On the Order Dimension of Outerplanar Maps

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Abstract

Schnyder characterized planar graphs in terms of order dimension. Brightwell and Trotter proved that the dimension of the vertex-edge-face poset $\mathbf{P}_M$ of a planar map M is at most four. In this paper we investigate cases where $\dim(\mathbf{P}_M) \leq 3$ and also where $\dim(\mathbf{Q}_M) \leq 3$; here $\mathbf{Q}_M$ denotes the vertex-face poset of M. We show:

- If M contains a K_4-subdivision, then $\dim(\mathbf{P}_M) = \dim(\mathbf{Q}_M) = 4$.
- If M or the dual M^* contains a $K_{2,3}$-subdivision, then $\dim(\mathbf{P}_M) = 4$.

Hence, a map M with $\dim(\mathbf{P}_M) \leq 3$ must be outerplanar and have an outerplanar dual. We concentrate on the simplest class of such maps and prove that within this class $\dim(\mathbf{P}_M) \leq 3$ is equivalent to the existence of a certain oriented coloring of edges. This condition is easily checked and can be turned into a linear time algorithm returning a 3-realizer.

Additionally, we prove that if M is 2-connected and M and M^* are outerplanar, then $\dim(\mathbf{Q}_M) \leq 3$. There are, however, outerplanar maps with $\dim(\mathbf{Q}_M) = 4$. We construct the first such example.

1 Introduction

This paper is about planar maps and the order dimension of posets related to them. A *planar map* $M = (G, D)$ consists of a finite planar multigraph G and a plane drawing D of G. By a planar map M we mean the combinatorial data given by the set V of vertices, the set E of edges, the set F of faces and the incidence relations between these sets.

The dual map M^* of M is defined as usual: there is a vertex F^* in M^* for each face F in M, and an edge e^* in M^* for each edge e of M, joining

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the dual vertices corresponding to the faces in M separated by e (if e is a bridge, e^* is a loop). Each vertex in M will then correspond to a face of M^*.

Most of the maps we consider in this paper are outerplanar. We differentiate between two notions of outerplanar maps. A planar map $M = (G, D)$ is *weakly outerplanar* if G is outerplanar, and *strongly outerplanar* if G is outerplanar and D is an outerplane drawing of G, i.e., a plane drawing of G where all the vertices are on the boundary of the outer face. When it is clear from the context, the qualifiers weakly and strongly will be omitted.

The dimension is a widely studied parameter of posets. Since its introduction by Dushnik and Miller [5] in 1941, dimension has moved into the core of combinatorics. There are close connections and analogies with the chromatic number of graphs and hypergraphs. From the applications point of view, dimension is attractive because low dimension warrants a small encoding complexity of the poset. Trotter [10] provides an extensive introduction to the area.

The vertex-edge-face poset $\mathbf{P}_M$ of a planar map M is the poset on the vertices, edges and faces of M ordered by inclusion. The vertex-face poset $\mathbf{Q}_M$ of M is the subposet of $\mathbf{P}_M$ induced by the vertices and faces of M.

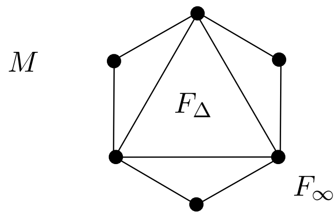

Figure 1: A planar map. Two faces are labeled, F_∞ is the *outer face*.

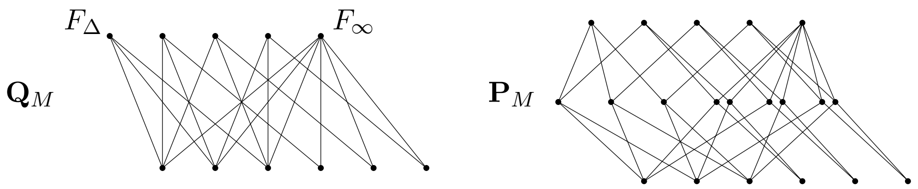

Figure 2: The vertex-face and the vertex-edge-face posets of the map from Figure 1.

Note that if M is connected, the vertex-edge-face poset $\mathbf{P}_{M^*}$ of the dual map, is just the dual poset $(\mathbf{P}_M)^*$ (i.e., $x < y$ in $(\mathbf{P}_M)^*$ if and only if $y < x$ in $\mathbf{P}_M$). The same observation is true for $\mathbf{Q}_M$.

The *dimension* $\dim(\mathbf{P})$ of a poset $\mathbf{P}$ is the minimum number t such that t is the intersection of t linear orders on the same ground set. Our investigations ground on Schnyder's characterization of planar graphs in terms of dimension [9] and the two theorems of Brightwell and Trotter stated below. A simpler proof of Theorem 2 can be found in [6].

Theorem 1 (Brightwell and Trotter [3])**.** *If M is a planar map, then* $\dim(\mathbf{P}_M) \leq 4$*.*

Theorem 2 (Brightwell and Trotter [2])**.** *If M is a 3-connected planar, then* $\dim(\mathbf{Q}_M) = 4$*.*

The cases where the dimension is 2 are well-studied. There are fast algorithms to test whether a poset $\mathbf{P}$ is of dimension 2, see e.g. [8]. Moreover, Brightwell and Trotter [3] have presented a complete characterization of the planar maps with $\dim(\mathbf{P}_M) = 2$. This characterization can be turned into a linear time recognition algorithm. Surprisingly the characterization of $\dim(\mathbf{Q}_M) = 2$ is less satisfying. It seems that the most compact way of stating the characterization is to refer to the list of 3-irreducible posets, see e.g., [10, Table 2]. A map that contains $\mathbf{A}_k$ for some $k \geq 4$ in its vertex-face poset also contains another forbidden subposet. Therefore, it is enough to forbid $\mathbf{A}_3$, $\mathbf{B}$, $\mathbf{CX}_2$ and $\mathbf{EX}_1$ and their duals, i.e, subposets with six or seven elements. Another option is to refer to characterizations of bipartite permutation graphs, e.g, as being exactly the bipartite AT-free graphs, see [1] for more on this topic. Again, these characterizations lead to linear time recognition algorithms, see [1, 4].

For $\dim(\mathbf{P}_M) \geq 3$ it is sufficient that M has a vertex of degree 3. To test for dimension 3 is known to be NP-complete [11]. A major open problem in the area is to determine the complexity of the dimension 3 problem for orders of height 2. Even the special case where the order of height 2 is the vertex-face poset of a planar map remains open. Motivated by these algorithmic questions we approach the problems of characterizing the maps M with $\dim(\mathbf{P}_M) \leq 3$ and the maps with $\dim(\mathbf{Q}_M) \leq 3$. These characterization problems have earlier been posed by Brightwell and Trotter [3].

1.1 Our contributions

In Section 2 we prove that for $\dim(\mathbf{Q}_M) \leq 3$ it is necessary that M is K_4-subdivision free. For $\dim(\mathbf{P}_M) \leq 3$ an additional necessary condition is that both M and M^* are $K_{2,3}$-subdivision free. This means that if $\dim(\mathbf{P}_M) \leq 3$, then both M and M^* are outerplanar.

In Section 3, we study the simplest class of maps M such that M and M^* are outerplanar. We call these maps *path-like*. For maximal path-like maps we prove that $\dim(\mathbf{P}_M) \leq 3$ is equivalent to the existence of a special oriented coloring of the interior edges and characterize the path-like maps which admit such a coloring. The characterization is turned into a linear time algorithm that generates a 3-realizer, i.e., three linear extensions whose intersection is $\mathbf{P}_M$, or returns the information that $\dim(\mathbf{P}_M) \geq 4$.

Finally, in Section 4, we prove that if M is 2-connected and M and M^* are outerplanar, then $\dim(\mathbf{Q}_M) \leq 3$. We also present a strongly outerplanar map with a vertex-face poset of dimension 4. The example, a maximal outerplanar graph with 21 vertices, is quite large. We provide some arguments which indicate that our example is not far from being as small as possible.

1.2 Tools from dimension theory

In this section we recall some facts from the dimension theory of finite posets. The reader is referred to Trotter's monograph [10] for additional background and references.

If $\mathbf{P}$ is a finite poset, a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of $\mathbf{P}$ is called a *realizer* of $\mathbf{P}$ if $P = \cap\mathcal{R}$, i.e. $x < y$ in $\mathbf{P}$ if and only if $x < y$ in L for all $L \in \mathcal{R}$. The dimension of $\mathbf{P}$ is the minimum cardinality of a realizer of $\mathbf{P}$.

A *critical pair* is a pair of incomparable elements (a, b) such that $x < b$ if $x < a$ and $y > a$ if $y > b$ for all $x, y \in \mathbf{P}$. A family of linear extensions $\mathcal{R}$ of P is a realizer of $\mathbf{P}$ if and only if each critical pair (a, b) is *reversed* in some linear extension $L \in \mathcal{R}$, i.e., $b < a$ in L. An incomparable min-max pair, i.e., a pair of incomparable elements (a, b) where a is a minimal element and b is a maximal element of $\mathbf{P}$, is always critical.

An alternating cycle is a sequence of critical pairs $(a_0, b_0), \ldots, (a_k, b_k)$ such that $a_i < b_{(i+1 \mod (k+1))}$ for all $i = 0, \ldots, k$. A fundamental result is that $\dim(\mathbf{P}) \leq t$ if and only if there exists a t-coloring of the critical pairs of $\mathbf{P}$ such that no alternating cycle is monochromatic.

In the following example we illustrate how these facts can be combined to determine the dimension of a specific incidence order.

Example: Let M be the planar map of the complete graph K_4. Every vertex has a single non-incident face, hence, there are these four incomparable min-max pairs in $\mathbf{Q}_M$. These are all the critical pairs. Any two of these critical pairs form an alternating cycle. Therefore, the hypergraph of alternating cycles is again a K_4 and has chromatic number 4. This shows that $\dim(\mathbf{Q}_M) = 4$.

2 Vertex-edge-face posets of dimension at most 3

From Theorem 2, we know that $\dim(\mathbf{Q}_M) = 4$ for every 3-connected map M. We show that this excludes K_4-subdivisions from being contained in M if $\dim(\mathbf{Q}_M) \leq 3$.

Theorem 3. *Let M be a planar map that contains a subdivision of K_4. Then* $\dim(\mathbf{Q}_M) > 3$.

Proof. We will prove that $\mathbf{Q}_M$ has the vertex-face poset of some 3-connected planar map as a subposet, and then apply the Brightwell-Trotter Theorem. This is essentially done in two steps: first 1-vertex cuts and then 2-vertex cuts are removed.

A K_4-subdivision in M will be contained in a 2-connected component of M. The vertex-face poset of a 2-connected component of M is an induced subposet of $\mathbf{Q}_M$. Hence, we can assume that M is 2-connected.

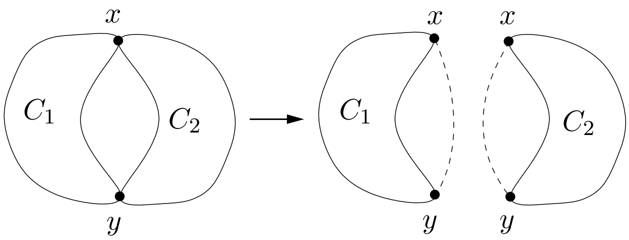

Figure 3: Removing a separating pair.

Now, consider a 2-vertex cut $\{x, y\}$. There must be two components C_1 and C_2 such that removing x and y separates C_1 from C_2. We create two new maps by replacing one of the two components by an edge $\{x, y\}$, see Figure 3. The vertex-face posets of these new maps are subposets of $\mathbf{Q}_M$. Furthermore, if M contains a K_4-subdivision, one of the new maps must contain a K_4-subdivision. □

For vertex-edge-face posets, we provide another criterion which forces dimension 4. If M contains a $K_{2,3}$-subdivision, then $\dim(\mathbf{P}_M) = 4$. Before analyzing the general situation we deal with the simple case where M is, actually, a subdivision of $K_{2,3}$.

Proposition 4. *Let M be a planar drawing of a subdivision of $K_{2,3}$. Then* $\dim(\mathbf{P}_M) > 3$.

Proof. Let x and y be the two vertices of degree 3 in M, and let P_1, P_2 and P_3 be the three x-y paths. The map has three faces F_1, F_2 and F_3. In Figure 4 face F_i is labeled R_i. The vertex closest to y in the path P_i is denoted v_i.

Suppose $\{L_1, L_2, L_3\}$ is a realizer of $\mathbf{P}_M$. By symmetry, we may assume that $y > x$ in L_1 and L_2 and $x > y$ in L_3. The edge $\{v_1, y\}$ can go below x only in L_3. In L_3 we thus have v_1 below $\{v_1, y\}$ below x below F_1, F_2 and F_3. In the same way, we obtain that v_2 and v_3 are below F_1, F_2 and F_3 in L_3. Hence, none of the three critical pairs of the the subposet $\mathbf{S}$ induced by v_1, v_2, v_3, F_1, F_2 and F_3 in $\mathbf{P}_M$ can be reversed in L_3. However, $\mathbf{S}$ is a crown with $\dim(\mathbf{S}) = 3$. This shows that $\{L_1, L_2, L_3\}$ is not a realizer of $\mathbf{P}_M$. Hence, $\dim(\mathbf{P}_M) > 3$. □

For the general case, where M only contains a $K_{2,3}$-subdivision, we have to use a more sophisticated technique. We illustrate this technique with an alternative proof of the simple case.

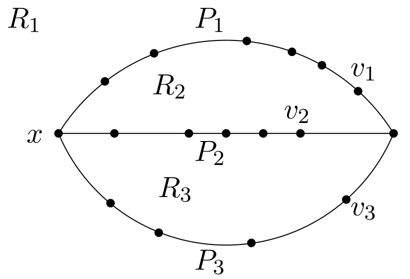

Figure 4: The three paths P_1, P_2 and P_3 partitions the map into 3 regions.

Second proof of Proposition 4. The vertices and edges of path P_i all belong to $F_i \cap F_{i+1}$ (cyclically). Hence, each path P_i induces a fence of the form $x < e_0 > v_1 < e_1 > \ldots v_s < e_s > y$ between x and y in $\mathbf{P}_M$ such that all maximal elements are below F_i and F_{i+1}. These three fences are mutually disjoint.

Suppose $\{L_1, L_2, L_3\}$ is a realizer of $\mathbf{P}_M$. By symmetry, we may assume that $y > x$ in L_1 and L_2 and $x > y$ in L_3. Now, consider the fence induced by P_i, $i \in 1, 2, 3$, see Figure 5.

The edge $\{v_i, y\}$ must be below x in L_3, hence v_i is below x in L_3. Let w_i be the last vertex encountered when traversing the path P_i from y to x which is below x in L_3, and let e_i be the edge leaving w_i in direction of x. The choice of w_i implies that e_i is above x and y in L_3. Since e_i has to go below y somewhere there is an index $j_i \in \{1, 2\}$ such that e_i and thus w_i go below y in L_{j_i}.

Two of the three indices j_1, j_2, j_3 must be equal, so we can w.l.o.g. assume that w_1 and w_2 are below all faces that contain both x and y in in L_2 and L_3.

Now, none of the critical pairs of the subposet **2+2** of $\mathbf{P}_M$ induced by w_1, w_2, F_1 and F_3 are reversed in L_2 or L_3. But $\dim(\mathbf{2+2}) = 2$, so the critical pairs of $\mathbf{Q}$ cannot be reversed in L_1 alone. Hence $\{L_1, L_2, L_3\}$ cannot be a realizer of $\mathbf{P}_M$. □

We now move on to the slightly more complicated case where M only contains a subdivision of $K_{2,3}$.

Theorem 5. *Let M be a planar map such that M contains a subdivision of $K_{2,3}$. Then* $\dim(\mathbf{P}_M) > 3$.

Proof. If M contains a subdivision of K_4, the conclusion of the lemma follows from Theorem 3. We thus can assume that M contains no subdivision of K_4.

Let x and y be the degree 3 vertices in the subdivision of $K_{2,3}$. Our goal is to find at least three mutually disjoint fences $\mathbf{T}_i$ between x and y, and a set of faces F_i such that $x, y \in F_i$ and each minimal element in $\mathbf{T}_i$ is below F_i and F_{i+1}.

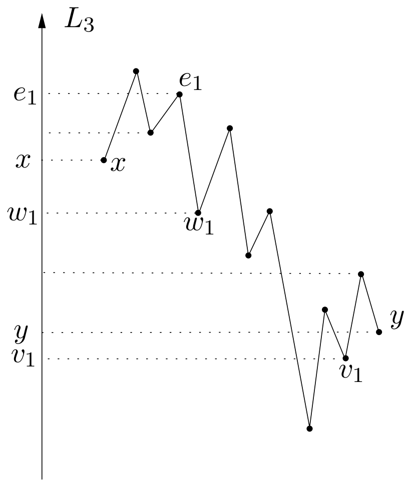

Figure 5: The fence of path P_1 and L_3.

Given fences $\mathbf{T}_i$ and faces F_i as described we can continue as in the previous proof: Assume a realizer $\{L_1, L_2, L_3\}$ such that $y > x$ in L_1 and L_2 and $x > y$ in L_3. In each fence $\mathbf{T}_i$ we find a minimal element w_i which is below x in L_3 and below y in some L_{j_i}, $j_i \in \{1, 2\}$. Since $i \geq 3$ there are indices a and b with $j_a = j_b$, and we can w.l.o.g. let $j_a = j_b = 1$. Let $a' \in \{a, a+1\}$ and $b' \in \{b, b+1\}$ be such that $w_b \notin F_{a'}$ and $w_a \notin F_{b'}$. Hence, $w_a, F_{a'}, w_b, F_{b'}$ induce a **2+2**. The critical pairs of this **2+2** are not reversed in L_3 nor in L_1, and they can't be both be reversed in L_2. This is in contradiction to the assumption that $\{L_1, L_2, L_3\}$ is a realizer. Hence, $\dim(\mathbf{P}_M) > 3$.

It remains to show how to determine appropriate fences $\mathbf{T}_i$. Consider a maximal set $P_0, P_1, \ldots, P_k$ of pairwise internally disjoint paths from x to y. Clearly, $k \geq 2$. The numbering should correspond to the cyclic order of their first edges at x. Let R_i be the bounded area between P_{i-1} and P_i. The maximality of the family $P_0, P_1, \ldots, P_k$ implies that in R_i there is a face F_i that has a nonempty intersection with the interior of both, P_{i-1} and P_i. Next we prove that this face F_i contains x and y.

Claim A. In R_i there is a face F_i containing x and y.

Otherwise, the cycle consisting of P_i and P_{i+1} has a chordal path and this path, together with P_i, P_{i+1} and some P_j, $j \notin \{i, i+1\}$ is a subdivision of K_4 in M. △

Let u and w be vertices of P_i such that u is closer to x than w and $(u, w) \neq (x, y)$. A *shortcut* between u and w over P_i is a path from u to w which is internally disjoint from P_i. Two shortcuts, $\{u_1, w_1\}$ and $\{u_2, w_2\}$ are *crossing* if their order along P_1 is either u_1, u_2, w_1, w_2 or u_2, u_1, w_2, u_1. In particular this requires the four vertices to be pairwise different.

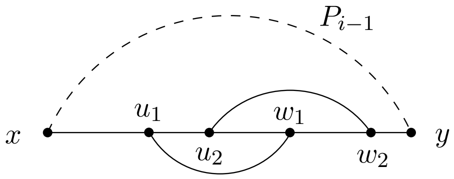

Figure 6: Crossing shortcuts.

Claim B. There is no crossing pair of shortcuts on P_i.

Otherwise, the four vertices of the two shortcuts are the degree three vertices of a subdivision of K_4. This subdivision of K_4 is formed by the shortcuts together with P_i and P_{i-1}. See Figure 6. △

Let V_i be the set of all vertices of P_i that are contained in $F_i \cap F_{i+1}$.

Claim C. Two consecutive vertices u and w in V_i either are the two endpoints of an edge or there exists a face F such that $F \cap V_i = \{u, w\}$.

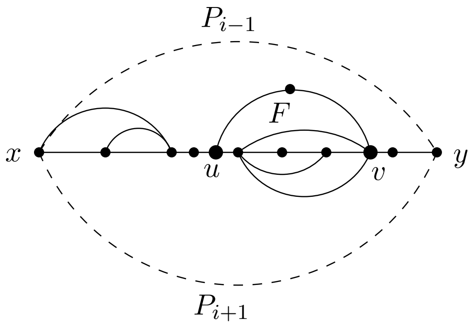

Figure 7: The common face F of u and w.

Suppose $\{u, w\}$ is not an edge. From Claim B it follows that there is a shortcut $\{u, w\}$ over P_i. Essentially the same proof as for Claim A shows that the subregion bounded by P_i and the shortcut between u and w contains a face F with $u, w \in F$; otherwise, there is a K_4-subdivision. △.

The fence $\mathbf{T}_i$ consists of V_i (the set of minimal elements) and edges, respectively faces over consecutive pairs of vertices in V_i. The existence of a $K_{2,3}$-subdivision between x and y implies that at least three of the fences $\mathbf{T}_0, \mathbf{T}_1, \ldots, \mathbf{T}_k$ are nontrivial, i.e., have minima different from x and y. These fences can be used to conclude the proof. □

Theorem 6. *If* $\dim(\mathbf{P}_M) \leq 3$*, then* M *and* M^* *are both weakly outerplanar.*

Proof. From Theorem 3 and Theorem 5, we know that if $\dim(\mathbf{P}_M) \leq 3$ then M contains neither a K_4-subdivision nor a $K_{2,3}$-subdivision. This is equivalent to saying that the graph G corresponding to the map M is outerplanar. Since $\mathbf{P}_M$ and $\mathbf{P}_{M^*}$ are dual orders and, hence, have the same dimension the same necessary condition for $\dim(\mathbf{P}_M) \leq 3$ applies to M^*. □

Note that testing if M and M^* are weakly outerplanar, i.e., if the corresponding graphs are outerplanar, can be done in linear time [7].

3 Path-like maps and permissible colorings

From Theorem 6 we know that if $\dim(\mathbf{P}_M) \leq 3$, both M and M^* are weakly outerplanar. In this section we study the order dimension of 2-connected maps M, such that M is strongly outerplanar and M^* is weakly outerplanar.

A 2-connected component of an outerplanar map M has a Hamilton cycle. If the graph of M is simple, the Hamilton cycle is unique. This yields a natural partition of the edges of M into *cycle edges* and *chordal edges*. The restriction of the dual graph to the graph induced by the vertices corresponding to bounded faces is called the *interior dual*. For a strongly outerplanar map, the edges of the interior dual are just the dual edges of the chordal edges.

We say that a simple 2-connected outerplanar map M is *path-like* if and only if the interior dual of M is a simple path. Note that this implies that the Hamilton cycle is the boundary of the outer face F_∞, i.e. that M is strongly outerplanar. Since the interior dual is a path, it follows that M^* is weakly outerplanar. On the other hand, if M is a 2-connected strongly outerplanar map and M^* is weakly outerplanar, the interior dual of M must be a simple path. Hence, M is path-like iff M is a 2-connected outerplanar map such that M^* is weakly outerplanar.

Path-like maps are in some sense the simplest ones with $\dim(\mathbf{P}_M) \leq 3$. From Theorem 6 it follows that if M is a 2-connected strongly outerplanar map with $\dim(\mathbf{P}_M) \leq 3$, M must be path-like. We can also prove something slightly stronger.

Proposition 7. *Let M be a simple 2-connected planar map with* $\dim(\mathbf{P}_M) \leq 3$*. The map M' obtained by moving all the chordal edges of M into the interior of the Hamilton cycle is path-like.*

Proof. Suppose not. Then the interior dual of M' contains a vertex of degree at least 3, and hence its dual $(M')^*$ contains a subdivision of $K_{2,3}$ with the dual of one degree 3 vertex inside the Hamilton cycle H and the dual of the other outside. We proceed to show that we can move the necessary chordal edges outside one by one to create M without destroying the $K_{2,3}$-subdivision in the dual.

We do this as follows: let $M' = M_0, M_1, \ldots, M_k = M$ be a sequence of maps such that M_{j+1} is obtained from M_j by moving a chordal edge from the inside to the outside of the Hamilton cycle. The proposition follows from the following claim.

Claim A. For each map M_i, $i = 0, 1, \ldots, k$, the dual map M_i^* contains a $K_{2,3}$-subdivision such that H^* separates the two vertices of degree 3.

We prove the claim by induction on i. We have already seen that the statement is true for $M_0 = M'$.

Suppose the claim is true for M_i. Let $e = \{u, v\}$ be the edge that has to be moved to the outside of H to get from M_i to M_{i+1}. Let F^* and G^* be the degree 3 vertices in the $K_{2,3}$-subdivision in M_i^*, where F is inside H. We construct a new map M_i', by adding the edge $e' = \{u, v\}$ to M_i outside H, see Figure 8. Note that F and G must be on the same side of the cycle $\{e, e'\}$, since otherwise $\{e^*, (e')^*\}$ is a 2-edge cut in $(M_i')^*$ separating F^* from G^*; a contradiction.

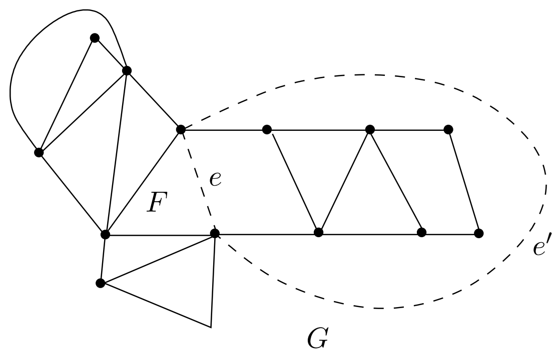

Figure 8: The map M_i' is constructed by adding e' to M_i.

Claim B. Let P^* be a simple F^*-G^* path in $(M_i')^*$. If $e^* \in P^*$, then P^* consists of at least 3 edges.

Suppose $e^* \in P^*$. Since P^* is simple, and F and G are on the same side of the cycle $\{e, e'\}$, it follows that $(e')^* \in P^*$. Moreover, the dual H^* of the Hamilton cycle H separates F^* from G^*, so P^* must contain the dual of a cycle edge. But neither e nor e' are cycle edges, so the claim follows. △

Now, M_{i+1} is obtained by removing e from M_i'. In $(M_i')^*$, this corresponds to the contraction of edge e^*. If e^* is not on any F^*-G^* path, then M_{i+1}^* contains a $K_{2,3}$-subdivision. On the other hand, an F^*-G^* path containing e^* has at least 3 edges, hence, the contraction of e^* cannot destroy the $K_{2,3}$-subdivision. □

Corollary 8. *Let M' be obtained from a simple weakly outerplanar map M by flipping all chordal edges to the interior of the Hamilton cycle. If $(M')^*$ contains a $K_{2,3}$-subdivision, then so does M^*.*

In the rest of this section, we consider *maximal path-like maps*, i.e., path-like maps where all interior faces are triangles. Consider a triangle of a maximal path-like map M. Each of the three vertices forms a critical pair with a face or edge that is above the other two vertices of the triangle. In Figure 9 these critical pairs are (u, F_u), (v, F_v), (w, e_w). Note that one of F_u or F_v can be an edge, if the triangle has two edges in the Hamilton cycle. Now, if $\dim(\mathbf{P}_M) = 3$, we can color each of these critical pairs with the linear extension it is reversed in to obtain a 3-coloring of the angles. For conveniance we interchangably use red, green and blue or 1,2 and 3 as the of colors.

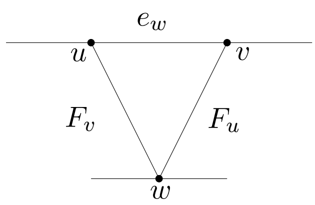

Figure 9: The critical pairs of a triangle.

We go on to prove some properties of such an angle 3-coloring for a maximal path-like map M with $\dim(\mathbf{P}_M) \leq 3$.

Lemma 9. *No two angles in a triangle can have the same color.*

Proof. Consider a triangle with the angle coloring described above. Any two of the critical pairs (u, F_u), (v, F_v), (w, e_w) form an alternating cycle. Hence, no two pairs can be reversed in the same linear extension. □

Lemma 10. *Let $e = \{a, b\}$ be a chordal edge. The four angles α_ℓ, α_r, β_ℓ and β_r incident on e at a and b are colored such that all three colors are used, and one of the pairs (α_ℓ, α_r) or (β_ℓ, β_r) is monochromatic.*

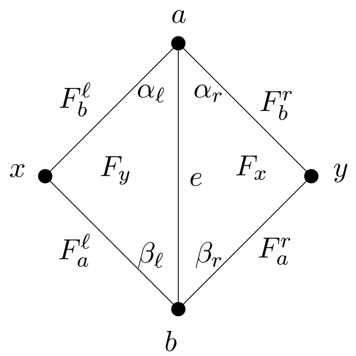

Figure 10: Colors and critical pairs around a chordal edge.

Proof. We refer to Figure 10. Suppose $\alpha_\ell = 1$ and $\alpha_r = 2$. This implies that in L_1 we have F_b^ℓ and F_b^r above a above F_a^ℓ above b. In L_2 we have the same order with F_a^r taking the role of F_a^ℓ. Hence b has to be above both F_b^ℓ and F_b^r in L_3 which is equivalent to $\beta_\ell = \beta_r = 3$.

Suppose both pairs of angles have the same colors, say $\alpha_\ell = \alpha_r = 1$ and $\beta_\ell = \beta_r = 2$ Then the third angle in both triangles (at x and y, respectively) must have color 3. This induces a monochromatic alternating cycle (x, F_x), (y, F_y), a contradiction. □

By Lemma 10 we can encode the angle coloring as an oriented coloring of the chordal edges: each chordal edge gets the color that appears twice around it and is oriented towards the endpoint where this happens.

The *orientation* of an interior triangle is is either *clockwise* or *counter-clockwise* depending on the cyclic reading which shows the colors 1,2,3 in this order. Lemma 9 implies that the orientation of interior triangles is defined.

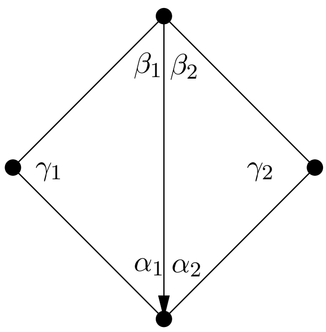

Figure 11: The triangles must have the same orientation.

Lemma 11. *All interior triangles have the same orientation.*

Proof. This is a direct consequence of Lemma 10 and Lemma 9. Referring to Figure 11 we discuss one of the cases: Suppose the left triangle in the figure is counterclockwise, i,e., $(\alpha_1, \beta_1, \gamma_1) = (i, i+1, i+2)$. The orientation of the edge implies $\alpha_2 = i$, hence from Lemma 10 we get $\beta_2 = i + 2$. This shows that the right triangle is counterclockwise as well. □

Lemma 12. *Let c be the color of the chordal edge e. Then $e > F_\infty$ in L_c.*

Proof. Again referring to Figure 11 we observe that $\gamma_1 \neq \gamma_2$ and $c \neq \gamma_1, \gamma_2$ by Lemma 11. Hence, $e < F_\infty$ in L_{γ_1} and L_{γ_2}. Therefore $e > F_\infty$ in L_c. □

Lemma 13. *A vertex is either a sink or a source w.r.t. the orientation of the chordal edges.*

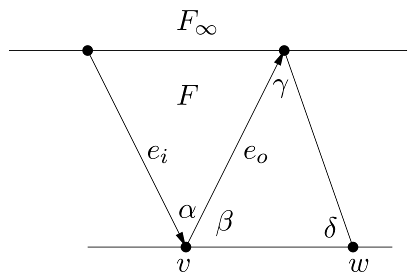

Figure 12: A vertex must be either a sink or a source.

Proof. Suppose that there is a triangle F and a vertex v such that F has two chordal edges e_i and e_o meeting at v, such that e_i is incoming and e_o is outgoing at v. See Figure 12. The colors of the angles α, β and γ must be pairwise different by Lemma 10. Hence, α and δ must have the same color (Lemma 9). But α has the same color as e_i. From Lemma 12 it follows that the alternating cycle (e_i, F_∞),(w, F) is monochromatic; contradiction. □

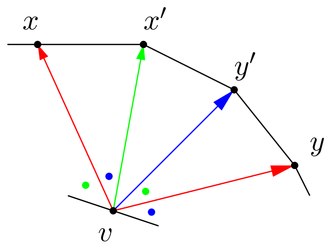

Figure 13: No two outgoing edges can have the same color.

Proposition 14. *No two outgoing edges from a vertex have the same color.*

Proof. Suppose that v has two outgoing edges of the same color and that all triangles are clockwise. From the coloring of angles it follows that edges sharing an angle have different colors. Even more, the colors of the outgoing edges at v in clockwise order cycle through 1,2,3. Hence, we find a sequence x, x', y', y of vertices, such that vx and vy have the same color, see Figure 13. Now, v is above x in blue and green, so any face incomparable to x which contains v has to be below x in red. The same is true for y. In particular $x > \{v, y', y\}$ and $y > \{v, x, x'\}$ in red. This is a monochromatic red alternating cycle; contradiction. □

Corollary 15. *No vertex belongs to four or more outgoing chordal edges.*

We say that the colors of the chordal edges bounding a face are *the colors of the face.* A face with two colors is called *bicolored.*

Proposition 16. *No two bicolored faces have the same colors.*

Proof. Suppose F and F' are two such faces. Suppose the two colors are red and green. Then F_∞ is below F and F' in red and green (Lemma 12). Therefore, F and F' can be below any vertex only in blue. Let x be vertex in $F \setminus F'$ and y be a vertex in $F' \setminus F$. Then (x, F'), (y, F) is a monochromatic blue alternating cycle; contradiction. □

We say that an oriented coloring satisfying Lemma 11, Proposition 14 and Proposition 16 is *permissible*. The map of Figure 14 is shown with a permissible coloring. We call it the *canonical map*. The vertices (edges) in the top of the figure are called q-vertices (edges) and the ones in the bottom of the figure are called p-vertices.

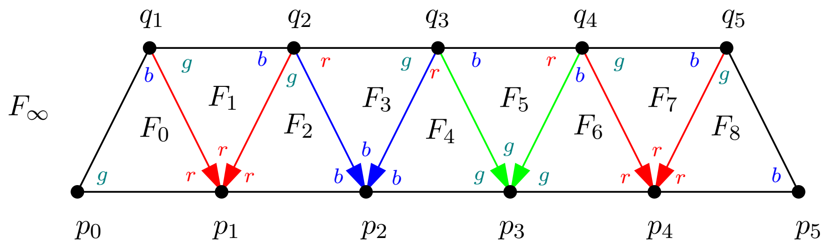

Figure 14: The canonical map.

Lemma 17. *Any maximal path-like map with a permissible coloring of the chordal edges can be constructed from the canonical map by a sequence of the following three operations:*

(i) Contracting a q-edge.

(ii) Subdividing a q-edge. Chordal edges between the new vertices and the p-vertex in the triangle are inserted with the same color and orientation as the old edges.

(iii) Deleting all the vertices, edges and faces on one side of a chordal edge.

Proof. Let M be a maxmimal path-like map with a permissible coloring of the chordal edges. If M has n sinks, then there are $n-1$ bicolored faces in M. From Proposition 16 and Lemma 11 it follows that the canonical map has the maximum possible number of sinks.

Again from Proposition 16 it follows that M has at most one vertex with outdegree 3. We can contract $\{q_2, q_3\}$ or $\{q_3, q_4\}$ in the canonical map to get such a vertex. Furthermore, any sinks or sources in the canonical map that are not in M can easily be removed using operation (iii). Hence, what remains is to possibly add some sources of degree one to the map M' we have constructed. But this is easy, since the new inserted chordal edges must have the same colors as the faces they split, i.e., we can just subdivide the q-edges as in (ii). □

Lemma 18. *Let M be the canonical map. Then* $\dim(\mathbf{P}_M) = 3$.

Proof. Consider the vertical symmetry through q_3 and the edge $\{p_2, p_3\}$. This partitions the vertex set into a left part $V_l = \{p_0, p_1, p_2, q_1, q_2\}$ a right part $V_r = \{\{p_5, p_4, p_3, q_5, q_4\}$ and $\{q_3\}$.

We construct three partial orders, one for each color. We start with the order of the vertices. In the green order we want $V_r > q_3 > V_l$. The ordering of the vertices in V_l and V_r is such that it allows to reverse all critical pairs corresponding to green angles, i.e., on $V_l : p_0 > q_1 > q_2 > \{p_2, p_1\}$ and on $V_r : p_3 > q_4 > q_5 > \{p_4, p_5\}$. We add the relations $p_2 > p_1$ and $p_5 > p_4$ so that the order on V_l and V_r conforms with the clockwise ordering around the outer face with start in p_0 and p_3, respectively.

The blue partial order is created symmetrically. I.e., it is obtained from the green order with the mappings $p_i \rightarrow p_{5-i}$ and $q_j \rightarrow q_{6-j}$. Note that the order on V_l and V_r conforms with the counterclockwise ordering around the outer face with start in p_2 and p_5, respectively.

For the red partial order, we construct two linear orders on $V_l \cup \{q_3\}$ and $V_r \cup \{q_3\}$. These linear orders gives us a partial order on $V_l \cup V_r \cup \{q_3\}$. In the linear order on $V_l \cup \{q_3\}$, the vertices come in the clockwise ordering around the outer face boundary with p_1 as maximal element, i.e, $p_1 > p_0 > q_1 > q_2 > q_3 > p_2$. The right part is done symmetrically, $p_4 > p_5 > q_5 > q_4 > q_3 > p_3$.

We now have three partial orders on the vertices. We extend these to partial orders on the vertices, edges and faces in three steps. First, we insert the Hamilton cycle edges and the outer face as low as possible in each of the three orders. Then the chordal edges are put above the outer face in their color, and as low as possible in the other two colors. Finally, the interior faces are inserted as low as possible.

Claim R. Every critical pair is reversed in one of the partial orders.

The lemma clearly follows from this claim; any three linear extensions of the partial orders constructed will then form a realizer.

There are three types of critical pairs: edge-face pairs, vertex-edge pairs and vertex-face pairs. All edge-face pairs are of the form (chordal edge, outer face), so they are reversed in the color of the edge.

Consider a vertex-edge critical pair (v, e). For the pair to be critical v and e must belong to a triangle and e has to be an edge of the Hamilton cycle. Such a critical pair corresponds to a colored angle at v. Since the order of each color reverses all critical pairs corresponding to this color each critical pair of this class is reversed.

It remains to prove that all vertex-face pairs (v, F) are reversed. Note that F is an interior face. If $v \in V_l$ and $F \subset V_r \cup \{q_3\}$, then (v, F) is reversed in blue. Similarly, if $v \in V_r$ and $F \subset V_l \cup \{q_3\}$, then (v, F) is reversed in green. All the vertices that are incomparable to F_4 are above F_4 in red, and q_3 is above all incomparable faces in green or blue. Hence, we only have to show that (v, F) is reversed when F and v are either both left or both right.

Suppose $v \in V_l$. The critical pairs (v, F_0) and $v = p_2, q_2$ are reversed in blue. The pair (p_0, F_1) is reversed in green and (p_2, F_1) in blue. The two pairs involving F_2 are reversed in green and all three pairs with F_3 in red. The cases where $v \in V_r$ are symmetric. □

Theorem 19. *Let M be a maximal path-like map. Then $\dim(\mathbf{P}_M) = 3$ if and only if there is a permissible coloring of the chordal edges.*

Proof. We have to prove that none of the operations of Lemma 17 increases the dimension.

(i): Only the q-edges $\{q_2, q_3\}$ and $\{q_3, q_4\}$ can be contracted (contracting one of the other q-edge is equivalent to the deletion of a part of the map). These two cases are symmetric, so we only consider the contraction of $\{q_2, q_3\}$. The new merged vertex $q_{2,3}$ takes the place of q_2 in green and blue, and the place of q_3 in red.

The vertex-edge pairs involving $q_{2,3}$ are reversed in green and red. Now, $q_{2,3}$ is only below the old position of q_2 in red. But the only vertex-face critical pair with q_2 that was reversed in red is (q_2, F_4), and $q_{2,3} \in F_4$, so all critical pairs involving $q_{2,3}$ are reversed.

Now, the position of a face in a partial order can only change if it contains q_2 or q_3 as its highest vertex. The new vertex $q_{2,3}$ is as high as q_2 in green and blue, and as high as q_3 in red, so the only affected face is F_5 in blue. In the blue partial order, the critical pairs (v, F_5), $v \in \{p_0, p_1, q_1\}$, are not reversed anymore. This is taken care of by moving q_1 (and hence p_1 and p_0) above q_4 in red. Since none of p_0, p_1 and q_1 where comparable to q_4 in red before, all previously reversed critical pairs are still reversed.

Suppose that both $\{q_2, q_3\}$ and $\{q_3, q_4\}$ are contracted. In this case it is enough to place the new vertex $q_{2,3,4}$ at the position of q_3 in all three vertex orders. The edges and face then are inserted by the above rules. This yields a realizer.

(ii): The partial orders are constructed as before (with possible changes resulting from a q-edge contraction). By an argument similar to the proof of Lemma 18, $\dim(\mathbf{P}_M) = 3$.

(iii): The only incidence that changes among the remaining elements of $\mathbf{P}_M$ is that one chordal edge e now is on the outer face. The edge e is moved below F_∞ in its color. The only new critical pair is (v, e), where v is the vertex in the same interior face as e that is not in e. But previously, there was a critical pair (v, F), $e \in F$, which was reversed, so (v, e) must be reversed. $\square$

3.1 Algorithmic aspects

Theorem 19 can easily be turned into an algorithm for testing if $\dim(\mathbf{P}_M) \leq 3$ for a maximal path-like map M. Start by fixing the orientation and the color of one chordal edge. This induces an angle coloring in the adjacent triangles (Lemma 10, Lemma 9). Lemma 11 now gives us the colors of the angles in all the interior triangles in M, which in turns induces an oriented coloring of the chordal edges. Hence, given a fixed orientation of one chordal edge, any permissible coloring is unique up to permutations of the colors. To test for $\dim(\mathbf{P}_M) \leq 3$, we check if any vertex has four outgoing edges or if any two bicolored faces have the same colors. This can be done in linear time.

Once we have a permissible coloring of the chordal edges of M a 3-realizer can be generated: Since we now know which vertices are sinks, we know the p-edges and q-edges of the colored map M and identify the operations of Lemma 17 (contracting a q-edge, subdividing a q-edge and removing a part of the map) that have to be applied to get M from the canonical map. The proof of Lemma 18 gives us a 3-realizer of the canonical map, and the proof of Theorem 19 demonstrates how to modify the 3-realizer for each of the operations. This yiels an algorithm to produce a 3-realizer of $\mathbf{P}_M$ from a permissible coloring of M. It is clear from the proofs Lemma 18 and Theorem 19 that the running time of this algorithm can be bounded by some constant times the number of elements in $\mathbf{P}_M$. Since M is a planar map, we have the following theorem.

Theorem 20. *There is an algorithm running in time $O(n)$, which takes as input a maximal path-like map M with n vertices and either returns a 3-realizer of $\mathbf{P}_M$, or asserts that* $\dim(\mathbf{P}_M) = 4$.

4 Vertex-face posets of dimension at most 3

From Theorem 3 we know that if $\dim(\mathbf{Q}_M) \leq 3$, then M does not contain a subdivision of K_4. Figure 15 shows an example of a planar map which contains no K_4-subdivision but still $\dim(\mathbf{Q}_M) = 4$. This example from [3], has a dual map which is a $K_{2,3}$, with each edge replaced by a 2-face. More generally every map M where we can find fences of vertices and faces like in the proof of Theorem 5 must have $\dim(\mathbf{Q}_M) = 4$.

However, unlike in the vertex-edge-face case, there are no 2-connected maps M of dimension 4 such that both M and M^* are weakly outerplanar.

Theorem 21. *Let M be a simple 2-connected planar map such both M and its dual M^* are weakly outerplanar. Then* $\dim(\mathbf{Q}_M) \leq 3$.

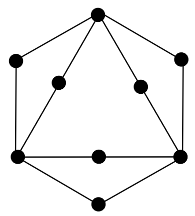

Figure 15: A planar map with vertex-face poset dimension 4.

Proof. We may assume that no two vertices v, w of degree 2 are neighbors in M: Two such vertices are twin elements in $\mathbf{Q}_M$, hence contracting the edge v, w in the graph does not affect $\dim(\mathbf{Q}_M)$.

From Corollary 8, we know that if all chordal edges of M are moved inside the Hamilton cycle, the dual of the resulting map M' is $K_{2,3}$-subdivision free. Hence, M' is a path-like map. We will inductively construct two linear extensions, L_1 and L_2, of $\mathbf{Q}_M$, in which all vertex-face pairs are reversed. Similar to the proof of Proposition 7, we start with the path-like map $M_0 = M'$ and then move the required chordal edges outside the Hamilton cycle one by one, creating a series of maps $M_0, M_1, \ldots, M_k = M$.

Since M_0 is path-like, there must be exactly two faces that contains only one chordal edge. Each of these two faces contains a vertex of degree 2. Let these two vertices be ℓ and r.

Given a face F and a vertex x we say that *x is left of F* iff there is a x-ℓ path avoiding F. Symmetrically, *x is right of F* if there is a x-r path avoiding F. This definition of left of and right of coincides with the intuition of left and right based on a drawing where the Hamilton cycle is a circle, and ℓ and r are its left and right extreme points. Note that x is neither right, nor left of F if and only if $x \in F$. We define a vertex to be to the left (right) of a chordal edge in the same way.

Next, we inductively construct two linear extensions L_1^i and L_2^i of the vertex-face poset of M_i, for $i = 0, 1, \ldots, k$, such that $L_1^k = L_1$ and $L_2^k = L_2$. Since M_0 is path-like, the interior dual of M_0 is a path. In the linear extensions L_1^0 and L_2^0 we order the interior faces by their position in this path, with the face containing ℓ highest in L_1^0 and the face containing r highest in L_2^0. The vertices and the outer face are inserted as high as possible.

Let e be the chordal edge that is moved outside the Hamilton cycle when M_i is changed to M_{i+1}. Before it is moved, it is contained in the two faces F' and F'' that are inside the Hamilton cycle. Let F' be the left one, i.e., let F' contain a vertex u that is left of e. When e is moved outside, some face G outside the Hamilton cycle is split into two faces G' and G'', let G' contain u. The faces F' and F'' are merged into a new face F^+ (see Figure 16).

We can now create L_1^{j+1} and L_2^{j+1} from L_1^j and L_2^j. In L_1^{j+1}, F^+ is inserted in the position of F' in L_1^j, G' is inserted in the position of G and

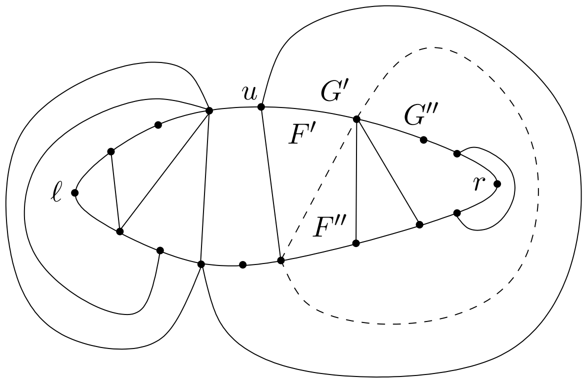

Figure 16: Moving the dashed edge from the inside to the outside.

G'' is inserted in the position of F''. In L_2^{j+1}, F^+ is inserted in the position of F'' in L_2^j, and G'' and G' are inserted in the positions of G and F', respectively.

Claim A. For $i = 0, 1, \ldots, k$, L_1^i and L_2^i are linear extensions of $\mathbf{Q}_{M_i}$. If a vertex v is to the left of a face F in the map M_i, then $v > F$ in L_1^i, and if v is to the right of F, $v > F$ in L_2^i.

The claim can be verified by induction on i. From the construction of L_1^0 and L_2^0, it is clear that the claim is true for $i = 0$.

Suppose the claim is true for i. The map M^{i+1} is constructed by moving the edge e in M_i outside the Hamilton cycle. To verify that L_1^{i+1} and L_2^{i+1} are linear extensions of $\mathbf{Q}_M$ it is enough to check that F, G' and G'' are above all the vertices contained in them. This is immediate from the construction.

It remains to prove the second part of the claim. By induction and symmetry it is enough to consider the case where v is to the left of e and F is one of F^+, G' and G''. If v is to the left of e, either $v \in F'$ or v is to left of F'. In the latter case v is above F' in L_1^i, so $v > F^+$ in L_1^{i+1}. If v is to the left of G, then v is also to the left of G' and G''. Since $v > G$ in $L_1 i$, $v > G' > G''$ in L_1^{i+1}. On the other hand, if $v \in G$, v must also be in G' and to the left of G''. But v is to the left of F'', so $v > F''$ in L_1^i by construction, and hence $v > G''$ in L_1^{i+1}. Hence, the claim is true for $i + 1$. △

Claim A implies that all vertex-face critical pairs of $\mathbf{Q}_M$ are reversed in L_1 and L_2. It remains is to find a linear extension L_3 of $\mathbf{Q}_M$ which reverses all vertex-vertex and face-face critical pairs. A vertex-vertex pair (v, w) is critical when $w \in F$ only if $v \in F$ for all faces F, and a face-face pair (F, G) is critical when $v \in F$ only if $v \in G$ for all vertices v. Hence there is no alternating cycle containing only vertex-vertex pairs and face-face pairs. This implies that there is a linear extension L_3 reversing all these pairs. Together L_1, L_2, L_3 reverse all critical pairs. Hence, they form a realizer and $\dim(\mathbf{Q}_M) \leq 3$. □

In a strongly outerplanar map M it is never the case that we can find fences like in the proof of Theorem 5. The interior dual is a tree, and F_∞ contains all the vertices. Hence, F_∞^* has to be one of the degree 3-vertices of any $K_{2,3}$-subdivision in M^* (M contains no $K_{2,3}$-subdivision since it is outerplanar). Therefore, the existence of a strongly outerplanar map of vertex-face dimension 4 is not obvious.

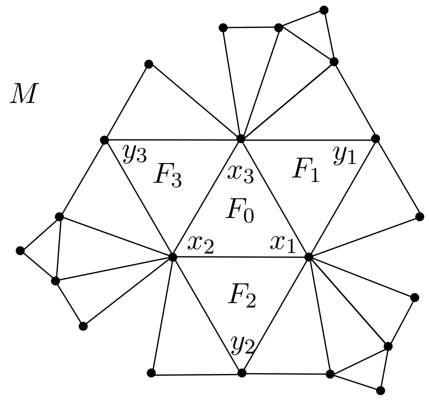

Figure 17: An outerplanar map with $\dim(\mathbf{Q}_M) = 4$.

Theorem 22. *The strongly outerplanar map M shown in Figure 17 has a vertex-face poset of dimension four.*

Proof. Suppose $\dim(\mathbf{Q}_M) \leq 3$. Again we identify the three linear extensions of a realizer with three colors and use these to color critical pairs. Our main focus will be on the coloring of critical pairs involving a vertex and an interior face.

Fact A. For an interior face F only two colors appear at critical pairs (v, F).

Suppose that the critical pair (F, F_∞) is reversed in color i. This forces all vertices below F in color i, and hence all critical pairs (v, F) are reversed in the other two colors. △

Fact B. If a triangular face $\Delta = \{v_1, v_2, v_3\}$ is surrounded by interior faces F_1, F_2, F_3 such that $v_i \notin F_i$, then the three critical pairs (v_i, F_i) use all three colors.

Any two of the three critical pairs form an alternating cycle and, hence, require different colors. Equivalently, the order induced by $v_1, v_2, v_3, F_1, F_2, F_3$ is a 3-crown. △

These two facts are applied to the central face F_0 of the map M: Fact B implies that the three critical pairs (x_i, F_{i-1}) use all three colors. Symmetry among the colors allows us to assume that (x_2, F_1) is red, (x_3, F_2) green and (x_1, F_3) blue. Fact A implies that two of the three critical pairs (y_i, F_0)

have the same color. The symmetry of the graph allows to assume that this duplicated color is blue. It is infeasible to have (y_3, F_0) in blue, because it forms an alternating cycle with the blue pair (x_1, F_3). Hence, (y_1, F_0) and (y_2, F_0) are both blue.

To reach a contradiction we can from now on concentrate on the submap of M shown in Figure 18. The colors of critical pairs which have already been fixed are indicated by the colored arrows in the figure.

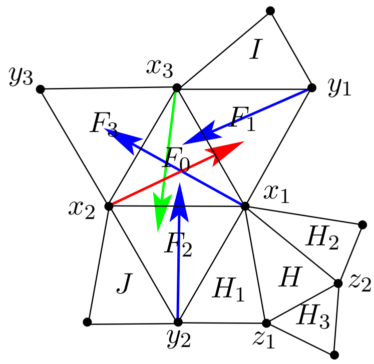

Figure 18: Having fixed the colors of some critical pairs we concentrate on a submap M' of M.

To avoid a monochromatic alternating cycle with the blue pair (y_1, F_0) and with the green pair (x_3, F_2) the color of (x_1, I) has to be red. Similarly, the colors of (y_2, F_0) and (x_2, F_1) imply that (x_1, J) is green.

From the critical pairs (x_1, J) and (x_1, F_3) we know that $x_1 > x_2$ in green and blue, so all critical pairs (x_2, F), where $x_1 \in F$ must be red. Similarly, all critical pairs (x_3, F), where $x_1 \in F$ must be green. In particular we have (x_2, H_i) red and (x_3, H_i) green for $i = 1, 2$.

From Fact A applied to H_1 and H_2 we can conclude that neither (z_1, H_2) not (z_2, H_1) can be blue. Applying Fact B to face H we can conclude that (x_1, H_3) is blue. See Figure 19.

Consider the critical pair (z_1, F_0). It forms an alternating cycle with (x_1, H_3), hence it can't be blue. It forms an alternating cycle with (x_2, H_1), hence it can't be red. It forms an alternating cycle with (x_3, H_1), hence it can't be green. Consequently there is no legal 3-coloring of the hypergraph of critical pairs of $\mathbf{Q}_M$. □

The maximal outerplanar map T_4 shown in Figure 20 has a vertex-face poset of dimension 3 (a 3-realizer is listed in Table 4). Therefore, the example of a strongly outerplanar map M with $\dim(\mathbf{Q}_M) = 4$ given in Theorem 22 is close to a minimal example. Figure 21 shows a map where all 2-connected components are submaps of T_4 and hence have vertex-face poset dimension

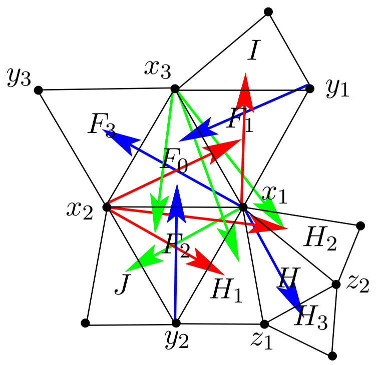

Figure 19: Colored critical pairs on M'.

3. Still an argument as in Theorem 22 shows that the map in Figure 21 has a 4-dimensional vertex-face poset.

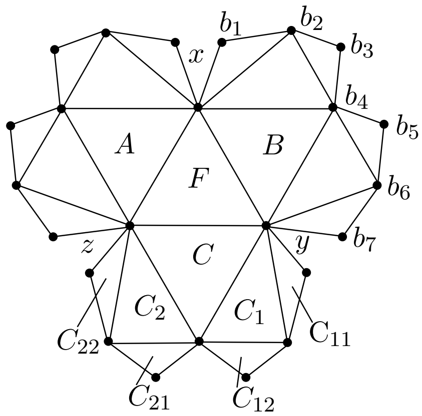

Figure 20: A 2-connected outerplanar map with vertex-face dimension 3. The naming scheme of the faces and vertices of the map is indicated in the figure.

5 Concluding remarks

When we started our investigations we set out to characterize the planar maps with vertex-edge-face posets of dimension at most 3. We proved that for all such maps M, both M and its dual M^* must be weakly outerplanar. In the case of maximal path-like maps, we found necessary and sufficient conditions for dimension at most 3 using an oriented coloring of the chordal edges. What remains open are four cases:

- M is path-like but not maximal, i.e., has non-triangular faces.

If M can be obtained from a maximal path-like map M_0 by subdividing

L_1^1	L_2^1	L_3^1	L_1^2	L_2^2	L_3^2	L_1^3	L_2^3	L_3^3
A_{22}	C	C_{11}	B_2	F	c_7	c_7	a_1	b_7
A_2	B_{12}	C_1	b_4	C_{11}	a_2	C_2	A_1	B_2
A	B_{21}	C_{12}	B_{22}	c_1	a_1	C	A	B
F	B_2	C_{21}	b_6	C_1	a_3	z	z	y
B	B_{22}	C_2	b_7	y	a_5	C_{21}	A_{12}	B_{21}
B_1	F_∞	C_{22}	A_{21}	C_{12}	C	c_6	a_2	b_6
B_{11}	b_6	A_{11}	a_6	c_2	c_4	c_5	a_3	b_5
F_∞	b_3	A_1	a_5	c_3	A_{22}	C_{12}	A_{21}	B_{12}
x	b_5	A_{12}	A_{12}	C_{21}	a_7	c_3	a_5	b_3
a_7	b_7	A_{21}	a_3	c_5	A_2	C_1	A_2	B_1
b_1	B_{11}	F_∞	A_1	C_2	a_6	c_4	a_4	b_4
B_{12}	b_1	c_2	a_4	c_4	A	C_{11}	A_{22}	B_{11}
b_2	B_1	c_6	A_{11}	C_{22}	a_4	y	x	x
b_3	b_2	c_1	a_2	c_6	F	c_2	a_6	b_2
B_{21}	B	c_3	a_1	c_7	z	c_1	a_7	b_1
b_5	b_4	c_5	C_{22}	A_{11}	B_{22}			

Table 1: A 3-realizer of the map in Figure 20. The order L_i is obtained from the concatenation $L_i = L_i^1 \oplus L_i^2 \oplus L_i^3$.

some edges of the Hamilton cycle, then $\dim(\mathbf{P}_M) = \dim(\mathbf{P}_{M_0})$. Now, let M be a 2-connected map with $\dim(\mathbf{P}_M) \leq 3$. If a cycle edge $e = \{u, v\}$ is subdivided by a vertex w incident on the new cycle edges $e_1 = \{u, w\}$ and $e_2 = \{w, v\}$ to create a new map M', we can change a 3-realizer of $\mathbf{P}_M$ into a realizer of $\mathbf{P}_{M'}$ in the following way. In each linear extension, insert e_1 in the old position of e if $v < u$, and right below v if $v > u$. Symmetrically, e_2 is inserted into the position of e if $u < v$, and right below u if $u > v$. The vertex w is inserted right below $\min\{e_1, e_2\}$. This produces a 3-realizer of $\mathbf{P}_{M'}$. The general case remains a challenge.

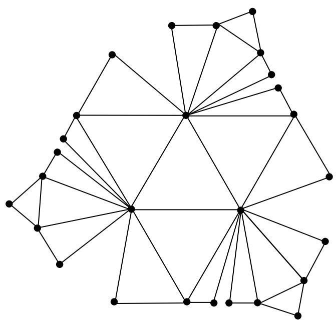

Figure 21: A map M with $\dim(\mathbf{Q}_M) = 4$, where each 2-connected component C has $\dim(\mathbf{Q}_C) = 3$

- M is simple, and both M and M^* are weakly outerplanar, but neither of them is strongly outerplanar.

We can extend the coloring approach from Section 3 to get a set of necessary conditions for $\dim(\mathbf{P}_M) \leq 3$ in this case. There is a natural way to get an oriented coloring as in the case of path-like maps, if M is a different drawing of the graph of some maximal path-like map. Instead of coloring the angles of triangular faces, we color the angles of triangles in the strongly outerplanar drawing of the graph of M. Again, this angle coloring can be encoded as an oriented coloring of the chordal edges. Instead of each vertex being a sink or a source, each vertex will now be a sink on one side of the Hamilton cycle and a source on the other side. The proof of this is similar to the proof of Lemma 13.

If F is a face in the 2-connected map M with $\dim(\mathbf{P}_M) \leq 3$, the submap M_F induced by the vertices in F must be path-like by Theorem 5. In the same way, the submap $M^*_{v^*}$ of the dual map M^* induced by the dual vertices in the dual face v^* is also path-like. Since (x, y) is a critical pair in $\mathbf{P}_M$ if and only if (y^*, x^*) is a critical pair in $\mathbf{P}_{M^*}$, the primal oriented coloring induces an oriented coloring in the dual map. Hence, the oriented coloring of a map M with $\dim(\mathbf{P}_M) \leq 3$ must be permissible "locally" around each vertex and face. The question remains whether the existence of such a locally permissible coloring is also a sufficient condition for $\dim(\mathbf{P}_M) \leq 3$, or if there are some non-local effects that force $\dim(\mathbf{P}_M) = 4$.

- M is not simple.
- M is not 2-connected.

Suppose M is not 2-connected. From $\dim(\mathbf{P}_C) \leq 3$ for each 2-connected component C it can not be concluded that $\dim(\mathbf{P}_M) \leq 3$. The conclusion is not even possible if all components are maximal path-like maps and have a common outer face. Consider the map M constructed by taking two maximal path-like maps C_1 and C_2 and identifying two vertices $v_1 \in C_1$ and $v_2 \in C_2$ and the outer faces of each map. We choose C_1 and C_2 such that $\dim(\mathbf{P}_{C_1}) = \dim(\mathbf{P}_{C_2}) = 3$ and that in any permissible coloring of the chordal edges in each map C_i there will be two outgoing edges from v_i. Such maps clearly exist. A straightforward modification of Proposition 14 will now show that $\dim(\mathbf{P}_M) = 4$.

Vertex-face posets and posets of height 2

For vertex-face posets, we saw that it seems hard to characterize even the strongly outerplanar 2-connected maps with dimension at most 3. This relates to the long-standing open question if it is NP-hard to determine if the dimension of a height 2 poset is at most 3. Yannakakis [11] proved in 1982 that it is NP-hard to determine if $\dim(\mathbf{P}) \leq 3$ for posets $\mathbf{P}$ of height at least 3. Brightwell and Trotter [3] refined the question and asked if it is

NP-hard to recognize planar maps with $\dim(\mathbf{Q}_M) \leq 3$. Given our results, it makes sense to ask this question even for 2-connected strongly outerplanar maps.

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