $n = 5$ by hand, we now have:

Proposition 5 *The arrangement $\mathscr{N}_5$ shown in Figure 4(a) is the unique non-circularizable arrangement among the 278 equivalence classes of intersecting arrangements of $n = 5$ pseudocircles.*

Proof (Sketch). Since we have realizations of all 278 intersecting arrangements of $n = 5$ pseudocircles except $\mathscr{N}_5$, it remains to show that $\mathscr{N}_5$ is not circularizable. Suppose for a contradiction that there is

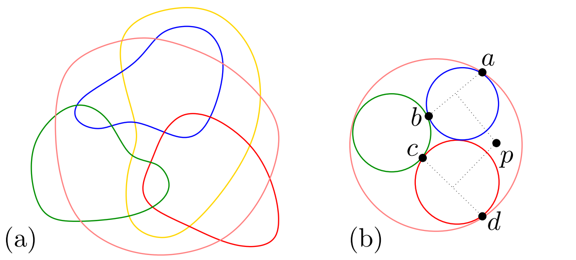

Figure 4: (a) The unique intersecting non-circularizable arrangement $\mathcal{N}_5$ of 5 pseudocircles. (b) An illustration of the proof of Proposition 5.

an equivalent arrangement $\mathscr{A}$ of circles. Shrink the red, green, and blue circle into their interior so that they touch each other and they all touch the pink circle; see Figure 4(b). Four of the touching points have been labeled. The bisectors of the chords ab and cd intersect in a point p, which is equidistant to a, b, c, and d. Hence, there is a circle C with center p, which is incident to each of the four points. Since the four labeled points are in four of the digons of $\mathscr{A}$, we know that the yellow circle of $\mathscr{A}$ has a and c in its interior but b and d in its exterior. Since on C, the counter-clockwise order of the four points is a, b, c, d, there is no circle with the properties needed for the yellow circle of $\mathscr{A}$. A contradiction. ☐

The proposition together with the work of Kang and Müller implies that all digon-free intersecting arrangements of at most 5 pseudocircles are circularizable. For $n = 6$ there are digon-free intersecting arrangements, which are non-circularizable. Figure 5 shows such an arrangement, which we denote as $\mathcal{N}_6$. The arrangement $\mathcal{N}_6$ is the unique arrangement for $n = 6$ minimizing the number of triangles, and, since $\mathcal{N}_6$ occurs as a subarrangement of every triangle-minimizing arrangement for $n = 7, 8, 9$, also neither of those arrangements is circularizable. From the 2131 digon-free intersecting arrangements of 6 pseudocircles 2128 are circularizable and 3 are not. In the following we sketch the proof of the non-circularizability of the arrangement $\mathcal{N}_6$. All realizations and the two additional non-circularizable arrangements can be found at [6].

Proposition 6 *The arrangement $\mathcal{N}_6$, as depicted in Figure 5, is non-circularizable.*

Proof (Sketch). Suppose for a contradiction that there is an equivalent arrangement of circles on the unit sphere. Choose a point in each of the eight triangles on the sphere and label them with letters as in Figure 5. Now embed $\mathbb{R}^3$ as an affine subspace into $\mathbb{R}^4$ such that a, b, c, d are mapped onto the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ in this order. On the sphere, the circle through a, b, c separates d and z. This implies that the first three components z_1, z_2, z_3 of z are positive and that z_4 is negative. Similarly, the unique negative components of w, x, y are w_1, x_2, and y_3, respectively. Next, for each circle of the arrangement, we consider the determinant of the four points in the incident triangles. E.g., for the green circle, we look at $\det(abwx)$. Geometric considerations allow us to argue that $\det(abwx)$ is positive. Therefore, $w_3x_4 > w_4x_3$. In an analogous manner, we obtain:

$$\begin{aligned}
\text{green}: &\quad \det(abwx) > 0\,; \quad w_3x_4 > w_4x_3\\
\text{red}: &\quad \det(cayw) > 0\,; \quad w_4y_2 > w_2y_4\\
\text{light blue}: &\quad \det(adwz) > 0\,; \quad w_2z_3 > w_3z_2\\
\text{pink}: &\quad \det(cbyx) > 0\,; \quad x_1y_4 > x_4y_1\\
\text{blue}: &\quad \det(bdxz) > 0\,; \quad x_3z_1 > x_1z_3\\
\text{yellow}: &\quad \det(dczy) > 0\,; \quad y_1z_2 > y_2z_1
\end{aligned}$$

The negative values do not show up in the inequalities. Moreover, if we take the product of the left-hand-sides and right-hand-sides, resp., we obtain the same value on both sides of the inequality – a contradiction. ☐

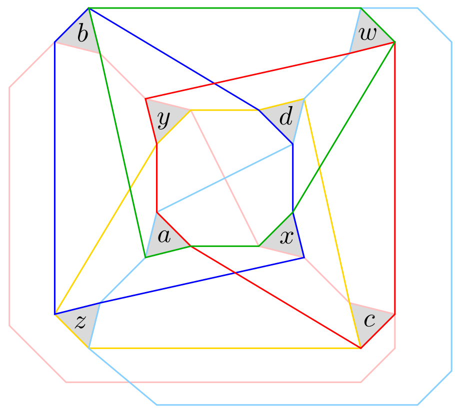

Figure 5: The unique digon-free intersecting arrangement $\mathcal{N}_6$ of 6 pseudocircles with 8 triangles. This arrangement is non-circularizable.

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