

Covering Partial Cubes with Zones

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Abstract. A partial cube is a graph having an isometric embedding in a hypercube. Partial cubes are characterized by a natural equivalence relation on the edges, whose classes are called *zones*. The number of zones determines the minimal dimension of a hypercube in which the graph can be embedded. We consider the problem of covering the vertices of a partial cube with the minimum number of zones. The problem admits several special cases, among which are the problem of covering the cells of a line arrangement with a minimum number of lines, and the problem of finding a minimum-size fibre in a bipartite poset. For several such special cases, we give upper and lower bounds on the minimum size of a covering by zones. We also consider the computational complexity of those problems, and establish some hardness results.

1 Introduction

As an introduction and motivation to the problems we consider, let us look at two puzzles.

Hitting a consecutive pair. Given a set of n elements, how many pairs of them must be chosen so that every permutation of the n elements has two consecutive elements forming a chosen pair?

Guarding cells of a line arrangement. Given an arrangement of n straight lines in the plane, how many lines must be chosen so that every cell of the arrangement is bounded by at least one of the chosen line?

While different, the two problems can be cast as special cases of a general problem involving *partial cubes*. The n -dimensional hypercube Q_n is the graph with the set $\{0, 1\}^n$ of binary words of length n as vertex set, and an edge between every pair of vertices that differ on exactly one bit. A *partial cube* G of dimension n is a

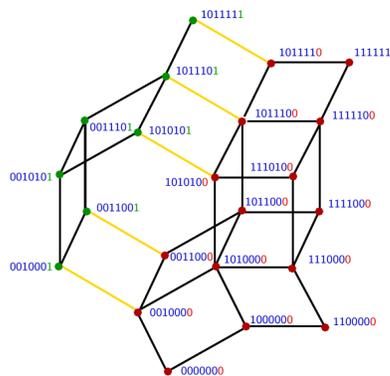


Fig. 1. A partial cube with vertex labels representing an isometric embedding in Q_7 and a highlighted zone.

A *partial cube* G of dimension n is a

subgraph of the n -dimensional hypercube with the property that the distance between two vertices in G is equal to their Hamming distance, i.e., their distance in Q_n . In general, a graph G is said to have an *isometric embedding* in another graph H whenever G is a subgraph of H and the distance between any two vertices in G is equal to their distance in H . Hence partial cubes are the graphs admitting an isometric embedding in Q_n , for some n . The edges of a partial cube can be partitioned into at most n equivalence classes called *zones*, each corresponding to one of the n directions of the edges of Q_n .

We consider the following problem:

Partial Cube Covering: *Given a partial cube, find a smallest subset S of its zones such that every vertex is incident to an edge of one of the zones in S .*

If we refer to the labeling of the vertices by words in $\{0, 1\}^n$, the problem amounts to finding a smallest subset I of $[n]$ such that for every vertex v of the input partial cube, there is at least one $i \in I$ such that flipping bit i of the word of v yields another vertex.

2 Classes of Partial Cubes

The reader is referred to the books of Ovchinnikov [16] and Imrich and Klavžar [14] for known results on partial cubes. Let us also mention that some other structures previously defined in the literature are essentially equivalent to partial cubes. Among them are *well-graded families of sets* defined by Doignon and Falmagne [7], and *Media*, defined by Eppstein, Falmagne, and Ovchinnikov [10].

We are interested in giving bounds on the minimum number of zones required to cover the vertices of a partial cube, but also in the computational complexity of the problem of finding such a minimum cover. Regarding bounds, it would have been nice to have a general nontrivial result holding for every partial cube. Unfortunately, only trivial bounds hold in general.

In one extreme case, the partial cube $G = (V, E)$ is a star, consisting of one central vertex connected to $|V| - 1$ other vertices of degree one. This is indeed a partial cube in dimension $n = |V| - 1$, every zone of which consists of a single edge. Since there are n vertices of degree one, all n zones must be chosen to cover all vertices. In the other extreme case, the partial cube is such that there exists a single zone covering all vertices. This lower bound is attained by the hypercube Q_n .

Table 1 gives a summary of our results for the various families of partial cubes that we considered. For each family, we consider upper and lower bounds and complexity results. We now briefly describe the various families we studied. The proofs of the new results are in the following sections.

2.1 Hyperplane Arrangements

The dual graph of a simple arrangement of n hyperplanes in \mathbb{R}^d is a partial cube of dimension n . The partial cube covering problem becomes the following: given

a simple hyperplane arrangement, find a smallest subset S of the hyperplanes such that every cell of the arrangement is bounded by at least one hyperplane in S .

The case of line arrangements was first considered in [3]. In fact it was this problem that motivated us to investigate the generalization to partial cubes. The best known lower and upper bounds for line arrangements on $|S|$ are of order $n/6$ and $n - O(\sqrt{n \log n})$ (see [1]). The complexity status of the optimization problem is unknown.

Instead of lines and hyperplanes it is also possible to consider Euclidean or spherical arrangements of pseudo-lines and pseudo-hyperplanes, their duals are still partial cubes. Actually, all the cited results apply to the case of arrangements of pseudo-lines.

Spherical arrangements of pseudo-hyperplanes are equivalent to oriented matroids, this is the Topological Representation Theorem of Folkman and Lawrence. The pseudo-hyperplanes correspond to the elements of the oriented matroid and the cells of the arrangement correspond to the topes of the oriented matroid. Hence our covering problem asks for a minimum size set C of elements such that for every tope T there is an element $c \in C$ such that $T \oplus c$ is another tope. For more on oriented matroids we refer to [2].

2.2 Acyclic Orientations

From a graph $G = (V, E)$, we can define a partial cube H in which every vertex is an acyclic orientation of G , and two orientations are adjacent whenever they differ by a single edge reversal (*flip*). This partial cube has dimension equal to the number of edges of G . It is also the dual graph of the arrangement of the $|E|$ hyperplanes of equation $x_i = x_j$ for $ij \in E$ in $\mathbb{R}^{|V|}$. Every cell of this arrangement corresponds to an acyclic orientation, and adjacent cells are exactly those that differ by a single edge. This observation, in particular, shows that H is connected. The graph H and the notion of edge flippability have been studied by Fukuda et al. [11] and more recently by Cordovil and Forge [6].

The partial cube covering problem now becomes the following: given a graph $G = (V, E)$, find a smallest subset $S \subseteq E$ such that for every acyclic orientation of G , there exists $e \in S$ such that flipping the orientation of e does not create a cycle.

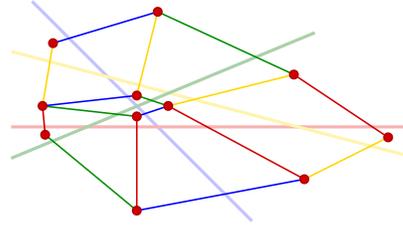


Fig. 2. An arrangement of 4 lines superimposed with its dual.

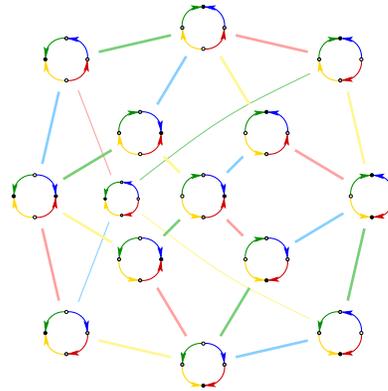


Fig. 3. Flip graph of acyclic orientations of the 4-cycle.

2.3 Median Graphs

A median graph is an undirected graph in which every three vertices x , y , and z have a unique *median*, i.e., a vertex $\mu(x, y, z)$ that belongs to shortest paths between each pair of x , y , and z . Median graphs form structured subclass of partial cubes that has been studied extensively, see [5] and references therein. Since the star and the hypercube are median graph there are no non-trivial upper or lower bounds for the zone cover problem on this class. We use a construction of median graphs to prove hardness of the zone cover problem.

2.4 Distributive Lattices

Cover graphs of distributive lattices are partial cubes. From Birkhoff's representation theorem (a.k.a. Fundamental Theorem of Finite Distributive Lattices) we know that there is a poset P such that the vertices of the partial cube are the downsets of P , the zones of the partial cube in turn correspond to the elements of P .

The problem becomes: given a poset P , find a smallest subset S of its elements such that for every downset D of P , there exists $v \in S$ such that either $D \cup \{v\}$ or $D \setminus \{v\}$ is a downset, distinct from D .

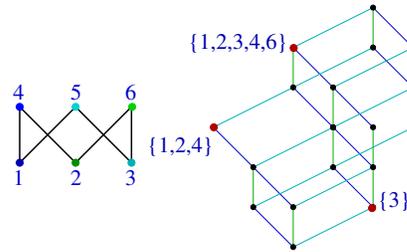


Fig. 4. A poset (left) and its lattice of down-sets (right).

2.5 Trees

Trees with n edges are partial cubes of dimension exactly n . Since every zone contains exactly one edge, the partial cube covering problem on trees boils down to the edge cover problem on trees. There are instances attaining the lower and upper bounds of $n/2$ and $n-1$, moreover, there is a simple dynamic programming algorithm that computes an optimal cover in linear time.

3 Line Arrangements

We first give a lower bound on the size of a guarding set for an arrangement of lines.

Theorem 1. *The minimum number of lines needed to guard the cells of any arrangement of n lines is $n - o(n)$.*

Proof. The proof is a direct consequence of known results on the following problem from Erdős: given a set of points in the plane with no four points on a line, find the largest subset in general position, that is, with no three points on a line. Let $\alpha(n)$ be the minimum size of such a set over all arrangements of n points.

Partial Cube	Lower bound	Upper bound	Complexity
Arrangements of n lines in \mathbb{R}^2	$n - o(n)$ (Thm. 1)	$n - \Omega(\sqrt{n \log n})$ [1]	–
Arrangements of n hyperplanes in \mathbb{R}^d	–	$n - \Omega(n^{1/d})$ (Thm. 2)	–
Acyclic orientations	–	Minimum edge cut (Thm. 4)	Recognition is coNP-complete, even for complete graphs (Thm. 3)
Median graphs	1	n	NP-complete (Cor. 2), APX-hard (Cor. 3)
Distributive lattices with representative poset of n elements	–	$2n/3$ (Cor. 4)	Recognition is coNP-complete (Cor. 5) Σ_2^P -complete (Cor. 6)
Trees with n edges	$n/2$	$n - 1$	P (min edge cover)

Table 1. Worst-case bounds and complexities for the special cases of the partial cube covering problem. When used, n denotes the dimension of the partial cube.

Füredi observed [12] that $\alpha(n) = o(n)$ follows from the density version of the Hales-Jewett theorem [13]. But this directly proves that we need at least $n - o(n)$ lines to guard all cells of an arrangement. The reduction is the one observed by Ackerman et al. [1]: consider the line arrangement that is dual to the point set, and slightly perturb it so that each triple of concurrent lines forms a cell of size three. Now the complement of any guarding set is in general position in the primal point set. \square

We now show that for arrangements of hyperplanes in \mathbb{R}^d , with $d = O(1)$, there always exists a guarding set of size at most $n - \Omega(n^{1/d})$. This generalizes a result for $d = 2$ obtained in [3].

Theorem 2. *In every arrangement of hyperplanes in general position in \mathbb{R}^d (with $d = O(1)$), there exists a subset of the hyperplanes of size at most $n - \Omega(n^{1/d})$ such that every cell is bounded by at least one hyperplane in the subset.*

Proof. We will prove that every such arrangement has an independent set of size $\Omega(n^{1/d})$, where an independent set is defined as the complement of a guarding set, that is, a subset of the hyperplanes such that no cell is bounded by hyperplanes of the subset only.

Let H be a set of n hyperplanes, and consider an arbitrary, inclusionwise maximal independent set I . For each hyperplane $h \in H \setminus I$, there must be a cell c_h of the arrangement that is bounded by a set of hyperplanes $C \cup \{h\}$ with $C \subseteq I$, since otherwise we could add h to I , and I would not be maximal. If c_h has size at least $d + 1$, there must be a vertex of the arrangement induced by I that is also a vertex of c_h . We charge h to this vertex. Note that each vertex can only be charged $2^d = O(1)$ times. If c_h has size d , we charge h to the remaining

$(d - 1)$ -tuple of hyperplanes of I bounding c_h . This tuple can also be charged at most $O(1)$ times. Therefore

$$|H \setminus I| = n - |I| \leq 2^d \cdot O(|I|^d) + O(|I|^{d-1}) \quad (1)$$

$$|I| = \Omega(n^{1/d}). \quad (2)$$

□

4 Acyclic Orientations

Given a graph $G = (V, E)$ we wish to find a subset $S \subseteq E$ such that for every acyclic orientation of G , there exists a *flippable* edge $e \in S$, that is, an edge $e \in S$ such that changing the orientation of e does not create any cycle. Let us call such a set a *guarding set* for G . Note that an oriented edge $e = uv$ in an acyclic orientation is flippable if and only if it is not *transitive*, that is, if and only if uv is the only oriented path from u to v .

4.1 Complexity

Theorem 3. *Given a graph $G = (V, E)$ and a subset $S \subseteq E$ of its edges, the problem of deciding whether S is a guarding set is coNP-complete, even if G is a complete graph.*

Proof. The set S is not a guarding set if and only if there exists an acyclic orientation of G in which all edges $e \in S$ are transitive.

Consider a simple graph H on the vertex set V , and define G as the complete graph on V , and S as the set of non-edges of H . We claim that S is a guarding set for G if and only if H does not have a Hamilton path. Since deciding the existence of a Hamilton path is NP-complete, this proves the result.

To prove the claim, first suppose that H has a Hamilton path, and consider the acyclic orientation of G that corresponds to the order of the vertices in the path. Then by definition, no edge of S is in the path, hence all of them are transitive, and S is not a guarding set. Conversely, suppose that S is not guarding. Then there exists an acyclic orientation of G in which all edges of S are transitive. The corresponding ordering of the vertices in V yields a Hamilton path in H . □

4.2 Special cases and an upper bound

Lemma 1. *The minimum size of a guarding set in a complete graph on n vertices is $n - 1$.*

Proof. First note that there always exists a guarding set of size $n - 1$ that consists of all edges incident to one vertex.

Now we need to show that any other edge subset of smaller size is not a guarding set. This amounts to stating that every graph with at least $\binom{n}{2} - (n - 2)$

edges has a Hamilton path. To see this, proceed by induction on n . Suppose this holds for graphs with less than n vertices. Consider a set S of at most $n - 2$ edges of the complete graph. Let u, v be two vertices with $uv \in S$. One of the two vertices, say v , is incident to at most $\lfloor (n - 2)/2 \rfloor$ edges of S . Consider a Hamilton path on the $n - 1$ vertices $\neq v$, which exists by induction hypothesis. Then vertex v must have one or two incident edges that do not belong to S and connect v to the first, or the last, or two consecutive vertices of this path. Hence we can integrate v in the Hamilton path. \square

Since acyclic orientations of the complete graph K_n correspond to the permutations of S_n , this is in fact the solution to the puzzle “hitting a consecutive pair” in the introduction. The dual graph of the arrangements of hyperplanes corresponding to the complete graph is known as the permutohedron. Hence the above result can also be stated in the following form.

Corollary 1. *The minimum size of a set of zones covering the vertices of the n -dimensional permutohedron is $n - 1$.*

We now give a simple, polynomial-time computable upper bound on the size of a guarding set. A set $C \subseteq E$ is an *edge cut* whenever the graph $(V, E \setminus C)$ is not connected.

Theorem 4. *Every edge cut of G is a guarding set of G .*

Proof. Consider an edge cut $C \subseteq E$ and an acyclic orientation A_G of G . This acyclic orientation can be used to define a partial order on V . Let us consider a total ordering of V that extends this partial order, and pick an edge $e = uv \in C$ that minimizes the rank difference between u and v . We claim that e is flippable. Suppose for the sake of contradiction that e is not flippable. Then e must be transitive and there exists a directed path P in A_G between u and v that does not use e . Since C is a cut, u and v belong to distinct connected components of $(V, E \setminus C)$, and P must use another edge $e' \in C$. By definition, the endpoints of e' have a rank difference that is smaller than that of u and v , contradicting the choice of e . \square

An even shorter proof of the above can be obtained by reusing the following observation from Cordovil and Forge [6]: for every acyclic orientation of G , the set of flippable edges is a spanning set of edges. Therefore, every such set must intersect every edge cut.

While guarding sets are hard to recognize, even for complete graphs, we show that a minimum-size guarding set can be found in polynomial time whenever the input graph is chordal. In that case, the upper bound given by the minimum edge cut is tight, and the result generalizes Lemma 1.

Theorem 5. *The minimum size of a guarding set in a chordal graph is the size of a minimum edge cut.*

Proof. We need to show that whenever a set S of edges of a chordal graph G has size strictly less than the edge connectivity of G , there exists an acyclic orientation of G in which all edges of S are transitive. Let us denote by k the edge connectivity of G , and n its number of vertices.

We proceed by induction. The base case consists of a complete graph on $k+1$ vertices. From Lemma 1, the minimum size of a guarding set is k . Now suppose the statement holds for every chordal graph with $n-1$ vertices, and that there exists a k edge connected chordal graph G on n vertices with a guarding set S of size $k-1$. Since G is chordal, it has at least two nonadjacent simplicial vertices u and v . The degree of both u and v is at least k . Hence one of them, say v , is incident to at most $\lfloor (k-1)/2 \rfloor \leq \lfloor (d(v)-1)/2 \rfloor$ edges of S . Now remove v and consider, using the induction hypothesis, a suitable acyclic orientation of the remaining graph. This orientation induces a total order, i.e., a path p with $d(v)$ vertices, on the neighbors $N(v)$ of v . Then vertex v must have one or two incident edges that do not belong to S and connect v to the first, or the last, or two consecutive vertices of path p . Hence we can integrate v in the path such that all the edges of S that are incident to v are transitive. This yields a suitable acyclic orientation for G and completes the induction step. \square

When the graph is not chordal, it may happen that the minimum size of a guarding set is arbitrarily small compared to the edge connectivity. We can in fact construct a large family of such examples.

Theorem 6. *For every natural number $t \geq 2$ and odd natural number g such that $3 \leq g \leq t$, there exists a graph G with edge connectivity t and a guarding set of size g .*

Proof. The graph G is constructed by considering a wheel with $g+1$ vertices and center c , and replacing every edge incident to the center c by a copy of the complete graph K_{t-1} such that each vertex of the complete graph is connected to both endpoints of the original wheel edge. Figure 5 shows an example. Let us call C_g the cycle induced by the non-center vertices.

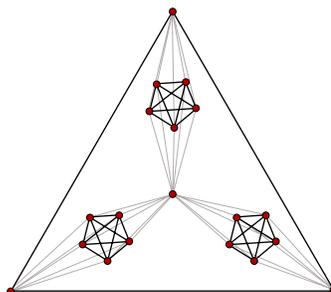


Fig. 5. A graph G with edge connectivity 6 and a guarding set of size 3.

Let us first look at the edge connectivity of G . Vertices belonging to one of the K_{t-1} have t incident edges. On the other hand we easily construct t edge-disjoint paths between every pair of vertices. Hence, the edge connectivity of G equals t .

We now show that the set of vertices of C_g is a guarding set.

Suppose the opposite: there exists an acyclic orientation for which no edge of the cycle is flippable, hence all of them are transitive. Since C_g is odd, there must exist two consecutive edges of C_g with the same orientation, say uv and vw , with uv oriented from u to v , and vw from v to w .

The only way to make vw transitive is to construct an oriented path from v to w going through the center c . Hence there must exist an oriented path of the form vPc , where P is a path in the complete graph K_{t-1} attached to v . To make uw transitive we need a directed path $cP'v$. The oriented cycle $vPcP'v$ is in contradiction to the acyclicity. Therefore, some edge of C_g is always flippable. \square

5 Median Graphs

A median graph is an undirected graph in which every three vertices x, y , and z have a unique *median*, i.e., a vertex $\mu(x, y, z)$ that belongs to shortest paths between each pair of x, y , and z . Imrich, Klavžar, and Mulder [15] proposed the following construction of median graphs: Start with any triangle-free graph $G = (V, E)$. First add an apex vertex x adjacent to all vertices of V , then subdivide every edge of E once, and let G' be the resulting graph. Figure 6 illustrates the construction.

We only need that G' is a partial cube. This can be shown with the explicit construction of an isometric embedding into Q_n . Let $V = \{v_1, \dots, v_n\}$ and map these vertices bijectively to the standard basis, i.e., $v_i \rightarrow e_i$. Apex x is mapped to 0 and the subdivision vertex w_{ij} of an edge $v_i v_j$ is mapped to $e_i + e_j$. The embedding shows that the zones are in one-to-one correspondence with the vertices of G , where the zone of v_i consists of the edge xv_i together with all edges $w_{ij}v_j$. Hence, a zone cover of G' corresponds to a subset of $V(G)$. Let us call w_e the vertex of G' subdividing the edge $e \in E(G)$. The construction of the embedding into Q_n shows that the transformation $G \rightarrow G'$ yields a partial cube even if G contains triangles, only membership in the smaller class of median graphs is lost in this case.

Lemma 2. *If G has no isolated vertices, then the minimum size of a zone cover of G' equals the minimum size of a minimum vertex cover of G .*

Proof. We first show that a vertex cover S of G corresponds to a zone cover of G' of the same size. For a vertex w_{ij} at least one of v_i and v_j is in S , hence w_{ij} is covered. If v_i belongs to S then it is covered by its own zone. Otherwise there is an edge $v_i v_j$ and necessarily $v_j \in S$, therefore in this case v_i is covered by the zone of v_j . The apex x is covered by every zone.

The other direction is straightforward. A zone cover of G' must in particular cover all subdivision vertices $w_e \in V(G')$. Hence the zone corresponding to at least one of the endpoint of e must be selected, yielding a vertex cover in G . \square

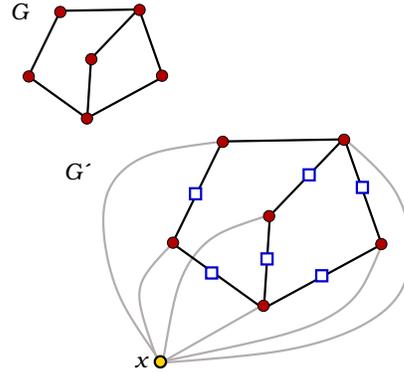


Fig. 6. A graph G and the G' obtained by the construction.

Given a connected triangle-free graph G , one can construct the median graph G' in polynomial time. Since deciding whether G has a vertex cover of size at most k is NP-complete, even on triangle-free graphs, we obtain:

Corollary 2. *Given a median graph G' and an integer k , deciding whether there exists a zone cover of size at most k of G' is NP-complete.*

The minimum vertex cover problem is hard to approximate, even on triangle-free graphs. This directly yields the following corollary.

Corollary 3. *Given a median graph G' , finding a minimum size zone cover of G' is APX-hard.*

6 Distributive Lattices

In this special case of the partial cube covering problem, we are given a poset P , and we wish to find a subset S of its elements such that the following holds: for every downset D of P , there exists $x \in S$ such that either $D \cup \{x\}$ or $D \setminus \{x\}$ is a downset, distinct from D . Given a poset P , we refer to a suitable set S as a *guarding set* for P , and let $g(P)$ be the size of a smallest guarding set. Figure 4 gives an example of poset and the corresponding partial cube.

6.1 Relation to poset fibres

We first establish a connection between guarding sets and fibres. A *fibre* of a poset is a subset of its elements that meets every nontrivial maximal antichain. Let $f(P)$ be the size of a smallest fibre of P .

Lemma 3. *Every fibre is a guarding set. In particular, $g(P) \leq f(P)$.*

Proof. Consider a poset $P = (V, \leq)$ and one of its downset $D \subseteq V$. Let $F := V \setminus D$, $A := \max(D)$, and $B := \min(F)$. By definition, a guarding set is a hitting set for the collection of subsets $A \cup B$ constructed in this way.

Let us now consider the subset $A \cup B' \subseteq A \cup B$, where $B' := \{b \in B : b \text{ is incomparable to } a, \forall a \in A\}$. This set is easily shown to be a maximal antichain. Hence, it must be hit by any fibre. \square

Duffus, Kierstead, and Trotter [8] have shown that every poset on n elements has a fibre of size at most $2n/3$. This directly yields the following.

Corollary 4. *For every n -element poset P , $g(P) \leq 2n/3$.*

We now consider a special case for which the notions of guarding set and fibre coincide. A poset is bipartite if $P = \min(P) \cup \max(P)$, i.e., if the height of P is at most 2.

Lemma 4. *For a bipartite poset P , a set S is a guarding set for P if and only if it is a fibre of P .*

Proof. We know from Lemma 3 that every fibre is a guarding set. The other direction is as follows.

Consider a guarding set S and let A be any maximal antichain of P . Let $T := A \cap \max(P)$, and let D be the downset generated by T . As a downset D is guarded, hence either an element of T is hit by S , or an element of $\min(P \setminus D)$ is hit, but since A is maximal $A = T \cup \min(P \setminus D)$, hence A is hit. Therefore, S is a fibre. \square

6.2 Complexity

Lemma 4 yields two interesting corollaries on the complexity of recognizing and finding guarding sets.

Corollary 5. *Given a poset P and a subset S of its elements, the problem of deciding whether S is a guarding set is coNP-complete. This holds even if P is bipartite.*

Proof. Recognition of fibres in bipartite posets has been proved coNP-complete by Duffus et al. [8]. From Lemma 4, this is the same problem as recognizing guarding sets. \square

Corollary 6. *Given a poset P and an integer k , the problem of deciding whether there exists a guarding set of size at most k is Σ_2^P -complete. This holds even if P is bipartite.*

Proof. Again, this is a consequence of Lemma 4 and a recent result of Cardinal and Joret [4] showing that the corresponding problem for fibres in bipartite posets is Σ_2^P -complete. \square

Duffus et al. [9] mention that it is possible to construct posets on $15n + 2$ elements, every fibre of which must contain at least $8n + 1$ elements, which gives a lower bound with a factor $8/15$. The guarding sets for these examples do not seem to require as many elements.

Open Problems

We left a number of problems open. For instance, we do not know the complexity status of the problem of deciding whether there exists a guarding set of edges of size at most k in a given graph. A natural candidate class would be Σ_2^P . For the same problem, we do not have any nontrivial lower bound on the minimum size of a guarding set. It would also be interesting to give tighter lower and upper bounds on the minimum size of a guarding set in a poset and in a line arrangement. Finally, questions involving partial cubes derived from antimatroids, such as elimination orderings in chordal graphs, are currently under investigation.

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