The graphs that can be drawn with one bend per edge

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Abstract

We provide a precise description of the class of graphs that admit a one-bend drawing, i.e., an orthogonal drawing with at most one bend per edge. The main tools for the proof are Eulerian orientations of graphs and discrete harmonic functions.

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1 Introduction

The term *d-dimensional orthogonal drawing* traditionally denotes a drawing of a graph in which vertices are placed at distinct points of the d-dimensional integer lattice and edges are represented by chains of axis-parallel segments. Orthogonal drawings and variations are classical topics in graph drawing. The discrete nature of the model makes orthogonal drawings accessible for tools from combinatorial optimization. Orthogonal drawings are also related to various applications ranging from circuit layout to information visualization.

Planar graphs with $\Delta \leq 4$ admit crossing-free 2-dimensional orthogonal drawings. Tamassia's seminal paper [19] is about the minimization of bends of such a drawing. For non-planar graphs, authors mainly worked on area minimization allowing a constant number of bends

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per edge [14]. Generalizations have been made in various directions, e.g. for higher degree graphs by representing vertices as boxes [1, 2, 7], incremental drawings [13, 6] and simple faces [9, 16, 15]. On 3-dimensional orthogonal drawings, there is a less extensive literature, most prominent are [4, 3].

Here we are interested in 2-dimensional orthogonal drawings of graphs with maximal degree $\Delta \leq 4$ and vertices represented as points. We only deal with 2-dimensional orthogonal drawings. For simplicity we refer to them as *orthogonal drawings*.

Every graph with $\Delta \leq 4$ has a orthogonal drawing. A *good* drawing, however, should be compact and readable. Therefore drawing algorithms are usually compared with respect to the drawing area and the number of bends. Optimizing either of these two parameters is NP-hard. This was shown for the area in [11, 5] and for the bend number in [18]. Several constructions of orthogonal drawings have been proposed e.g. by Schäffter [17] or by Eades, et al. [4] who describe an algorithm which draws a graph of maximal degree 4 in a box of dimensions $O(n) \times O(n) \times O(1)$ with 3 bends per edge. From work of Biedl and Kant [1] and Papakostas and Tollis [14] it follows that graphs with n vertices and $\Delta \leq 4$ admit orthogonal drawings with area $0.76n^2$, at most two bends per edge and a total of at most $2n + 2$ bends. The problem has been reconsidered recently from a different view point, namely from the requirement of orthogeodesic edge routing, where all edges are required to be monotone [10, 8].

We investigate which graphs admit an orthogonal drawing with at most one bend per edge, for the sake of brevity we call such drawings *one-bend drawings*. The paper by Biedl and Kaufmann [2] provides a general framework for such one-bend drawings but it focuses on the more general case of higher degrees where the vertices are represented by boxes. In [20], their algorithm has been analyzed more closely and extended by several heuristics. Here we focus on the case of vertices represented as points, which restricts the graphs to be of maximal degree 4. Theorem 2.3 gives a full characterization of maximal graphs admitting a one-bend drawing, The construction of the embedding is based on discrete harmonic functions, a tool that has not been used before in this area. Theorem 3.1 extends the characterization so that it includes non-maximal graphs.

Let $G = (V, E)$ be a graph with maximum degree $\Delta = 4$. Suppose that there is an orthogonal drawing of G such that every edge has at most one bend. If v is a vertex and edge (v, w) is leaving v towards the north, i.e., constant x-coordinate and increasing y-coordinate, then w is further to the north than v, i.e., $v_x < w_x$. Corresponding statements hold for edges leaving a vertex towards the other directions.

Let $S \subset V$ be a set of vertices. Consider the subgraph $G[S] = (V, E[S])$ of G induced by this set and its induced one-bend drawing. From the above we see that in $G[S]$ the northernmost vertex of S has no edge towards the north. With the respective statements for the south, east and west extreme vertices we obtain that the sum of degrees of vertices in $G[S]$ can be at most $4|S| - 4$. This yields the following necessary condition for the existence of one-bend drawings:

Proposition 1.1. *If $G = (V, E)$ has a one-bend drawing and $S \subseteq V$, then $|E[S]| \leq 2|S| - 2$.*

The assertion of Theorem 3.1 will be that the degree condition $\Delta \leq 4$ together with the condition on edge densities given in Proposition 1.1 yields a sufficient set of conditions for the existence of a one-bend drawing.

2 The four-regular case

Let $G = (V, E)$ be a connected four-regular graph. We even allow multiple edges but Proposition 1.1 implies that in interesting cases the multiplicity of edges is at most two.

From Proposition 1.1 we know that there is no one-bend drawing for G. However if we specify any vertex v_∞ let $G' = G[V \setminus \{v_\infty\}]$, then G' may have a one-bend drawing. Such a drawing of G' can also be interpreted as a one-bend drawing of G with v_∞ placed at ∞. We refer to such a drawing of G as an *extended one-bend drawing*. Figure 1 shows an example.

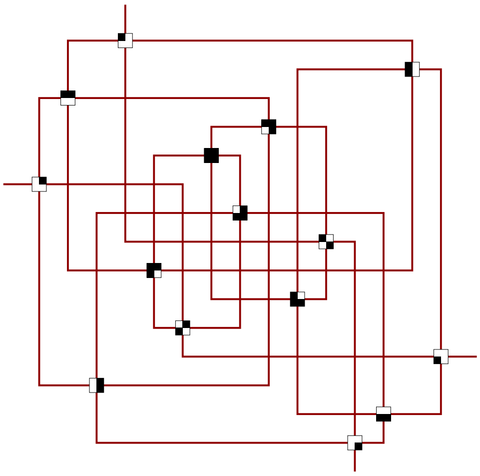

Figure 1: An extended one-bend representation of the four dimensional cube.

We can use an extended one-bend drawing of G which has the extra property that every edge has exactly one bend to define an orientation D_G of G according to the rule:

- edge vw is oriented as $v \to w$ iff in the drawing the edge is leaving v horizontally, i.e., to the east or to the west.

Every edge has a unique orientation in D_G because by assumption every edge has exactly one bend in the drawing. Moreover, the same orientation can be defined in terms of vertical segments as: edge vw is oriented as $v \to w$ iff in the drawing the edge is leaving w vertically. Together the rules imply a unique orientation for the edges incident to v_∞.

The orientation D_G has the property that $\text{in-deg}(v) = 2$ and $\text{out-deg}(v) = 2$ for every vertex v of G, hence it is an Eulerian orientation.

Our construction of extended one-bend drawings starts with an Eulerian orientation O_G of a four-regular graph G with a special vertex v_∞. We then aim at constructing a one-bend drawing such that the orientation D_G defined on G according to the above rule equals the Eulerian orientation O_G.

Given an Eulerian orientation O_G we identify

- v_L and v_R, these are the two vertices with edges $v_L \to v_\infty$ and $v_R \to v_\infty$, the indices L and R are assigned arbitrarily. We call v_L and v_R the *horizontal poles*.
- For $v \in V \setminus \{v_L, v_R, v_\infty\}$ we let B_v be the set of vertices w such that in O_G there is an edge $v \to w$. Clearly $|B_v| = 2$ and in an one-bend drawing corresponding to O_G vertex

v has to be horizontally between the elements of B_v, i.e., if $B_v = \{w', w''\}$ the either $w'_x < v_x < w''_x$ or $w'_x > v_x > w''_x$. This is the *horizontal betweenness condition* for v.

The Eulerian orientation also provides two vertical poles v_T and v_B and a vertical betweenness condition for all $v \in V \setminus \{v_T, v_B, v_\infty\}$. We now focus on solving the horizontal betweenness problem.

Although the adequate representation of a solution of a betweenness problem is a permutation of the vertices we model the problem as a continuous one. The advantage is that in the continuous formulation we can use ideas from spring-embedding and solve the problem with the aid of discrete harmonic functions (cf. [12]). Consider the following system of linear equations in the variables x_v:

(H) $x_v = \frac{1}{2}(x_{w'} + x_{w''})$ whenever $B_v = \{w', w''\}$, and $x_{v_L} = 0$, and $x_{v_R} = 1$.

Lemma 2.1. *The system (H) has a unique solution.*

Proof. The system has as many equations as it has variables. We claim that the homogeneous system only has the trivial solution. From the claim it follows that there exists a unique solution for any right hand side.

Suppose that (z_v) is a non-trivial solution of the corresponding homogeneous system. Consider a variable of maximal absolute value $|z_u| \neq 0$ and let $A = \{v \in V : |z_v| = |z_u|\}$. Since G is connected there is some edge connecting A to $V \setminus A \supseteq \{v_L\}$. Since O_G is Eulerian there is an edge $v \to w'$ with $v \in A$ and $w' \in V \setminus A$. If $v \to w''$ is the other out-edge of v, then $|z_v| > \frac{1}{2}(|z_{w'}| + |z_{w''}|) \geq |\frac{1}{2}(z_{w'} + z_{w''})|$, a contradiction. □

The solution of system (H) does not necessarily give solution of the betweenness problem. Indeed a solution of the system may clump a set of vertices at a single position. Such a clumping can be accidental (resolvable by a perturbation $x_v = \frac{1}{2}((1+\varepsilon)x_{w'} + (1-\varepsilon)x_{w''})$ of the equations) or the clumping can be essential (this happens e.g. if v_L is a cut vertex and one component is fixed at 0). To exclude essential clumpings we need a little more than mere connectivity.

Let $S \subset V$ be a set of vertices. A *pole* of S is a vertex $v \in S$ such that in O_G there is an edge $v \to w$ with $w \notin S$. Note that v_L, v_R are the poles of $V' = V \setminus \{v_\infty\}$.

Proposition 2.2. *The betweenness problem has a solution* $\iff$ *every subset* $S \subset V'$ *with* $|S| > 1$ *has at least two poles.*

Proof. Suppose a permutation π of V is a solution to the betweenness problem. It follows immediately from the definitions that the leftmost and the rightmost vertex of S with respect to π are poles of S.

Now suppose that every subset $S \subset V'$ with $|S| > 1$ has at least two poles. We consider solutions of perturbed systems (H_ϵ) were the equations of the vertices perturbed by independent parameters ε. Consider a solution (x_v) of a perturbed problem where the number of pairs of vertices sharing a position is minimized. Suppose that this solution has a clump A at a, i.e., $A = \{v : x_v = a\}$ and $|A| \geq 2$. Since A has at least two poles and at least one of v_L, v_R is not in A there is an edge $v \to w$ leaving A. If for all edges $v \to w$ leaving A we have $x_v > x_w$ we get a contradiction because if $v \to w''$ is the other out-edge of v, then $\frac{1}{2}((1+\varepsilon)x_w + (1-\varepsilon)x_{w''}) > a = x_v$. Basically the same argument shows that if $v \in A$ has an edge $v \to w$ with $x_v > x_w$, then the other out-edge $v \to w''$ has $x_v < x_{w''}$. Increasing the parameter ε of the equation of v by a small $\delta > 0$ will move v slightly to the left. This resolves the clump A. By choosing δ small enough we can make sure that the effect of moving v on other vertices will not form new clumps. The contradiction shows that there is a perturbed system (H_ϵ) such that the solution has no clumps. $\square$

Theorem 2.3. *A four-regular graph $G = (V, E)$ with a designated vertex $v_\infty \in V$ admits a extended one-bend drawing iff $|E[S]| \leq 2|S| - 2$ for all $S \subset V$.*

Proof. The necessity of the density condition for $S \subseteq V'$ was shown in Proposition 1.1. Since G is 4-regular the condition for S with $v_\infty \in S$ is implied by the condition for the complement.

To prove sufficiency we choose an Eulerian orientation O_G of G such that with respect to O_G every $S \subset V'$ with $|S| > 1$ has at least two poles. In fact we want that the two poles condition also holds for the reverse orientation $\overline{O_G}$. In Proposition 2.4 below we show that such an Eulerian orientation O_G of G exists.

Proposition 2.2 implies that the horizontal betweenness problem associated with O_G has a solution. Let $\pi_x : V' \to \{1, .., |V'|\}$ be the ordering of the vertices obtained by sorting them according to the x_v values.

The two poles condition for $\overline{O_G}$ allows to use Proposition 2.2 again and infer the existence of a solution y_v of the vertical betweenness associated with O_G. Let π_y be the ordering of the vertices obtained by sorting them according to the y_v values.

Let $n = |V'|$. We now can safely place the vertices of G' on integral points of the $n \times n$ grid $v \to (\pi_x(v), \pi_y(v))$ and draw the edges with one bend. The horizontal and vertical betweenness conditions guarantee that there is no conflict of direction. $\square$

Proposition 2.4. *A four-regular multi-graph G with $|E[S]| \leq 2|S| - 2$ for all $S \subset V$ has an Eulerian orientation O_G such that with respect to O_G and $\overline{O_G}$ every $S \subset V$ with $|S| > 1$ has at least two poles.*

Proof. Note that the condition implies that G is 4-edge connected. Hence, if O is an Eulerian orientation and $S \subset V$, then there are at least two edges leaving S. If there are two such edges with different tail-vertices, then there are two poles in S. Conversely, if S with $|S| > 1$ has only one pole in O, then there are only four edges in the cut $E[S, \overline{S}]$ and the two edges in O pointing from S to $\overline{S}$ have the same tail $v \in S$. Note that in this case the other two edges incident to v belong to $E[S]$.

We call a vertex v *dangerous* if v has four different neighbors w_1, w_2, w_3, w_4 and there exists some $S \subset V$ with $|E[S, \overline{S}]| = 4$, $v, w_1, w_2 \in S$, and $w_3, w_2 \in \overline{S}$. In the first step of the construction of an Eulerian orientation with the desired properties we take dangerous vertices and duplicate them, i.e., if v is dangerous with neighbor set w_1, w_2, w_3, w_4 as above, then we replace v and its four edges by vertices v' and v'' and edges $(w_1, v'), (w_2, v'), (w_3, v''), (w_4, v'')$ and a double-edge connecting v' and v'', see Figure 2.

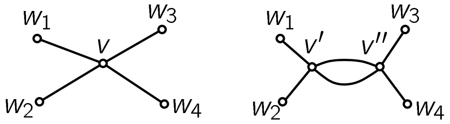

Figure 2: The replacement of a dangerous vertex.

The replacement of a dangerous vertex reduces the number of vertices with four different neighbors the step ends with a four-regular graph G_1^+ that has no dangerous vertex. Remove all double edges from G_1^+, this yields G_2^+. Since all vertices of G_2^+ are of degree 4 or 2 or 0 there is an Eulerian orientation O^+ of G_2^+. Merging pairs of vertices v', v'' that were created by the replacement of a dangerous vertex we recover the original graph G. Let O_G be the orientation of G inherited from O^+. The orientation O_G is Eulerian. Moreover, it has the property that every $S \subset V$ with $|S| > 1$ has at least two poles with respect to O_G as well as with respect to $\overline{O_G}$. For the proof of this property note that whether u is dangerous is not effected by the replacement of $v \neq u$ by v' and v''. Also if v is dangerous because of two sets S and T then either $(N(v) \cap S) = (N(v) \cap T)$ or $(N(v) \cap S) \cap (N(v) \cap T) = \emptyset$. □

3 The general case

Theorem 3.1. *If $G = (V, E)$ is a graph with $\Delta \leq 4$ and with $|E[S]| \leq 2|S| - 2$ for all $S \subseteq V$ then there is an extended one-bend drawing of G with respect to any designated vertex $v_\infty \in V$.*

Proof. We aim at using Theorem 2.3. To this end we define a 4-regular graph G^+ that has G as a subgraph. Let X be a set of $4|V| - 2|E|$ new vertices, i.e., X is disjoint from V. Let E_C be the set of edges of a bipartite graph with color classes X and V such that every vertex in X has degree 1 and a vertex $v \in V$ has degree $4 - \deg_G(v)$ in E_C. Finally, let E_X be the edge set of a 3-connected 3-regular graph on the vertex set X. Since $|X|$ is even and $|X| \geq 4$ we can e.g. take the dual of a plane triangulation as (X, E_X). Let $G^+ = (V \cup X, E \cup E_C \cup E_X)$. This graph is clearly 4-regular. It remains to verify the density condition of Theorem 2.3. For $S \subset V$ this is part of the assumption. If $S \subset V \cup X$ with $\emptyset \neq S \cap X \neq X$, then there are at least three edges in $E[S, \overline{S}] \cap E_X$ but since G^+ is 4-regular the size of the cut is even, hence at least four. □

The proof does not cover the case $|E| = 2|V| - 1$. However, as we show next, the density condition is also sufficient for this case.

If $|E| = 2|V| - 1$, then there may be a vertex of degree two in G. Note that this vertex is not an admissible candidate for v_∞. However, Theorem 2.3 applies after replacing the degree two vertex by an edge connecting its neighbors. Reinserting the vertex in the one-bend drawing is easy.

If there is no vertex of degree two but two vertices of degree three, then we add an edge to make the graph 4-regular. It is easy to check that the density condition fails for the graph with the added edge iff it already failed for the original graph. Hence again the case is covered by Theorem 2.3.

4 Additional comments and problems

1. We have shown that graphs with $\Delta \leq 4$ and $|E[S]| \leq 2|S| - 2$ for all $S \subseteq V$ admit one-bend drawings. The construction only places one vertex on each grid line. Improvements in the area requirement and in the total number of bends can be achieved through an obvius compaction that can be applied in a post processing phase. Is it possible to guide the algorithm so that the gain of this compaction can be controlled?

2. The technique of this paper clearly yields a polynomial algorithm for one-bend drawings. It might be worth investigating whether the technique can be used for a linear or near linear time algorithm.

3. 2-Bends Problem: Does every simple graph with maximum degree 6 have a 3-D orthogonal graph drawing with at most 2 bends per edge?

   According to Wood [21]: Eades, et al. [4] who first mentioned this problem, conjecture that the answer is false.

4. Extend the new technique using Eulerian orientation to graphs with higher degree and box representation for the vertices. Improve the unbalanced orientations used in Biedl/Kaufmann and prove better bounds for vertices with bounded aspect ratio.

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