Bend-optimal orthogonal graph drawing in the general position model*

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Abstract

We consider orthogonal drawings in the general position model, i.e., no two points share a coordinate. The drawings are also required to be bend minimal, i.e., each edge of a drawing in k dimensions has exactly one segment parallel to each coodinate direction that are glued together at $k-1$ bends.

We provide a precise description of the class of graphs that admit an orthogonal drawing in the general position model in the plane. The main tools for the proof are Eulerian orientations of graphs and discrete harmonic functions.

The tools developed for the planar case can also be applied in higher dimensions. We discuss two-bend drawings in three dimensions and show that K_{2k+2} admits a k-bend drawing in $k+1$ dimensions. If we allow that a vertex is placed at infinity, we can draw K_{2k+3} with k bends in $k+1$ dimensions.

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1 Introduction

The term *d-dimensional orthogonal drawing* traditionally denotes a drawing of a graph in which vertices are placed at distinct points of the d-dimensional integer lattice and edges are represented by chains of axis-parallel segments. Orthogonal drawings and variations are classical topics in graph drawing. The discrete nature of the model makes orthogonal drawings accessible for tools from combinatorial optimization. Orthogonal drawings are also related to various applications ranging from circuit layout to information visualization.

In this paper, we consider orthogonal drawings in the general position model, i.e., no two points share a coordinate. This model, which has not been studied very much has the advantage that all edges are treated the same. In k dimensions each edge is composed of k segments, one for each coodinate direction, hence, each edge has exactly $k-1$ bends.

Planar graphs with $\Delta \leq 4$ admit crossing-free 2-dimensional orthogonal drawings. Tamassia's seminal paper [20] is about the minimization of bends of such a drawing. For non-planar graphs, authors mainly worked on area minimization allowing a constant number of bends per edge [15]. Generalizations have been made in various directions, e.g. for higher degree graphs by representing vertices as boxes [1, 2, 7], incremental drawings [14, 6] and simple faces [9, 17, 16]. Orthogonal drawings in 3-dimensions have been studied less. Work in this area can be found in [4, 3].

In the first part of this paper we are interested in 2-dimensional orthogonal drawings of graphs with maximum degree $\Delta \leq 4$. If there is no restriction on the number of bends per edge, then every graph with $\Delta \leq 4$ has an orthogonal drawing in the plane. A *good* drawing, however, should be compact and readable. Therefore drawing algorithms are usually compared with respect to the drawing area and the number of bends. Optimizing either of these two parameters is NP-hard. This was shown for the area in [11, 5] and for the bend number in [19]. Several constructions of orthogonal drawings have been proposed e.g. by Schäffter [18] or by Eades, et al. [4] who describe an algorithm which draws a graph of maximum degree 4 in a box of dimensions $O(n) \times O(n) \times O(1)$ with 3 bends per edge. From work of Biedl and Kant [1] and Papakostas and Tollis [15] it follows that graphs with n vertices and $\Delta \leq 4$ admit orthogonal drawings with area $0.76n^2$, at most two bends per edge and a total of at most $2n+2$ bends. The problem has been reconsidered recently from a different view point, namely from the requirement of orthogeodesic edge routing, where all edges are required to be monotone [10, 8].

We investigate which graphs admit an orthogonal drawing in the general position model in the plane with at most one bend per edge. For the sake of brevity we call such drawings *one-bend drawings*. Since vertices are represented as points, graphs admitting a one-bend drawing are necessarily of maximum degree at most 4. Theorem 2.6 gives a full characterization of graphs admitting a one-bend drawing. For reasons of uniformity we allow an *anchor*, i.e., a vertex placed at infinity. In this model each vertex may be incident to four edges, i.e., 4-regular graphs can potentially be drawn with one bend. Theorem 2.4 shows that for this class of graphs the obvious necessary density condition (Proposition 2.1) is also sufficient. The construction of the embedding is based on discrete harmonic functions, a tool that has not been used before in this area.

Orthogonal drawings in three and higher dimensions have been studied by Eades, et al. [4] and intensively by D. Wood in his PhD thesis and in [21, 22, 23]. Eades et al. emphasize the

2-Bends Problem: *Does every simple graph with maximum degree 6 have a 3-D orthogonal graph drawing with at most 2 bends per edge?* They conjecture that the answer is false already for small graphs as K_7, however Wood found a 2 bend drawing of K_7. We surpass the 2 bend problem because we only deal with the general position model.

In Section 3 we discuss how to extend our results to higher dimensions. In Proposition 3.1 we illustrate our techniques at a class of graphs with maximum degree $\Delta \leq 5$ that can be drawn in three dimensions with two bends per edge. The emphasis in this part is on the techniques, because Wood [22] proved that all 5-regular graphs admit two-bend drawings in three dimensions. In Theorem 3.7 we show that K_{2k+3} can be drawn in $k+1$ dimensions with k bends per edge. Note that here again we need a vertex at infinity. Removing the vertex at infinity we obtain an again optimal (with respect to the dimension) drawing of K_{2k+2}.

2 One-Bend Drawings in the Plane

2.1 Preliminaries

Let $G = (V, E)$ be a graph with maximum degree $\Delta = 4$. Suppose that there is an orthogonal drawing of G such that every edge has at most one bend. If v is a vertex and edge (v, w) is leaving v towards the north, i.e., constant x-coordinate and increasing y-coordinate, then w is further to the north than v, i.e., $v_y < w_y$. Corresponding statements hold for edges leaving a vertex towards the other three directions.

Let $S \subset V$ be a set of vertices. Consider the subgraph $G[S] = (V, E[S])$ of G induced by this set and its induced one-bend drawing. From the above we see that in $G[S]$ the northernmost vertex of S has no edge towards the north. With the respective statements for the south, east and west extreme vertices we obtain that the sum of degrees of vertices in $G[S]$ can be at most $4|S| - 4$. This yields the following necessary condition for the existence of one-bend drawings:

Proposition 2.1. *If $G = (V, E)$ has a one-bend drawing and $S \subseteq V$, then $|E[S]| \leq 2|S| - 2$.*

The assertion of Theorem 2.6 will be that the degree condition $\Delta \leq 4$ together with the condition on edge densities given in Proposition 2.1 yields a sufficient set of conditions for the existence of a one-bend drawing. Before dealing with the general case we consider the 4-regular case.

Since a 4-regular graph has $2|V|$ edges we know from the proposition that there is no one-bend drawing for such a graph G. However, if we specify any vertex v_∞ and let $G' = G[V \setminus v_\infty]$, then G' may have a one-bend drawing. Such a drawing of G' can also be interpreted as a one-bend drawing of G with v_∞ placed at a point at ∞. We refer to v_∞ as the *anchor* of the drawing and call drawings with an anchor vertex *anchored drawing*s. Figure 1 shows an example.

2.2 The 4-regular case

Let $G = (V, E)$ be a connected 4-regular graph. We even allow multiple edges but Proposition 2.1 implies that in interesting cases the multiplicity of edges is at most two.

Note that in the 4-regular case $|E[S]| \leq 2|S| - 2$ for all proper subsets S of V is equivalent to the 4-edge connectivity of G.

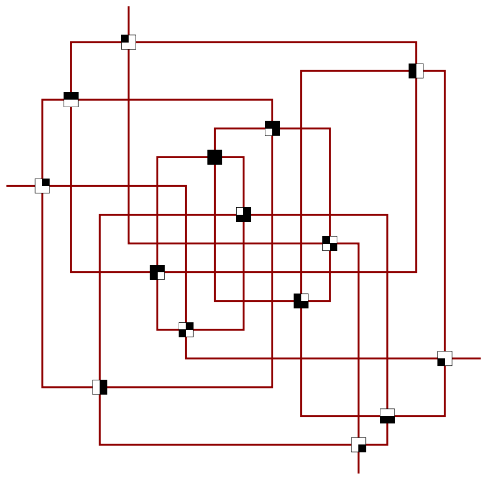

Figure 1: An anchored one-bend drawing of the 4-dimensional cube, the white and black sections of the squares representing vertices encode the 0/1 vectors.

Suppose that G has an anchored one-bend drawing with the property that every edge has exactly one bend. Using this drawing we define an orientation D_G of G according to the rule:

- edge vw is oriented as $v \to w$ if and only if in the drawing the edge is leaving v horizontally, i.e., to the east or to the west.

Every edge has a unique orientation in D_G because we assume that every edge has exactly one bend in the drawing. Moreover, the same orientation can be defined in terms of vertical segments as: edge vw is oriented as $v \to w$ if and only if in the drawing the edge is leaving w vertically. Together the two rules also imply a unique orientation for the edges incident to v_∞.

The orientation D_G has the property that $\textbf{in-deg}(v) = 2$ and $\textbf{out-deg}(v) = 2$ for every vertex v of G. We call an orientation with $\textbf{in-deg}(v) = \textbf{out-deg}(v)$ for all v an *Eulerian orientation*. We have thus observed that D_G is an Eulerian orientation.

Our construction of anchored one-bend drawings starts with an Eulerian orientation O_G of a 4-regular graph G with a special vertex v_∞. We then aim at constructing a one-bend drawing such that the orientation D_G defined on G according to the above rule equals the Eulerian orientation O_G.

Given an Eulerian orientation O_G we identify

- v_L and v_R, these are the two vertices with edges $v_L \to v_\infty$ and $v_R \to v_\infty$, the indices L and R are assigned arbitrarily. We call v_L and v_R the *horizontal poles*.
- For $v \in V \setminus \{v_L, v_R, v_\infty\}$ we let B_v be the set of vertices w such that in O_G there is an edge $v \to w$. Clearly $|B_v| = 2$ and in a one-bend drawing corresponding to O_G vertex v has to be horizontally between the elements of B_v, i.e., if $B_v = \{w', w''\}$ then either $w'_x < v_x < w''_x$ or $w'_x > v_x > w''_x$. This is the *horizontal betweenness condition* for v.

The Eulerian orientation also provides two vertical poles v_T and v_B and a vertical betweenness condition for all $v \in V \setminus \{v_T, v_B, v_\infty\}$. We now focus on solving the horizontal betweenness problem.

Although the adequate representation of a solution of a betweenness problem is a permutation of the vertices, we model the problem as a continuous one. The advantage is that in the continuous formulation we can use ideas from spring-embedding and solve the problem with the aid of discrete harmonic functions (cf. [12]). Up to Theorem 2.4 we develop some basics from the theory of discrete harmonic functions in a setting that fits the needs of our application.

Consider the following system of linear equations in the variables x_v:

(H) $x_v = \frac{1}{2}(x_{w'} + x_{w''})$ whenever $B_v = \{w', w''\}$, and $x_{v_L} = 0$, and $x_{v_R} = 1$.

Lemma 2.2. *The system (H) has a unique solution.*

Proof. The system has as many equations as it has variables. We claim that the corresponding homogeneous system only has the trivial solution. From the claim it follows that there exists a unique solution for any right hand side.

Suppose that (z_v) is a non-trivial solution of the homogeneous system. Consider a variable of maximal absolute value $|z_u| \neq 0$ and let $A = \{v \in V : |z_v| = |z_u|\}$. Since G is connected, there is some edge connecting A to $V \setminus A \supseteq \{v_L\}$. Since O_G is Eulerian, there is an edge $v \to w'$ with $v \in A$ and $w' \in V \setminus A$. If $v \to w''$ is the other out-edge of v, then $|z_v| > \frac{1}{2}(|z_{w'}| + |z_{w''}|) \geq |\frac{1}{2}(z_{w'} + z_{w''})|$, a contradiction. □

The solution of system (H) does not necessarily give a solution of the betweenness problem. Indeed a solution of the system may *clump* a set of vertices at a single point. Such a clumping can be accidental (resolvable by a perturbation $x_v = \frac{1}{2}((1+\varepsilon)x_{w'} + (1-\varepsilon)x_{w''})$ of the equations) or the clumping can be essential (this happens e.g. if v_L is a cut vertex and one component is clumped at the position $0 = x_{v_L}$). To exclude essential clumpings we need more than mere connectivity.

Let $S \subset V$ be a set of vertices. A *pole* of S is a vertex $v \in S$ such that in O_G there is an edge $v \to w$ with $w \notin S$. Note that v_L, v_R are the poles of $V' = V \setminus \{v_\infty\}$.

Proposition 2.3. *The betweenness problem has a solution if and only if every subset $S \subset V'$ with $|S| > 1$ has at least two poles.*

Proof. Suppose a permutation π of V is a solution to the betweenness problem. It follows immediately from the definitions that the leftmost and the rightmost vertex of S with respect to π are poles of S.

Now suppose that every subset $S \subset V'$ with $|S| > 1$ has at least two poles. We consider solutions of perturbed systems (H_ϵ) were the equations of the vertices are perturbed by independent parameters ε. Consider a solution (z_v) of a perturbed problem where the number of pairs of vertices sharing a position is minimized. Suppose that this solution has a clump A at a, i.e., $A = \{v : z_v = a\}$ and $|A| \geq 2$. Since A has at least two poles and at least one of v_L, v_R is not in A, there is an edge $v \to w$ leaving A. If for all edges $v \to w$ leaving A we have $z_w > a$, we get a contradiction because if $v \to w''$ is the other out-edge of v, then $\frac{1}{2}((1+\varepsilon)z_w + (1-\varepsilon)z_{w''}) > a = z_v$. Basically the same argument shows that if $v \in A$ has an edge $v \to w$ with $z_v > z_w$, then the other out-edge $v \to w''$ has $z_v < z_{w''}$. Increasing the parameter ε of the equation of v by a small $\delta > 0$ will move v slightly to the left. This resolves the clump A. By choosing δ small enough we can make sure that the effect of moving v on other vertices will not form new clumps. This shows that there is a system (H_ϵ) such that the solution has no clumps. □

Theorem 2.4. *A 4-regular graph $G = (V, E)$ with a designated vertex $v_\infty \in V$ admits an anchored one-bend drawing if and only if $|E[S]| \leq 2|S| - 2$ for all $S \subset V$.*

Proof. The necessity of the density condition for $S \subseteq V' = V \setminus \{v_\infty\}$ was shown in Proposition 2.1. Since G is 4-regular the condition for S with $v_\infty \in S$ is implied by the condition for the complement.

To prove sufficiency we choose an Eulerian orientation O_G of G such that with respect to O_G every $S \subset V'$ with $|S| > 1$ has at least two poles. In fact we want that the two poles condition also holds for the reverse orientation $\overline{O_G}$. In Proposition 2.5 below we show that such an Eulerian orientation O_G of G exists.

Proposition 2.3 implies that the horizontal betweenness problem associated with O_G has a solution (x_v). Let $\pi_x : V' \to \{1, \ldots, |V'|\}$ be the ordering of the vertices obtained by sorting them according to the x_v values.

The two poles condition for $\overline{O_G}$ allows to use Proposition 2.3 again and infer the existence of a solution y_v of the vertical betweenness associated with O_G. Let π_y be the ordering of the vertices obtained by sorting them according to the y_v values.

Let $n = |V'|$. We now can safely place the vertices of G' on integral points of the $n \times n$ grid $v \to (\pi_x(v), \pi_y(v))$ and draw the edges with one bend such that edges that are outgoing at v in the Eulerian orientation are attached to the horizontal ports of v while in-edges are attached at the vertical ports. The horizontal and vertical betweenness conditions guarantee that there is no conflict of direction, i.e, each of the four ports of every vertex is used by exactly one edge. □

Proposition 2.5. *A 4-regular multi-graph G with $|E[S]| \leq 2|S| - 2$ for all $S \subset V$ has an Eulerian orientation O_G such that with respect to O_G and $\overline{O_G}$ every $S \subset V$ with $|S| > 1$ has at least two poles.*

Proof. Note that the condition implies that G is 4-edge connected. Hence, if O is an Eulerian orientation and $S \subset V$, then there are at least two edges leaving S. If there are two such edges with different tail-vertices, then there are two poles in S. Conversely, if S with $|S| > 1$ has only one pole in O, then there are only four edges in the cut $E[S, \overline{S}]$ and the two edges in O pointing from S to $\overline{S}$ have the same tail $v \in S$.

To prove the existence of an Eulerian orientation without such a *bad* vertex we are going to identify all the vertices that have the potential of being bad. At each of these vertices we slightly change the graph so that an Eulerian orientation of the modified graph can be used to get an Eulerian orientation of G that has no bad vertex.

First note that if v is a bad vertex of O, i.e., $|E[S, \overline{S}]| = 4$ and $v \in S$ has two out-neighbors in $\overline{S}$, then the other two edges incident to v belong to $E[S]$. Otherwise the cut $E[S - x, \overline{S} + x]$ would only contain two edges, contradicting the edge connectivity of G.

We call a vertex v *dangerous* if v has four different neighbors w_1, w_2, w_3, w_4 and there exists some $S \subset V$ with $|E[S, \overline{S}]| = 4$, $v, w_1, w_2 \in S$, and $w_3, w_4 \in \overline{S}$. (Note that the existence of a dangerous vertex implies that the vertex-connectivity of G is ≤ 3).

We claim that if v is dangerous because of two different sets S and T then both sets induce the same partition $\{w_1, w_2\}, \{w_3, w_4\}$ on the neighbors $N(v)$ of v. Suppose not, then S and T cross as in Figure 2. Then $E[S, \overline{S}] \cup E[T, \overline{T}]$ consists of only 7 edges, however, each of the four non-empty parts $S \cap T$, $S \cap \overline{T}$, $\overline{S} \cap T$, and $\overline{S} \cap \overline{T}$ induces a cut that contains at least four edges and all these edges belong to $E[S, \overline{S}] \cup E[T, \overline{T}]$. Since each edge in the union can only contribute to two cuts this implies $|E[S, \overline{S}] \cup E[T, \overline{T}]| \geq 8$.

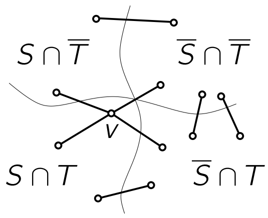

Figure 2: An impossible case.

We now come to the construction of an Eulerian orientation with the desired property. Identify dangerous vertices and duplicate them, i.e., if v is dangerous with neighbor set w_1, w_2, w_3, w_4 as above, then replace v and its four edges by vertices v' and v'', edges $(w_1, v'), (w_2, v'), (w_3, v''), (w_4, v'')$ and a double-edge connecting v' and v'', see Figure 3.

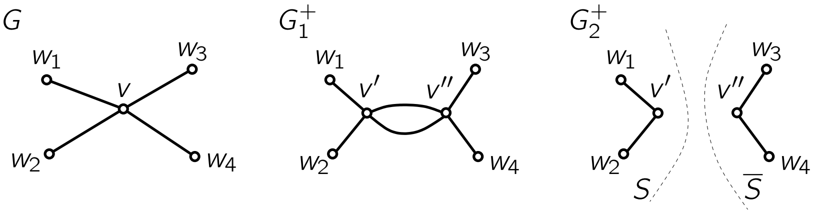

Figure 3: The replacement of a dangerous vertex.

The replacement of a dangerous vertex reduces the number of vertices with four different neighbors. Therefore, iterating the replacement we eventually obtain a 4-regular graph G_1^+ that has no dangerous vertex. Remove all double edges from G_1^+ and let G_2^+ be the resulting graph. Since all vertices of G_2^+ are of degree 4, 2 or 0 there is an Eulerian orientation O^+ of G_2^+. Merging pairs of vertices v', v'' that were created by the replacement of a dangerous vertex and adding back original double edges with opposite orientations, we recover the original graph G. Let O_G be the Eulerian orientation of G inherited from O^+. The orientation O_G has the property that every $S \subset V$ with $|S| > 1$ has at least two poles with respect to O_G as well as with respect to $\overline{O_G}$. For the proof of this property note that whether $u \neq v$ is dangerous is not affected by the replacement of v by v' and v''. □

2.3 The case $\Delta \leq 4$

Theorem 2.6. *If $G = (V, E)$ is a graph with $\Delta \leq 4$ and with $|E[S]| \leq 2|S| - 2$ for all $S \subseteq V$ then there is an anchored one-bend drawing of G with respect to any* designated vertex $v_\infty \in V$.*

Proof. We aim at using Theorem 2.4. To this end we define a 4-regular graph G^+ that has G as a subgraph.

Let $X \neq \emptyset$ be a set of $4|V| - 2|E|$ new vertices disjoint from V. Let E_C be the set of edges of a bipartite graph with color classes X and V such that every vertex in X has degree 1 and a

*The only exception is that when $\mathrm{deg}(v) = 2$ and $|E| = 2|V| - 1$.

vertex $v \in V$ has degree $4 - \deg_G(v)$ in E_C. Finally, let E_X be the edge set of a 3-connected 3-regular graph on the vertex set X. Suppose $|X| \geq 4$, since $|X|$ is even we can e.g. take the dual of a plane triangulation as (X, E_X). Let $G^+ = (V \cup X, E \cup E_C \cup E_X)$. This graph is clearly 4-regular. It remains to verify the density condition of Theorem 2.4. For $S \subset V$ this is part of the assumption. If $S \subset V \cup X$ with $\emptyset \neq S \cap X \neq X$, then there are at least three edges in $E[S, \overline{S}] \cap E_X$ but since G^+ is 4-regular the size of the cut is even, hence at least four.

It remains to consider the case $|X| < 4$, i.e., $|X| = 2$ and $|E| = 2|V| - 1$. Suppose G has a vertex degree 2. This vertex is not an admissible candidate for v_∞. However, Theorem 2.4 applies after replacing the degree 2 vertex by an edge connecting its neighbors. Reinserting the vertex in the one-bend drawing is easy.

If there is no vertex of degree 2 but two vertices of degree three, then we add an edge to make the graph 4-regular. It is easy to check that the density condition fails for the graph with the added edge if and only if it already failed for the original graph. Hence again the case is covered by Theorem 2.4. □

2.4 Comments and problems

1. We have shown that graphs with $\Delta \leq 4$ and $|E[S]| \leq 2|S| - 2$ for all $S \subseteq V$ admit one-bend drawings. The construction only places one vertex on each grid line. Improvements in the area requirement can be achieved through an obvious compaction that can be applied in a post processing phase. Is it possible to guide the algorithm so that the gain of this compaction can be controlled?

2. The technique of this section clearly yields a polynomial algorithm for one-bend drawings. It might be worth investigating whether this can be shown to yield a linear or near linear time algorithm.

3 k bends in $k + 1$ dimensions

We start into this section with Proposition 3.1 which is about a class of graphs with $\Delta \leq 5$ that admit two-bend drawings in three dimensions.

After that we discuss embeddings of $(2k + 2)$-regular graphs in $k + 1$ dimensions. In Theorem 3.4 we give a condition on a 2-factorization of such a graph that is sufficient for the existence of a k-bend drawing in $k + 1$ dimensions. The sufficiency of the condition is proved by showing that the 2-factorization induces $k + 1$ betweenness relations that comply with the 2-pole condition of Proposition 2.3. Finally we show that the complete graph K_{2k+3} admits a 2-factorizations with the required properties, so that in Theorem 3.7 we can state the existence of a k-bend drawing in $k + 1$ dimensions for this graph.

Proposition 3.1. *If $G = (V, E)$ is a 4-regular graph that admits a one-bend drawing in the plane and M is a matching on V, then there is a two-bend drawing of the graph $G_{+M} = (V, E \cup M)$ in 3 dimensions. (We allow G and G_{+M} to be multi-graphs.)*

Proof. Let Γ be a one-bend drawing of G in the plane. Draw the edges of M on top of Γ with only one bend on each edge. To draw an edge $\{u, v\} \in M$ we have two options, the edge may enter u vertically or horizontally. For our approach, however, it is irrelevant which of the two we choose. In the drawing Γ_+ obtained by drawing M on top of Γ, some horizontal or vertical ports at vertices are occupied by two edges. These pairs of edges are said to be in *conflict*.

In the following we find a two-bend drawing Γ_3 of G_{+M} in 3 dimensions such that its orthogonal projection to the xy-plane is exactly Γ_+. In Γ_3, each edge e of G_{+M} contains a segment parallel to the z-axis. This segment is projected to a single point in the xy-plane, further called *the z-point of e*.

Let $C \subseteq E \cup M$ be the set of edges that participate in a conflict. Since every conflict consists of an edge in M and an edge in E the set C decomposes into paths and cycles of even length. If $e \in C \cap E$ only participates in a conflict at a horizontal port, then we call e a *free* edge, note that a free edge is an extremal edge of a path in C.

Assign a z-point to every edge of $E \cup M$ as follows. If $e \notin C$ of if e is free then the z-point is the bend of edge e in the drawing Γ_+. If $e \in C \cap M$ the z-point is the vertex where e is using the horizontal port in Γ_+. If $e \in C \cap E$ and not free, then the z-point is the vertex where e is using the vertical port in Γ_+.

Assign arbitrarily distinct z-coordinates to the vertices. With these z-coordinates for vertices we obtain a two-bend drawing from Γ_+ by lifting and adjusting the edges as follows; see Figure **??**. If u, v is an edge with z-point at one of the end vertices, say at u, then we lift the edge to z_v where one of its ends connects to v and then add a segment of length $|z_v - z_u|$ in z-direction to connect the other end to u. If u, v is an edge with z-point at its bend we lift the incident segments to z_u respectively z_v and reconnect these two segment with a segment of length $|z_v - z_u|$ in z-direction over the bend.

Note that every vertex v is the z-point of at most one edge. Therefore, at most one z-port of each v is used by an edge. This shows that we have constructed a two-bend drawing of $G_{+M} = (V, E \cup M)$. □

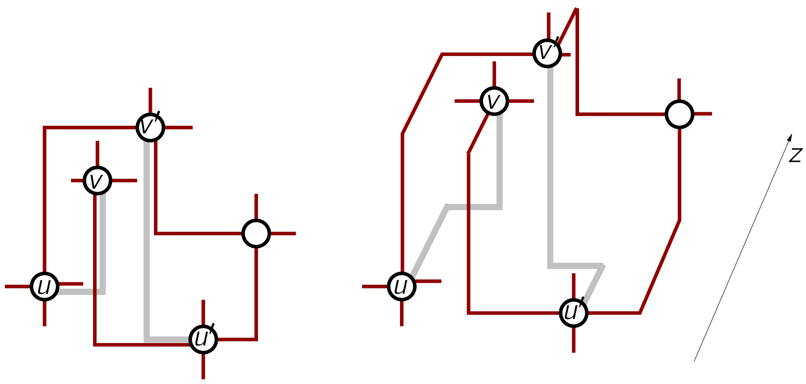

Figure 4: Some grey M-edges and some G edges in Γ_+. The lifting resolves the conflicts.

3.1 $(2k+2)$-regular graphs

Let G be a $(2k+2)$-regular graph. Suppose that G has an anchored k-bend drawing in $k+1$ dimensions in the general position model, i.e., the coordinate vectors of any two vertices differ in each component. The general position condition implies that the embedding of each edge consists of $(k+1)$ segments, one segment for each coordinate direction. Moreover, in the drawing each of the $(2k+2)$ ports of a vertex is used by one of its incident edges. Let the

$k+1$ coordinate directions be $x_0, x_1, \ldots, x_k$. If $e = \{v, w\}$ is an edge using one of the two x_i ports of v, then we color the incidence $[v, e]$, resp. the half-edge of e at v, with color i. This yields a coloring of the half-edges of the graph with colors $0, 1, \ldots, k$ respecting the following properties:

(1) Each vertex is incident to two half-edges of color i for each $i \in \{0, 1, \ldots, k\}$.

(2) The half-edges of each edge have different colors. This follows from the general position assumption.

Proposition 3.2. *Every $(2k+2)$-regular graph admits a coloring of half-edges with $k+1$ colors satisfying Properties (1) and (2).*

Proof. Petersen (1891) proved that that a $2r$-regular graph admits a 2-factorization, i.e., a partition $C_1, \ldots, C_r$ of the edges such that each C_i is a spanning collection of cycles, c.f. [13, Thm. 6.2.4].

Let G be $(2k+2)$-regular and $C_0, C_1, \ldots, C_k$ be a 2-factorization. Choose an arbitrary orientation for each of the cycles of each 2-factor. Color all out-going half-edges of cycles belonging to C_i with color i and all in-coming half-edges belonging to this 2-factor with color $i+1 \pmod{k+1}$. This coloring obeys the two properties. □

A coloring of half-edges of a $(2k+2)$-regular graph G induced by an anchored k-bend drawing in $k+1$ dimensions, however, satisfies another condition. Consider a subset $S \subset V$ with $v_\infty \notin S$ the two vertices of S with maximal and minimal x_i-coordinate both have a half-edge of color i that connects to a vertex in $\overline{S} = V \setminus S$. A counting argument shows that in the cut $E[S, \overline{S}]$ there also are at least two half-edges of color i incident to a vertex of $\overline{S}$.

(3) If S is a subset of vertices with $|S| \geq 2$ and $v_\infty \notin S$, then for each i in $0, \ldots, k$ there are two different vertices of S that are incident to a half-edge of color i that belongs to an edge in the cut $E[S, \overline{S}]$.

Recall that in the 4-regular case the density condition of Proposition 2.1 is equivalent to the 4-edge connectivity of G. The third property of the coloring yields the same kind of necessary condition in the present case:

Proposition 3.3. *A $(2k+2)$-regular graph G admitting an anchored k-bend drawing in $k+1$ dimensions has edge connectivity $(2k+2)$.*

Assuming an anchored k-bend drawing of a $(2k+2)$-regular graph we observed that the induced coloring of half-edges obeys Properties (1), (2), and (3). It turns out that the existence of a *good coloring of half-edges*, i.e., of a coloring of half-edges that obeys all three conditions, is sufficient for the existence of an anchored k-bend drawing.

Theorem 3.4. *If a $(2k+2)$-regular graph G admits a good coloring of half-edges, then there is an anchored k-bend drawing of G in $k+1$ dimensions.*

Proof. The idea for the proof to make use of discrete harmonic functions is as in Subsection 2.2.

Select a vertex $v_\infty \in V$ that is going to be placed at infinity. From Property (3) of the coloring, whose existence is assumed, we find that there are two different vertices $v^+(i)$ and $v^-(i)$ that have an incident half-edge of color i that belongs to an edge connecting to v_∞. These two vertices are the *poles of color i*.

For $v \in V \setminus \{v^+(i), v^-(i), v_\infty\}$ we let $B_v(i) = \{w', w''\}$ if w' and w'' are the neighbors of v connected to v by edges whose half-edge at v has color i. From Property (1) we know that there are exactly two such vertices. If $B_v(i) = \{w', w''\}$ then the edges from v to w' and w'' are supposed to leave v at the two different ports of coordinate direction x_i, i.e., either $w'_i < v_i < w''_i$ or $w'_i > v_i > w''_i$. This is the *betweenness condition* in color i for v.

From the betweenness condition in color i we obtain a system $H(i)$ of linear equations that is uniquely solvable (Lemma 2.2). The solution of system $H(i)$ can be converted into a solution of the betweenness problem in color i exactly if every subset S of $V' = V \setminus \{v_\infty\}$ with $|S| > 1$ has at least two poles (Proposition 2.3). The existence of the two poles, however, is precisely the condition asserted by Property (3).

A solution of the betweenness problem in color i can be converted into a permutation $\pi_i : V' \to \{1, .., |V'|\}$ respecting the betweenness condition.

Let $n = |V'|$. We now can safely place the vertex v at the point $(\pi_0(v), \pi_1(v), \ldots, \pi_k(v))$ in the $[n]^{k+1}$ grid and draw the edges with k bends such that edges whose half-edge at v is colored i are attached to ports of v that correspond to coordinate direction x_i. The betweenness conditions guarantee that there is no conflict of direction, i.e, each of the $2k+2$ ports of every vertex is used by exactly one edge. □

Note that along an edge v, w we have only fixed the directions of the first and the last segment in the drawing. The edge consist of k segments, the segment in coordinate-direction i has length $|\pi_i(v) - \pi_i(w)|$. Sice only the first and the last segment have been fixed the order of the remaining $k - 2$ segments can be chosen arbitrarily. Because no two vertices share a coordinate (general position) there can be no intersection of edges except at endpoints. This observation directly implies the following proposition.

Proposition 3.5. *Let G and H be graphs on the same vertex set V. If G admits an anchored r-bend drawing in $r+1$ dimensions, and H admits an anchored s-bend drawing in $s+1$ dimensions, then there is an anchored $(r+s+1)$-bend drawing of the (multi)graph $(V, E_G \cup E_H)$ in $r+s+2$ dimensions.*

3.2 Complete graphs

Orthogonal drawings of complete graphs have been studied by Wood [21]. He was interested in minimizing bends. This clearly leads to drawings that are not in the general position model. We show that in the general position model, respectively, the anchored general position model the obvious lower bound for the dimension that comes from the degrees is sufficient (Theorem 3.7).

As an application of Petersen's 2-Factorization Theorem we constructed a coloring of half-edges of a $(2k + 2)$-regular graph respecting Properties (1) and (2) (Proposition 3.2). It is not difficult to come up with conditions that guarantee that a 2-factorization $C_0, C_1, \ldots, C_k$ yields a good coloring of half-edges. However, to construct a 2-factorization meeting these conditions seems to be quite hard. In fact we have no general method for constructing good colorings.

We now show that the complete graph K_{2k+3} has a 2-factorization that yields a good coloring of half-edges and, hence, by Theorem 3.4 an anchored k-bend embedding in $k + 1$ dimensions. By removing the anchor vertex of the drawing we obtain a drawing of K_{2k+2} with k bends in $k + 1$ dimensions.

Let $\{0, 1, \ldots, 2k+2\}$ represent the vertices of K_{2k+3}. For each $i \in 0, \ldots, k$ we define the 2-factor C_i as the set of all edges $a \to a+i+1 \mod (2k+3)$. This definition already induces the orientation of cycles of C_i that is needed to distinguish the half-edges of colors i and $i+1$ defined on the basis of C_i.

Lemma 3.6. *For each $i \in 0, \ldots, k$ the half-edges of color i defined by the oriented 2-factors C_i and $C_{(i-1 \mod k+1)}$ obey Property (3).*

Proof. For notational simplicity we assume $i > 0$. Let S be a set of vertices, $1 < |S| < 2k+3$ and let $c_i = |C_i \cap E[S, \overline{S}]|$ and $c_{i-1} = |C_{i-1} \cap E[S, \overline{S}]|$. Clearly c_i and c_{i-1} are both even. We distinguish some cases.

Case $c_i \geq 4$. Consider two edges of C_i in the cut $E[S, \overline{S}]|$ that are oriented from S to $\overline{S}$. They contribute two half-edges of color i that are incident to different vertices of S.

Case $c_i = 0$. This case is illustrated in Figure 5. To have $c_i = 0$ there has to be at least one cycle of C_i that belongs entirely to S and one that consists only of vertices in $\overline{S}$. Let $\gamma_0, \gamma_1, \ldots, \gamma_s$ be an enumeration of the cycles of C_i such that $a \in \gamma_j$ implies that $a + i \mod (2k+3)$ belongs to $\gamma_{(j+1 \mod s+1)}$, i.e., the cycles of C_{i-1} loop through the γ_j. Since each cycle γ_j belongs to S or to $\overline{S}$ we find a j such that $\gamma_j \subseteq \overline{S}$ and $\gamma_{j+1} \subseteq S$. It follows that every vertex of γ_{j+1} is incident to a half-edge of color i that belongs to an edge in the cut.

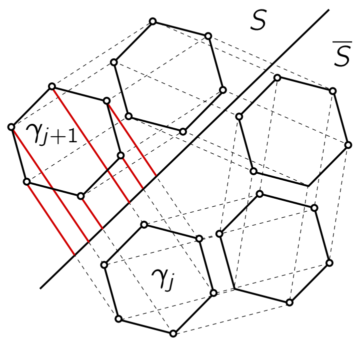

Figure 5: A sketch for the case $c_i = 0$. The cycles of C_i are bold, the cycles of C_{i-1} dashed. Half-edges of color i are emphasized.

Case $c_i = 2$. Again consider the cycles of C_i. Exactly one of them, say γ_0 contains vertices of S and $\overline{S}$.

Subcase $\gcd(i+1, 2k+3) = 1$. In this case γ_0 contains all vertices. Since $c_i = 2$ we know that S is a contiguous piece $v_1 \to v_2 \to \ldots \to v_s$ of γ_0. Now consider $w_j = v_j - i \mod 2k+3$ and note that $w_1 \to w_2 \to \ldots \to w_s$ is again a contiguous piece of γ_0 and $w_1 \neq v_1, v_2$. Therefore, there is a $j < s$ with $w_j \in \overline{S}$. This yields half-edges of color i at v_j and v_s.

Subcase $\gcd(i+1, 2k+3) = s+1 > 1$. Let $\gamma_0, \gamma_1, \ldots, \gamma_s$ be an enumeration of the cycles of C_i as in the case $c_i = 0$. From that case we know that if we find a j such that $\gamma_j \subseteq \overline{S}$ and $\gamma_{j+1} \subseteq S$, then every vertex of γ_{j+1} contributes an i-colored edge to the cut. If $\gamma_1 \subset S$, then we have an edge in C_{i-1} oriented from a vertex of $\overline{S}$ in γ_0 to a vertex in γ_1 this yields a half-edge of color i incident to a vertex in γ_1 in addition to the contribution of C_i in γ_0. It remains to consider the case that $\gamma_1, \ldots, \gamma_s$ all belong to $\overline{S}$. Since $|S| > 2$ there is a vertex $v \in \gamma_0$ such that that $v + i - 1 \in S$, this vertex has an half-edge of color i in the cut because $v - i \in \gamma_s \subset \overline{S}$. □

Theorem 3.7. *The complete graph K_{2k+3} admits an anchored k-bend embedding in $k+1$ dimensions. Removing the anchor vertex yields a k-bend embedding of K_{2k+2} in $k+1$ dimensions.*

Proof. Define the 2-factorization of the complete graph K_{2k+3} consisting of the 2-factors $C_i = \{(a, b) : b = (a + i + 1 \mod 2k + 3)\}$ for $i = 0, \ldots, k$. In Lemma 3.6 we have shown that the coloring of half-edges induced by these 2-factors (c.f. Proposition 3.2) is a good coloring of half-edges. From Theorem 3.4 we know that a good coloring of half-edges can be used for the construction of an anchored k-bend embedding in $k+1$ dimensions. □

3.3 Comments and problems

1. We could not resolve the question whether the connectivity condition (Proposition 3.3) is sufficient for two-bend embeddings of 6-regular graphs. Our feeling is that if this can be resolved (positively), then the corresponding problem in higher dimension can be answered in the same sense.

2. With the aid of discrete harmonic functions we could provide a nice combinatorial condition (good coloring of half-edges) for the existence of k-bend drawings of $(2k+2)$-regular graphs. We believe that the notion of good colorings of half-edges has the potential of being of more general interest since it provides decompositions of well connected graphs such that pieces and unions of the pieces also show good connectivity.

3. In the one-bend situation we anchored the characterization from the 4-regular case to the case $\Delta \leq 4$ by adding vertices and edges to get a 4-regular supergraph with the required connectivity. Such an extension becomes an interesting question as soon as the existence of k-bend drawings of $(2k+2)$-regular graphs with the necessary connectivity is established.

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