Mixing Times of Markov Chains on Degree Constrained Orientations of Planar Graphs

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Abstract. We study Markov chains for α-orientations of plane graphs, these are orientations where the outdegree of each vertex is prescribed by the value of a given function α. The set of α-orientations of a plane graph has a natural distributive lattice structure. The moves of the up-down Markov chain on this distributive lattice corresponds to reversals of directed facial cycles in the α-orientation. We have a positive and several negative results regarding the mixing time of such Markov chains.

A 2-orientation of a plane quadrangulation is an orientation where every inner vertex has outdegree 2. We show that there is a class of plane quadrangulations such that the up-down Markov chain on the 2-orientations of these quadrangulations is slowly mixing. On the other hand the chain is rapidly mixing on 2-orientations of quadrangulations with maximum degree at most 4.

Regarding examples for slow mixing we also revisit the case of 3-orientations of triangulations which has been studied before by Miracle et al. [20]. Our examples for slow mixing are simpler and have a smaller maximum degree, Finally we present the first example of a function α and a class of plane triangulations of constant maximum degree such that the up-down Markov chain on the α-orientations of these graphs is slowly mixing.

1 Introduction

Let $G = (V, E)$ be a graph and let $\alpha : V \to \mathbb{N}$ be a function, an α-orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all vertices $v \in V$. A variety of interesting combinatorial structures on planar graphs can be modeled as α-orientations. Examples are spanning trees, Eulerian orientations, Schnyder woods of triangulations, separating decompositions of quadrangulations. These and further examples are discussed in [10] and [14]. In this paper we are interested in Markov chains to sample uniformly from the α-orientations of a given planar graph G for a fixed α.

A uniform sampler may be used to get data for a statistical approach to typical properties of α-orientations. Under certain conditions such a chain can be used for approximate counting of α-orientations. Counting α-orientations is #P-complete in general. Mihail and Winkler [18] have shown that counting Eulerian orientations is #P-complete. Creed [4] has shown that this counting problem remains #P-complete on planar graphs. Further examples of #P-complete variants of counting α-orientations are given in [14]. In [14, Section 6.2] it is shown that counting α-orientations can be reduced to counting perfect matchings of a related bipartite graph. The latter problem can be approximately solved using the celebrated algorithm of Jerrum, Sinclair and Vigoda [15] or its improved version by Bezáková et al. [3]. These algorithms build on random sampling.

For sampling α-orientations of plane graphs, however, there is a more natural Markov chain. The reversal of the orientation of a directed cycle in an α-orientation yields another α-orientation. If G

is a plane graph and $\vec{G}, \vec{G}'$ are α-orientations of G, then we define $\vec{G} < \vec{G}'$ whenever $\vec{G}'$ is obtained by reverting a clockwise cycle of $\vec{G}$. In [10] it has been shown that this order relation makes the set of α-orientations of G into a distributive lattice.

A finite distributive lattice is the lattice of down-sets (also known as ideals) of some poset P. Let a 'step' consist in adding/removing a random element of P to/from the down-set. These step yield the *up-down Markov chain* on the distributive lattice. A nice feature of the up-down Markov chain is that it is monotone, see [22]. A monotone Markov chain is suited for using *coupling from the past*, see [23]. This method allows to sample exactly from the uniform distribution on the elements of a distributive lattice.

The challenge in applications of the up-down Markov chain is to analyze its mixing time. In [22] some examples of distributive lattices are described where this chain is rapidly mixing but there are examples where the mixing is slow. Miracle, Randall, Streib and Tetali [20] have investigated the mixing time of the up-down Markov chain for 3-orientations, a class of α-orientations intimately related to Schnyder woods. They show that there is a class of plane triangulations such that the up-down Markov chain on the 3-orientations of these triangulations is slowly mixing. For positive they show that the chain is rapidly mixing on 3-orientations of plane triangulations with maximum degree at most 6.

In this paper we present similar results for the up-down Markov chain on the 2-orientations of plane quadrangulations. These special 2-orientations are of interest because they are related to separating decompositions, a structure with many applications in floor-planning and graph drawing. We refer to [5, 12, 11] and references given there for literature on the subject. Specifically we show that there is a class of plane quadrangulations such that the up-down Markov chain on the 2-orientations of these quadrangulations is slowly mixing. On the other hand the chain is rapidly mixing on 2-orientations of quadrangulations with maximum degree at most 4.

Regarding examples for slow mixing we also revisit the case of 3-orientations, here we have somewhat simpler examples, compared to those from [20]. Our examples also have a smaller maximum degree, $O(\sqrt{n})$ instead of $O(n)$ on graphs with n vertices. We also exhibit a function α and a class of plane graphs of maximum degree 6 such that the up-down Markov chain on the α-orientations of these graphs is slowly mixing.

2 Preliminaries

In the first part of this section we give some background on the up-down Markov chain on general α-orientations. Then we discuss 2-orientations and the associated separating decompositions. Finally we provide some background on mixing times for Markov chains.

2.1 The up-down Markov chain of α-orientations

Let G be a plane graph and $\alpha : V \to \mathbb{N}$ be such that G admits α-orientations. For α-orientations $\vec{G}, \vec{G}'$ of G we define $\vec{G} < \vec{G}'$ whenever $\vec{G}'$ is obtained by reverting a simple clockwise cycle of $\vec{G}$. This order relation makes the set of α-orientations of G into a distributive lattice, see [10] or [13].

The steps of the up-down Markov chain on a distributive lattice $\mathcal{L} = (X, <)$ correspond to changes $x \leftrightarrow x'$ for covering pairs $x < x'$, i.e., pairs $x < x'$ such that there is no $y \in X$ with $x < y < x'$. In other words the up-down Markov chain performs a random walk on the diagram of the lattice. The transition probabilities are (usually) chosen uniformly with a nonzero probability for staying in a state. Since the diagram of a lattice is connected the chain is ergodic. It is also symmetric, hence, the unique stationary distribution is the uniform distribution on the set of α-orientations.

The steps of the up-down Markov chain of α-orientations are given by certain reversals of cycles. For a clean description we need the notion of a *rigid edge*. An edge of $G = (V, E)$ is *α-rigid* if it has the same direction in every α-orientation of G. Let $R \subseteq E$ be the set of α-rigid edges. Since directed cycles of an α-orientation $\vec{G}$ can be reversed, rigid edges never belong to directed cycles. Define $r(v)$ as the number of rigid edges that have v as a tail and let $\alpha'(v) = \alpha(v) - r(v)$. Now α-orientations of G and α'-orientation of $G' = (V, E - R)$ are in bijection. And with the inherited plane embedding of G' the distributive lattices are isomorphic. *rigid edge*

If G' is disconnected then we can shift connected components of G' to get a plane drawing $G^{\#}$ without nested components. Since the orientation, clockwise or counterclockwise, of a directed cycle in G' and $G^{\#}$ is identical the distributive lattices of α'-orientations are isomorphic. The steps of the up-down Markov chain of α'-orientations of $G^{\#}$ are easy to describe, they correspond to the reversal of cycles that form the boundary of bounded faces, the face boundaries of $G^{\#}$ are the *essential cycles* for the up-down Markov chain of α-orientations of G. In slight abuse of notation we also refer to the up-down Markov chain of α-orientations of G as the *face flip Markov chain*, after all the essential cycles of G are faces in $G^{\#}$. *essential cycles* *face flip Markov chain*

A more algebraic description of the lattice for a disconnected G is as follows: Let H be a component of G, then $\mathcal{L}_\alpha(G)$ can be obtained as the product of lattices $\mathcal{L}_{\alpha_1}(G - H) \times \mathcal{L}_{\alpha_2}(H)$, where α_1 and α_2 are the restrictions of α to the vertex sets of $G - H$ and H respectively.

From the previous description it follows that the elements of the poset P_α whose down-sets correspond to elements of $\mathcal{L}_\alpha(G)$, i.e., to α-orientations of G, are essential cycles. It is important to keep the following in mind:

Fact A. An essential cycle can correspond to several elements of the poset P_α.

This fact is best illustrated with an example. Figure 1(left) shows the octahedron graph G_{oct} with an Eulerian orientation, this is an α orientation with $\alpha(v) = 2$ for all v. The orientation is the minimal one in the lattice, it has no counterclockwise oriented cycle. Figure 1(middle) depicts the poset P_α the labels of the elements of P_α refer to the corresponding faces of G_{oct}. The elements $1, 1', 1''$ all correspond to the same face of G_{oct}, this face has to be reversed three times in a sequence of face flips that transforms the minimal Eulerian orientation into the maximal.

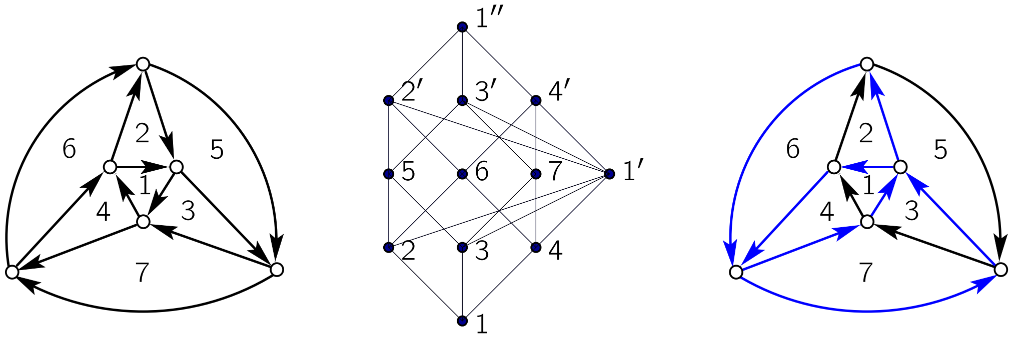

Figure 1: Left: A minimal α-orientation. Middle: The poset P_α. Right: The α-orientation corresponding to the down set $\{1, 2, 3, 4, 1', 6, 7, 4'\}$ of P_α.

The elements of P_α can be found as follows. Let $\vec{G}_{\text{min}}$ be the minimal α-orientation, i.e., the one without counterclockwise cycles. Starting from $\vec{G}_{\text{min}}$ perform *flips*, i.e., reversals of essential cycles from clockwise to counterclockwise, in any order until no further flip is possible. The unique α-orientation that admits no flip is the maximal one. The flips of a maximal flip-sequence S are the elements of P_α. Let $\hat{p}(f)$ be the number of times an essential cycle f has been flipped in S. Hence, the elements of P_α are $\{(f, i) : f \text{ essential cycle}, 1 \le i \le \hat{p}(f)\}$. *flip*

If essential cycles f and f' share an edge e then from observing the orientation of e we find that between any two appearances of f in a flip-sequence there is a appearance of f'. From this we obtain

Fact B. If essential cycles f and f' share an edge, then $|\hat{p}(f) - \hat{p}(f')| \leq 1$.

The above discussion is based on [10] where α-orientations of G have been analyzed via α-potentials, an encoding of down-sets of P_α. If $\vec{G}$ is an α-orientation, then we say that an essential cycle f is at *potential level* i in $\vec{G}$ if (f, i) belongs to the down-set $D_{\vec{G}}$ of P_α corresponding to $\vec{G}$ but $(f, i+1) \notin D_{\vec{G}}$. *potential level*

2.2 2-orientations and separating decompositions

A *quadrangulation* is a plane graphs whose faces are uniformly of degree 4. Equivalently quadrangulations are maximal bipartite plane graphs. *quadrangulation*

Let Q be a quadrangulation, we call the color classes of the bipartition white and black and name the two black vertices on the outer face s and t. A *2-orientation* of Q is an orientation of the edges such that $\text{outdeg}(v) = 2$ for all $v \neq s, t$. Since a quadrangulation with n vertices has $2n - 4$ edges it follows that s and t are sinks. *2-orientation*

A *separating decomposition* of Q is an orientation and coloring of the edges of Q with colors red and blue such that two conditions hold: *separating decomposition*

(1) All edges incident to s are ingoing red and all edges incident to t are ingoing blue.

(2) Every vertex $v \neq s, t$ is incident to a nonempty interval of red edges and a nonempty interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval of a color is outgoing and all the other edges of the interval are incoming. If v is black, the outgoing edge is the last one of its color in clockwise order (see Figure 2).

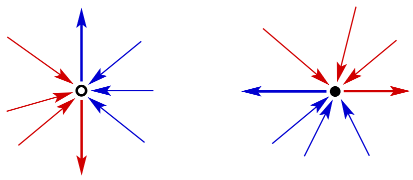

Figure 2: Edge orientations and colors at white and black vertices.

Separating decompositions have been studied in [2], [5], [11], and [12]. Relevant to us are the following two facts:

Fact 1. In a separating decomposition every vertex $v \neq s, t$ has a unique directed red path $v \to s$ and a unique blue path $v \to t$. The two paths only intersect at v.

Fact 2. The forget function that associates a 2-orientation with a separating decomposition is a bijection between the set of 2-orientations and the set of separating decompositions of a quadrangulation.

A proof of these facts can be found in [5].

2.3 Markov chains and mixing times

We refer to [16] for basics on Markov chains. In applications of Markov chains to sampling and approximate counting it is critical to determine how quickly a Markov chain M converges to its stationary distribution π. Let $M^t(x,y)$ be the probability that the chain started in x has moved to y in t steps. The *total variation distance* at time t is $\|M^t - \pi\|_{TV} = \max_{x\in\Omega} \frac{1}{2}\sum_y |M^t(x,y) - \pi(y)|$, here The *mixing time* of M is defined as $\tau_{\text{mix}} = \min_t(\|M^t - \pi\|_{TV} \le 1/4)$. The state space Ω of the Markov chains considered by us consists of sets of graphs on n vertices. Such a chain is *rapidly mixing* if τ_{mix} is upper bounded by a polynomial of n.

total variation distance

mixing time

rapidly mixing

A key tool for lower bounding the mixing time of an ergodic Markov chain is the *conductance* defined as $\Phi_M = \min_{S\subseteq\Omega,\pi(S)\le 1/2} \frac{1}{\pi(S)} \sum_{s_1\in S, s_2\notin S} \pi(s_1)\cdot M(s_1,s_2)$. The connection with τ_{mix} is given by

conductance

Fact T. $\tau_{\text{mix}} \ge (4\Phi_M)^{-1}$.

This is Theorem 7.3 from [16]. A similar result was already shown in [26]. We will use this inequality mainly in the context of *hour glass* shaped state spaces where we have a partition $\Omega^-, \Omega^0, \Omega^+$ of the state space with the property that all paths of the transition graph of the Markov chain that connect Ω^- and Ω^+ contain a vertex from Ω^0. The following lemma shows that if Ω^- and Ω^+ are large and Ω^0 is small with respect to π, then the conductance is small.

hour glass

Lemma 1 *If $\Omega^-, \Omega^0, \Omega^+$ is a partition of Ω such that $M(s_1,s_2) = 0$ for all $s_1 \in \Omega^-$ and $s_2 \in \Omega^+$, then*

$$\Phi_M \le \frac{\pi(\Omega^0)}{\min\{\pi(\Omega^-), \pi(\Omega^+)\}}.$$

Proof. We assume that $\pi(\Omega^-) \le \pi(\Omega^+)$ and hence $\pi(\Omega^-) \le \frac{1}{2}$. Now

$$\begin{aligned}
\Phi_M &= \min_{S\subseteq\Omega,\pi(S)\le\frac{1}{2}} \frac{1}{\pi(S)} \sum_{s_1\in S, s_2\notin S} \pi(s_1)\cdot M(s_1,s_2) \quad \le \quad \frac{1}{\pi(\Omega^-)} \sum_{s_1\in\Omega^-, s_2\notin\Omega^-} \pi(s_1)\cdot M(s_1,s_2) \\
&= \frac{1}{\pi(\Omega^-)} \left(\sum_{s_1\in\Omega^-, s_2\in\Omega^+} \pi(s_1)\cdot \underbrace{M(s_1,s_2)}_{=0} + \sum_{s_1\in\Omega^-, s_2\in\Omega^0} \pi(s_1)\cdot M(s_1,s_2) \right) \\
&\le \frac{1}{\pi(\Omega^-)} \left(\sum_{s_2\in\Omega^0} \sum_{s\in\Omega} \pi(s)\cdot M(s,s_2) \right) \quad = \quad \frac{1}{\pi(\Omega^-)} \sum_{s_2\in\Omega^0} \pi(s_2) \\
&= \frac{\pi(\Omega^0)}{\pi(\Omega^-)} \quad = \quad \frac{\pi(\Omega^0)}{\min\{\pi(\Omega^-), \pi(\Omega^+)\}}
\end{aligned}$$

□

3 Markov chains for 2-Orientations

In this section we study the Markov chain M_2 for 2-orientations of plane quadrangulations. This is a special instance of the up-down Markov chain for α-orientations. A step of the chain consists in the reversal of a directed essential cycle.

Lemma 2 *The essential cycles for the Markov chain M_2 of a plane quadrangulation are the four-cycles that contain no rigid edge.*

Proof. Let C be a four cycle with nonempty interior. We claim that all the edges incident to a vertex of C and a vertex from the interior of C are rigid. Let U be the set of vertices interior to C and $E[U]$ be the set of edges incident to a vertex of U. Since the cycle together with U induces a quadrangulation we have $|E[U] \cup E_C| = 2|U \cup C| - 4$, i.e., $|E[U]| = 2|U|$. Hence all edges connecting U to C are out-edges of U, this is the claim.

It follows that every four cycle that contains no rigid edge is a face boundary of a component of the non-rigid edges. This shows that such four-cycles are essential.

Now let C be a cycle of length more than 4 which is a directed cycle in some 2-orientation $\vec{Q}$. A simple counting argument as above shows that in $\vec{Q}$ there is an edge $\vec{e}$ oriented from C into the interior. From the correspondence between 2-orientations and separating decompositions together with Fact 1 we know that there is a directed path p starting with $\vec{e}$ and again hitting C. The path p together with a section of the directed cycle C is a directed cycle of $\vec{Q}$. Hence, the edges of p are not rigid and C is not a boundary of a face of a component of the non-rigid edges. □

The *Markov chain* M_2 can now be readily described. In each step it chooses a four-cycle C and $p \in [0, 1]$ uniformly at random. If C is directed in the current orientation $\vec{Q}$ and $p \leq 1/2$, then C is reversed, otherwise the new state equals the old one. The stationary distribution of M_2 is the uniform distribution. (The role of p and the threshold 1/2 is only to ensure that the Markov chain is aperiodic.)

Markov chain M_2

Fehrenbach and Rüschendorf [8] have shown that M_2 is rapidly mixing for certain subsets of the quadrangular grid. In Subsection 3.2 we generalize this result and prove rapid mixing for quadrangulations of maximum degree ≤ 4.

First, however, we show an exponential lower bound for the mixing time of M_2 on a certain family of quadrangulations.

3.1 Slow mixing for 2-orientations

Theorem 1 *Let Q_n be the quadrangulation on $5n+1$ vertices shown in Figure 3. The Markov chain M_2 on 2-orientations of Q_n has $\tau_{mix} > 3^{n-3}$.*

Proof. Let Ω be the set of 2-orientations of Q_n. We define a hour glass partition $\Omega_L, \Omega_c, \Omega_R$ of this set. The edge (x_0, s) is rigid, the second out-edge (x_0, a) of x_0 is called *left* if $a \in \{v_2, \ldots, v_n\}$, it is *right* if $a \in \{w_2, \ldots, w_n\}$ and it is *central* if $a = x_1$. Now $\Omega_L, \Omega_c, \Omega_R$ are the sets 2-orientations where the second out-edge of x_0 is left, central, and right respectively. With the next claim we show that this is a hour glass partition.

Claim 1. If $\vec{Q}_1 \in \Omega_L$ and $\vec{Q}_2 \in \Omega_R$, then $M_2(\vec{Q}_1, \vec{Q}_2) = 0$.

If $\vec{Q} \to \vec{Q}'$ is a step of M_2 which changes the second out-edge $\vec{e}$ of x_0, then the step corresponds to the reversal of an essential four-cycle containing $\vec{e}$. Any four-cycle of Q_n that contains x_0 either only contains edges from x_0 to vertices from $\{x_1, v_2, \ldots, v_n\}$ or it only contains edges from x_0 to vertices from $\{x_1, w_2, \ldots, w_n\}$. Hence, if $\vec{Q} \in \Omega_L$, then $\vec{Q}' \in \Omega_L \cup \Omega_c$. △

Claim 2. $|\Omega_c| = 1$ and Figure 3 shows the unique 2-orientation in this set.

Consider $\vec{Q} \in \Omega_c$. All the edges between $\{v_n, x_n, w_n\}$ and $\{\overline{v_n}, \overline{w_n}\}$ are oriented upward in $\vec{Q}$, they are rigid. Suppose all the edges between $\{v_k, x_k, w_k\}$ and $\{\overline{v_k}, \overline{w_k}\}$ are oriented upward in $\vec{Q}$. We also know the directed edges (v_k, x_0) and (w_k, x_0) in $\vec{Q}$. Together this accounts for all out-edges of v_k, x_k, and w_k. Hence all the edges between $\{\overline{v_{k-1}}, \overline{w_{k-1}}\}$ and $\{v_k, x_k, w_k\}$ are oriented upward in $\vec{Q}$. These edges cover all the out-edges of $\overline{v_{k-1}}$ and $\overline{w_{k-1}}$ whence all edges between $\{v_{k-1}, x_{k-1}, w_{k-1}\}$ and $\{\overline{v_{k-1}}, \overline{w_{k-1}}\}$ are oriented upward in $\vec{Q}$. With downward induction on k this shows that $\vec{Q}$ has to be the 2-orientation shown in Figure 3. △

Claim 3. $|\Omega_L| = |\Omega_R| \geq \frac{1}{2}(3^{n-1} - 1)$.

From the symmetry of Q_n we easily get that $|\Omega_L| = |\Omega_R|$. Now let P_k be the set of directed path from x_0 to v_k in $\vec{Q}$ from Figure 3. If $p \in P_k$ then (v_k, x_0) together with p forms a directed cycle

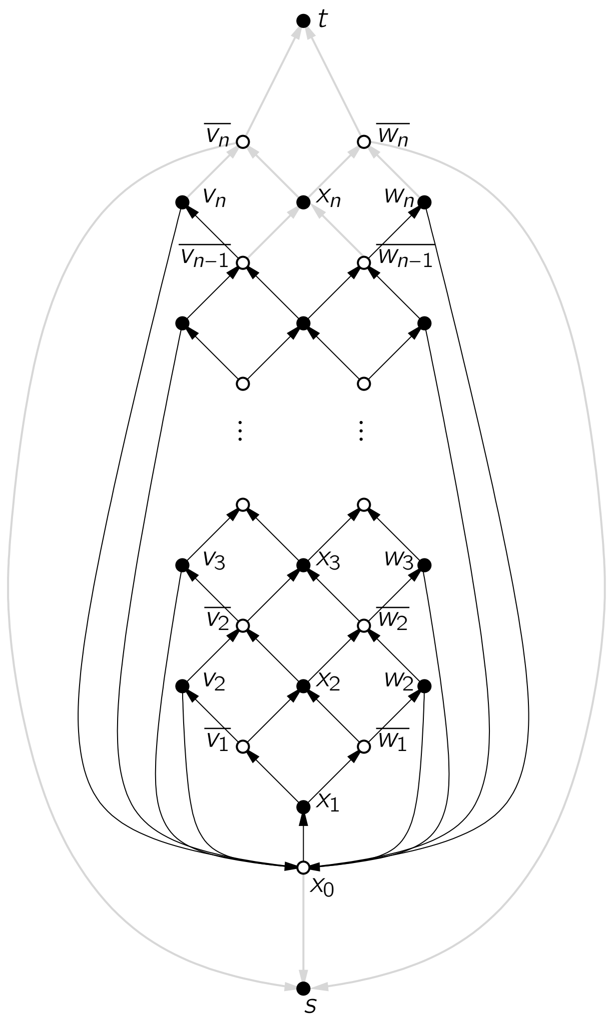

Figure 3: The graph Q_n with the unique 2-orientation containing the edge (x_0, x_1). Rigid edges are shown gray.

in $\vec{Q}$. Reverting this cycle yields a 2-orientation that contains the edge (x_0, v_k). This 2-orientation belongs to Ω_L. Different paths in P_k yield different orientations. Therefore, $|\Omega_L| \geq \sum_k |P_k|$ (in fact equality holds).

It remains to evaluate $|P_k|$. With induction we easily obtain that in $\vec{Q}$ there are exactly 3^{i-1} directed paths from x_0 to either of $\overline{v_i}$ and $\overline{w_i}$. Hence $|P_k| = 3^{k-2}$ and $|\Omega_L| \geq \sum_{2 \leq k \leq n} 3^{k-2} = \frac{1}{2}(3^{n-1} - 1)$. △

The three claims together with Lemma 1 yield $\Phi_{M_2(Q_n)} \leq \frac{2}{3^{n-1}-1}$. Which implies the theorem via Fact T. □

3.2 The tower chain for low degree quadrangulations

Following ideas originating from [17] we define a tower Markov chain M_{2T} that extends M_2. A single step of M_{2T} can combine several steps of M_2. Using a coupling argument we show that M_{2T} is rapidly mixing on quadrangulations of degree at most 4. With the comparison technique this positive result will then be extended to M_2.

The basic approach for our analysis of M_{2T} on low degree quadrangulations is similar to what Fehrenbach and Rüschendorf [8] did on a class of subgraphs of the quadrangular grid. In the context of 3-orientations of triangulations similar methods were applied by Creed [4] to certain subgraphs of the triangular grid and later by Miracle et al. [20] to general triangulations. As Creed [4] noted there is an inaccurate claim in the proof of [8]. Later Miracle et al. [19] stepped into the same trap (it has been corrected in the final version [20]). In 3.2.1 below we discuss these issues and show how to repair the proofs.

Let Q be a quadrangulation on n vertices and Ω be the set of 2-orientations of Q. From the considerations in Subsection 2.1 we know that there is a redrawing $Q^{\#}$ of the subgraph of non-rigid edges of Q such that the essential cycles of Q are the boundaries of bounded faces of $Q^{\#}$. From Lemma 2 we know that these faces are four-faces.

Let $\vec{Q}$ be a 2-orientation and C be a simple cycle. With $e^{+}(C)$ we denote the number of clockwise edges of C and with $e^{-}(C)$ the number of counterclockwise edges. If f is a four-cycle and $\nu(f) = |e^{+}(f) - e^{-}(f)|$, then $\nu(f)$ can take the values 0, 2, and 4. The face f is *oriented* if $\nu(f) = 4$, it is *scrambled* if $\nu(f) = 0$, and it is *blocked* is $\nu(f) = 2$. If f is blocked, then three edges have the same orientation and one edge does not. We call this the *blocking edge* of f.

scrambled
blocked

A *tower* of length k is a sequence $(f_1, f_2, \ldots, f_k)$ of four-cycles of $\vec{Q}$ such that each f_i for $i = 1, .., k-1$ is blocked and f_k is oriented. Moreover, in f_i the blocking edge of f_{i-1} is opposite to the blocking edge of f_i for $i = 1, .., k-1$. A tower of length 1 is just an oriented face, Figure 4 shows a tower of length 5.

tower

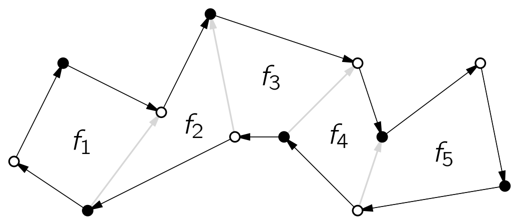

Figure 4: A tower of length 5.

Lemma 3 below implies that that removing all blocking edges from a tower T of length k we obtain a connected region whose boundary ∂T is an oriented cycle with $2k+2$ edges. This is the *boundary cycle* of the tower. The boundary cycle need not be simple but each edge of ∂T only belongs to a single face $f_i \in T$. Therefore, we can also obtain the effect of reverting ∂T by reverting $f_k, f_{k-1}, \ldots, f_1$ in this order.

boundary cycle

For the following arguments we assume that Q has no nested four-cycles. This is justified by the isomorphism between the lattices of 2-orientations of Q and of α'-orientations of $Q^{\#}$.

Lemma 3 *If f_i and f_j are different faces of a tower T and they share an edge, then $j \in \{i-1, i+1\}$ and the shared edge is the blocking edge of one of them.*

Proof. The construction sequence $f_1, \ldots, f_k$ of a tower $T = (f_1, \ldots, f_k)$ yields a directed walk in the dual. The edges of this walk are the duals of the blocking edges. Each f has at most one blocking edge and f_k has no blocking edge. Hence, there is no repetition of faces in a tower. It follows that ∂T is an oriented cycle. Two faces $f_i \neq f_j$ do not share an edge e of ∂T. This is because they would be the faces of the two sides of e whence e would be clockwise in one of them and counterclockwise in the other, however, ∂T is uniformly oriented. □

Lemma 4 *If f is a four-cycle, then there is at most one tower starting with $f = f_1$.*

Proof. Again we look at the construction sequence of a tower and the corresponding directed path in the dual. Each f_i has at most one blocking edge, hence, there is a unique candidate for f_{i+1}. If f_{i+1} is oriented it completes the tower. If f_{i+1} is blocked and the blocking edge is opposite to the edge shared with f_i the construction of the potential tower can be extended. Otherwise, there is no tower starting with f. □

tower Markov chain M_{2T}

We are ready to describe the *tower Markov chain* M_{2T}.

If M_{2T} is in state $\vec{X}$ then it performs the transition to the next step as follows: an essential four-cycle f, and a $p \in [0,1]$ are each chosen uniformly at random. If in $\vec{X}$ there is a tower T_f of length k starting with f then revert ∂T_f if

- ∂T_f is clockwise and either $k = 1$ and $p \le 1/2$ or $k > 1$ and $p \le 1/(4k)$,
- ∂T_f is counterclockwise and either $k = 1$ and $p < 1/2$ or $k > 1$ and $p \ge 1 - 1/(4k)$.

In all other cases the new state is again $\vec{X}$.

Since the steps of M_2 are also steps of M_{2T} the chain is connected. In the orientation obtained by reverting the tower $T = (f_1, \dots, f_k)$ there is the tower $T' = (f_k, \dots, f_1)$ whose reversal leads back to the original orientation. Since both towers have the same length the chain is symmetric and its stationary distribution is uniform.

The next lemma is where the degree condition is indispensable.

Lemma 5 *Let Q have maximum degree ≤ 4 and let $T = (f_1, \dots, f_k)$ be a tower and $\hat{f} \ne f_k$ be an oriented face in a 2-orientation $\vec{Q}$ of Q. If T and $\hat{f}$ share an edge e but $\hat{f}$ and f_1 share no edge, then e is the edge of f_k opposite to the blocking edge of f_{k-1}.*

Proof. Let (u_i, v_i) be the blocking edge of f_i. We extend the labeling of vertices of T such that ∂T is the directed cycle $v_0, v_1, \dots, v_{k-1}, v_k, u_k, u_{k-1}, \dots, u_1, u_0$.

If (u_{i+1}, u_i) with $i \ge 1$ is an edge of $\hat{f}$ and $u_{i-1} \notin \hat{f}$, then $\hat{f}$ contains an out-edge of u_i which is not part of T. However, u_i contains the out-edges (u_i, v_i) and (u_i, u_{i-1}). This contradicts $\text{outdeg}(u_i) = 2$.

If (v_i, v_{i+1}) with $i \ge 1$ is an edge of $\hat{f}$ and $v_{i-1} \notin \hat{f}$, then $\hat{f}$ contains an in-edge of v_i which is not part of T. Vertex v_i also contains the in-edges (u_i, v_i) and (v_{i-1}, v_i). Now v_i has in-degree ≥ 3, since $\text{outdeg}(v_i) = 2$ the degree is at least 5. A contradiction.

We are not interested in edges shared by $\hat{f}$ and f_1, i.e, in edges containing u_0 or v_0. Therefore, the only remaining candidate for e is the edge (v_k, u_k). □

Theorem 2 *Let Q be a plane quadrangulation with n vertices so that each inner vertex is adjacent to at most 4 edges. The mixing time of M_{2T} on 2-orientations of Q satisfies $\tau_{mix} \in O(n^5)$.*

The proof of Theorem 2 is based on the path coupling theorem of Dyer and Greenhill [7]. Before stating a simple version of the Dyer–Greenhill Theorem we need a definition. A *coupling* for a Markov chain M on a state space Ω is a pair (X_t, Y_t) of processes satisfying two conditions:

- Each of (X_t) and (Y_t) represents M, i.e., $\Pr(Z_{t+1} = j | Z_t = i) = M_{i,j}$, for $Z \in \{X, Y\}$ and all t.
- If $X_t = Y_t$ then $X_{t+1} = Y_{t+1}$.

Theorem 3 (Dyer–Greenhill) *Let M be a Markov chain with state space Ω. If there is a graph $\mathcal{G}_M$ with vertex set Ω and a coupling (X_t, Y_t) of M such that with the graph distance $d : \Omega \times \Omega \to \mathbb{N}$ based on $\mathcal{G}_M$ we have:*

$$\mathbb{E}[d(X_{t+1}, Y_{t+1})] \le d(X_t, Y_t) \quad \textit{and} \quad \Pr(d(X_{t+1}, Y_{t+1}) \ne d(X_t, Y_t)) \ge \rho$$

then $\tau_{mix}(M) \le 2\lceil e/\rho \rceil diam(\mathcal{G}_M)^2$.

The coupling of M_{2T} used for the proof of Theorem 2 is the trivial one, i.e., we run chains X_t and Y_t with the same choices of f and p in each step.

The graph $\mathcal{G}$ will be the transition graph of M_2, i.e, the distance between 2-orientations $\vec{X}$ and $\vec{Y}$ equals the number of four-cycles that have to be reversed to get from $\vec{X}$ to $\vec{Y}$.

Lemma 6 *The maximum potential $\hat{p}_{max} = \max_f \hat{p}(f)$ of an essential cycle is less than n.*

Proof. Let Q be the quadrangulation whose 2-orientations are in question. It is convenient to replace Q by $Q^{\#}$ so that essential cycles are just faces. Recall that $\hat{p}$ of the outer face is 0 and $|\hat{p}(f) - \hat{p}(f')| \le 1$ for any two adjacent faces (Fact B). Since a quadrangulation has $n-2$ faces we obtain $(n-3)$ as an upper bound for $\hat{p}_{\max}$. □

Lemma 7 *The diameter of $\mathcal{G}$ is at most $n^2/2$.*

Proof. The height of the lattice $\mathcal{L}_\alpha(Q^{\#})$ is the length of a maximal flip sequence, i.e., $\sum_f \hat{p}(f)$. Using (Fact B) as in the proof of the previous lemma we find that $\sum_f \hat{p}(f) \le 0 + 1 + \ldots + (n-3)$. This is $< n^2/2$.

In the diagram of a distributive lattice the diameter is attained by the distance between the zero and the one, i.e., between the global minimum and the global maximum. This distance is exactly the height of the lattice. Since $\mathcal{G}$ is the cover graph (undirected diagram) of the distributive lattice $\mathcal{L}_\alpha(Q)$ we obtain that the diameter of $\mathcal{G}$ is at most $n^2/2$. □

3.2.1 Finding an appropriate ρ

To get a reasonable ρ the following argument is tempting and was actually used in [8] and [19]: For given $\vec{X}$ and $\vec{Y}$ there is always at least one essential cycle f whose reversal in $\vec{X}$ reduces the distance to $\vec{Y}$. If $(X_t, Y_t) = (X, Y)$ and this cycle f is chosen by M_{2T}, then with probability 1/2 the distance is reduced. There are at most $n-3$ essential cycles. Hence we may set $\rho = 1/(2n)$.

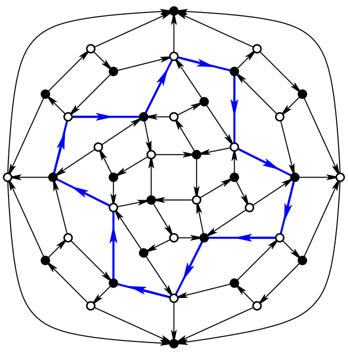

Figure 5: A quadrangulation Q with a 2-orientation $\vec{Q}$. The 2-orientation $\vec{Q}'$ obtained by reverting the blue cycles has the same oriented faces. Hence, all face reversals preserve the distance.

Indeed for up-down Markov chains on distributive such a statement holds. If I and J are down-sets of the poset P, then there is an $x \in P$ whose addition to or removal from I decreases the distance to J. In the context of α-orientations, however, an f whose reversal in $\vec{X}$ reduces the distance to $\vec{Y}$ may be oriented in $\vec{Y}$ with the same orientation as in $\vec{X}$. In that case if f is chosen by M_{2T} the reversal of f is applied to both or to none Figure 5 shows that there are cases where a pairs (X_t, Y_t) exists such that $\Pr(d(X_{t+1}, Y_{t+1}) \neq d(X_t, Y_t)) = 0$.

To overcome this problem we now define the *slow tower Markov chain* M_{S2T}. *slow tower Markov chain*

If M_{S2T} is in state $\vec{X}$ then it performs the transition to the next step as follows: an essential four-cycle f, a value i with $0 \leq i < n$ and a $p \in [0,1]$ are each chosen uniformly at random. If f is not at potential level i in $\vec{X}$, then nothing is done and $\vec{X}$ is the new state.

Otherwise, if there is a tower T_f of length k starting with f then revert ∂T_f if

- ∂T_f is clockwise and either $k = 1$ and $p \leq 1/2$ or $k > 1$ and $p \leq 1/(4k)$,
- ∂T_f is counterclockwise and either $k = 1$ and $p > 1/2$ or $k > 1$ and $p \geq 1 - 1/(4k)$.

In all other cases the new state is again $\vec{X}$.

Lemma 8 *If (X_t, Y_t) is a trivial coupling of the slow chain M_{S2T}, then*

$$\Pr(d(X_{t+1}, Y_{t+1}) \neq d(X_t, Y_t)) \geq 1/(2n^2).$$

Proof. For given $\vec{X}$ and $\vec{Y}$ there is always at least one essential cycle f_1 whose reversal in $\vec{X}$ reduces the distance to $\vec{Y}$. If f_1 appears in $\vec{X}$ and $\vec{Y}$ with the same orientation then the potential level of f_1 in $\vec{X}$ and $\vec{Y}$ is different. Hence, if for the step of M_{S2T} the triple (f, i, p) is chosen such that $f = f_1$ and i is the potential level of f in $\vec{X}$ and p is such that f is actually reversed, then the distance decreases.

The probability for choosing f is at least $1/n$. For i and p the probabilities are $1/n$ and $1/2$ respectively. Together this yields the claimed bound. □

3.2.2 Completing the proof of Theorem 2

In Lemma 9 we show that if $(\vec{X}, \vec{Y})$ is an edge of $\mathcal{G}$ and $(\vec{X}^+, \vec{Y}^+)$ is the pair obtained after a single coupled step of the tower chain M_{2T}, then $\mathbb{E}[d(\vec{X}^+, \vec{Y}^+) - d(\vec{X}, \vec{Y})] \leq 0$. Note that a step of the coupled slow chain M_{S2T} moves the pair $(\vec{X}, \vec{Y})$ to $(\vec{X}^+, \vec{Y}^+)$ with probability $1/n$ and otherwise stays at $(\vec{X}, \vec{Y})$. Hence Lemma 9 also applies to M_{S2T}.

Assuming Lemma 9 we get the following:

Proposition 1 *Let Q be a plane quadrangulation with n vertices so that each inner vertex is adjacent to at most 4 edges. The mixing time of M_{S2T} on 2-orientations of Q satisfies $\tau_{mix}(M_{S2T}) \in O(n^6)$*

Proof. For the condition $\mathbb{E}[d(X_{t+1}, Y_{t+1})] \leq d(X_t, Y_t)$ needed for the application of Theorem 3 we need Lemma 9. The inequality from the lemma is also true for M_{S2T} because this behaves like a slowed down version of M_{2T}. Linearity of expectation allows to transfer the inequality from single edges to paths.

An application of Theorem 3 with parameters $\rho = \frac{1}{2n^2}$ (Lemma 8) and $\mathrm{diam}(\mathcal{G}) \leq n^2/2$ (Lemma 7) yields $\tau_{\mathrm{mix}}(M_{S2T}) \leq en^6$. □

The mixing time of the slow chain could thus be proven with a coupling that allows an application of the theorem of Dyer and Greenhill. Now consider a single state $\vec{X}_t$ evolving according to the slow chain M_{S2T}. Note that this is exactly as if we would run the tower chain M_{2T} but only allow a transition to be conducted if an additional uniform random variable $q \in \{0, \ldots, n-1\}$ takes the value $q = 0$. It follows that the mixing times of M_{S2T} and of M_{2T} deviate by a factor of n. Therefore, $\tau_{\text{mix}}(M_{2T}) \le en^5$.

To complete the proof of Theorem 2 it remains to prove Lemma 9.

Lemma 9 *If $(\vec{X}, \vec{Y})$ is an edge of $\mathcal{G}$ and $(\vec{X}^+, \vec{Y}^+)$ is the pair obtained after a single coupled step of M_{2T}, then $\mathbb{E}[d(\vec{X}^+, \vec{Y}^+) - d(\vec{X}, \vec{Y})] \le 0$.*

Proof. Since $(\vec{X}, \vec{Y})$ is an edge of $\mathcal{G}$ they differ in the orientation of exactly one face $\hat{f}$. We assume w.l.o.g that $\hat{f}$ is oriented clockwise in $\vec{X}$ and counterclockwise in $\vec{Y}$.

Let f be the face chosen for the step of M_{2T}. Depending on f we analyze $d(\vec{X}^+, \vec{Y}^+)$ in three cases.

A. If $f = \hat{f}$, then depending on the value of p face f is reversed either in $\vec{X}$ or in $\vec{Y}$. After the step the orientations $\vec{X}^+$, $\vec{Y}^+$ coincide. The expected change of distance in this case is -1.

B. If f and $\hat{f}$ share an edge and $f \ne \hat{f}$ there are three options depending on the type of f in $\vec{Y}$.

1. Face f is oriented in $\vec{Y}$, necessarily clockwise. It follows that in $\vec{X}$ face f starts the clockwise tower $(f, \hat{f})$ of length two. In $\vec{Y}$ a face f is a clockwise tower of length 1. If $p \le 1/8$ both towers are reversed so that $\vec{X}^+$ and $\vec{Y}^+$ coincide. If $1/8 < p \le 1/2$, then f is reversed in $\vec{Y}$ while $\vec{X}^+ = \vec{X}$, in this case the distance increases by 1. If $p > 1/2$ both orientations remain unchanged. The expected change of distance in this case is $\frac{1}{8} \cdot (-1) + (\frac{1}{2} - \frac{1}{8}) \cdot (+1) + \frac{1}{2} \cdot 0 = \frac{1}{4}$.

2. Face f is scrambled in $\vec{Y}$. In this case f is blocked in $\vec{X}$ and it may start a tower of length k. If $p \le 1/(4k)$ this tower is reverted which results in a increase of distance by k. In all other cases the distance remains unchanged. Hence, the expected change of distance in this case is $\le 1/4$.

3. Face f is blocked in $\vec{Y}$. Then it is either oriented or scrambled in $\vec{X}$. After changing the role of $\vec{X}$ and $\vec{Y}$ we can use the analysis of the other two cases to conclude that the expected change of distance is again $\le 1/4$.

C. Finally, suppose that f and $\hat{f}$ have no edge in common.

1. If f starts a tower in $\vec{X}$ which has no edge in common with $\hat{f}$, then f starts the very same tower in $\vec{Y}$ and the coupled chain will either revert both towers or none of them. The distance remains unchanged.

2. Now let f start a tower $T = (f_1, \ldots, f_k)$ in $\vec{X}$ which has an edge in common with $\hat{f}$. The case where $\hat{f}$ and $f_1 = f$ share an edge was considered in **B**. Now Lemma 5 implies that either $\hat{f} = f_k$ or $\hat{f} \ne f_k$ and the shared edge is such that $(f_1, \ldots, f_k, \hat{f})$ is a tower in $\vec{Y}$. Hence, with T there is a tower T' in $\vec{Y}$ that starts in f and has length $k \pm 1$, moreover T and T' have the same orientation. Let ℓ be the larger of the lengths of T and T'. With a probability of $1/(4\ell)$ both towers are reversed and the distance decreases by 1. With a probability of $1/(4(\ell-1)) - 1/(4\ell)$ only the shorter of the two towers is reversed and the distance increases by $\ell - 1$. With the remaining probability both orientations remain unchanged. The expected change of distance in this case is $\frac{1}{4\ell} \cdot (-1) + (\frac{1}{4(\ell-1)} - \frac{1}{4\ell}) \cdot (\ell - 1) = 0$.

Let m be the number of essential four-cycles, i.e., the number of options for f. Combining the values for the change of distance in cases **A, B, C** and the probability of these cases we obtain:

$$\mathbb{E}[d(\vec{X}^+, \vec{Y}^+) - d(\vec{X}, \vec{Y})] \le \frac{1}{m}(-1) + \frac{4}{m}(1/4) + \frac{m-5}{m} 0 = 0.$$

□

3.3 Comparison of M_{2T} and M_2

The comparison of the mixing times of M_{2T} and M_2 is based on a technique developed by Diaconis and Saloff-Coste [6]. We will use Theorem 4 a variant due to Randall and Tetali [24].

Let M and $\widetilde{M}$ be two reversible Markov chains on the same state space Ω such that M and $\widetilde{M}$ have the same stationary distribution π. With $E(M)$ we denote the edges of the directed transition graph of M, i.e, $(x,y) \in E(M)$ whenever $M(x,y) > 0$. Define $E(\widetilde{M})$ alike. For each $(x,y) \in E(\widetilde{M})$ define a canonical path γ_{xy} as a sequence $x = v_0, v_1, \dots, v_k = y$ of transitions of M, i.e. $(v_i, v_{i+1}) \in E(M)$ for all i. Let $|\gamma_{xy}|$ be the length of γ_{xy} and for $(x,y) \in E(M)$ let $\Gamma(x,y) := \{(u,v) \in E(\widetilde{M}) : (x,y) \in \gamma_{uv}\}$. Further let

$$\mathcal{A} := \max_{(x,y)\in E(M)} \left\{ \frac{1}{\pi(x)M(x,y)} \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\pi(u)\widetilde{M}(u,v) \right\}$$

and let $\pi_\star := \min_{x\in\Omega} \pi(x)$.

Theorem 4 (Randall–Tetali) *In the above setting* $\tau_{mix}(M) \leq 4\log(4/\pi_\star)\,\mathcal{A}\,\tau_{mix}(\widetilde{M})$.

We are going to apply this theorem with $M = M_2$ and $\widetilde{M} = M_{2T}$. Both chains are symmetric, hence reversible, and have the uniform distribution π as stationary distribution.

The definition of the canonical paths comes quite natural. A transition $(\vec{U}, \vec{V})$ of M_{2T} corresponds to the reversal of ∂T for some tower T of $\vec{U}$. Suppose that $T = (f_1, \dots, f_k)$ and recall that the effect of reverting ∂T can also be obtained by reverting $f_k, f_{k-1}, \dots, f_1$ in this order. Reverting them one by one yields a path in $E(M)$, this path is chosen to be $\gamma_{\vec{U}\vec{V}}$.

If $|\gamma_{\vec{U}\vec{V}}| = k$, i.e., the transition $(\vec{U}, \vec{V})$ corresponds to a tower of length k, then $M_{2T}(\vec{U}, \vec{V}) = 1/(4k)$, hence, $|\gamma_{\vec{U}\vec{V}}| M_{2T}(\vec{U}, \vec{V}) = 1/4$. Also π is constant so that $\pi(\vec{U})/\pi(\vec{X}) = 1$. For an upper bound on $\mathcal{A}$ we therefore only have to estimate the number of tower moves that have a canonical path that contains the face flip at f that moves $\vec{X}$ to $\vec{Y}$. If $T = (f_1, \dots, f_k)$ is such a tower with $f = f_i$, then $(f_1, \dots, f_{i-1}, f)$ is a tower in $\vec{X}$ and $(f_k, \dots, f_{i+1}, f)$ is a tower in $\vec{Y}$. Since a tower is defined by its initial face each of $\vec{X}$ and $\vec{Y}$ has at most n towers, all the more each has at most n towers ending in f. This shows $|\Gamma(\vec{X}, \vec{Y})| \leq n^2$ and $\mathcal{A} \leq n^2/4$.

It remains to find $\pi_\star = \frac{1}{|\Omega|}$. Since a quadrangulation has $2n-4$ edges it has at most 2^{2n} orientations this would suffice for our purposes. However, a better upper bound of 1.9^n for the number of 2-orientations was obtained in [14].

Given the above ingredients for the comparison theorem and the mixing time of $\tau_{\text{mix}}(M_{2T}) \in O(n^5)$ from Theorem 2 we finally have shown rapid mixing for M_2 on certain quadrangulations.

Theorem 5 *Let Q be a plane quadrangulation with n vertices so that each inner vertex is adjacent to at most 4 edges. The mixing time of the face reversal Markov chain M_2 on 2-orientations of Q satisfies $\tau_{mix}(M_2) \in O(n^8)$.*

4 Slow mixing for 3-orientations

A *triangulation* is a plane graphs whose faces are uniformly of degree 3. Equivalently triangulations are maximal plane graphs. *triangulation*

A *3-orientation* of a triangulation T is an orientation of the internal edges, i.e., of the edges except the three edges of the outer face, such that $\text{outdeg}(v) = 3$ for all inner vertices v. Since a triangulation with n vertices has $3n-9$ inner edges it follows that the three outer vertices are sinks. *3-orientation*

A *Schnyder wood* of T is an orientation and coloring of the edges of T with colors red, green, and blue such that two conditions hold: Schnyder wood

(1) If the vertices of the outer face are colored red, green and blue in clockwise order, then all inner edges incident to a vertex s of the outer face are oriented towards s and colored in the color of s.

(2) Every inner vertex v has three outgoing edges colored red, green, and blue in clockwise order. Incoming edges in the sector between two outgoing edges are colored in the third color (see Figure 2).

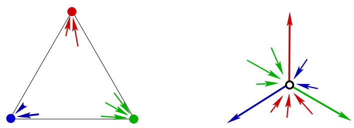

Figure 6: The two conditions for Schnyder woods.

Schnyder woods were introduced by Schnyder in [25]. We refer to [5, 21, 9, 1] and the references given there for properties, applications and generalizations of Schnyder woods. Relevant to us is the following fact, see [5]:

Fact 3. The forget function that associates a 3-orientation with a Schnyder wood is a bijection between the set of 3-orientations and the set of Schnyder woods of a triangulation.

From the correspondence between Schnyder woods and 3-orientations it follows that the triangle flip Markov chain can be used to sample from either of these structures. The mixing time of this Markov chain was studied by Creed [4] for certain subgraphs of the triangular grid and then By Miracle et al. [20] for general triangulations. Here we want to revisit the following negative result.

Theorem 6 (Miracle–Randall–Streib–Tetali) *There is a triangulation T_n' with $4n+1$ vertices with maximum degree $2n+3$ such that the triangle flip Markov chain M_3 on 3-orientations of T_n' has $\tau_{mix} > \frac{1}{16}2^{n/4}$.*

With Theorem 7 we prove a similar result with a larger exponential bound on τ_{mix}. Moreover, T_n is simpler than T_n'. This carries over to the simplicity of the proof. In fact the proof is very similar to the proof for Theorem 1. Below, in Proposition 2 we modify T_n to show that slow mixing of the triangle flip chain M_3 can also be observed for triangulations with maximum degree in the order of $\sqrt{n}$.

Theorem 7 *Let T_n be the triangulation on $3n+4$ vertices with maximum degree $2n+3$ shown in Figure 7. The triangle flip Markov chain M_3 on 3-orientations of T_n has $\tau_{mix} > (2+\sqrt{3})^{n-2} \approx 3.732^{n-2}$.*

Proof. Let Ω be the set of Schnyder woods of T_n. We define a hour glass partition $\Omega_L, \Omega_c, \Omega_R$ of this set. The edges (x_0, a_g) and (x_0, a_b) are rigid, the red out-edge (x_0, z) of x_0 is called *left* if $z \in \{v_1, \ldots, v_n\}$, it is *right* if $z \in \{w_1, \ldots, w_n\}$ and it is *central* if $z = x_1$. Now $\Omega_L, \Omega_c, \Omega_R$ are the sets Schnyder woods where the red edge of x_0 is left, central, and right respectively. With the next claim we show that this is a hour glass partition.

Claim 1. If $S_1 \in \Omega_L$ and $S_2 \in \Omega_R$, the $M_2(S_1, S_2) = 0$.

If $S \to S'$ is a step of M_3 which changes the red out-edge $\vec{e}$ of x_0, then the step corresponds to the reversal of a triangle containing $\vec{e}$. There is no triangle in T_n with vertices $\{x_0, v_i, w_j\}$ for $i, j \in [n]$. Hence, if $S \in \Omega_L$, then $S' \in \Omega_L \cup \Omega_c$. △

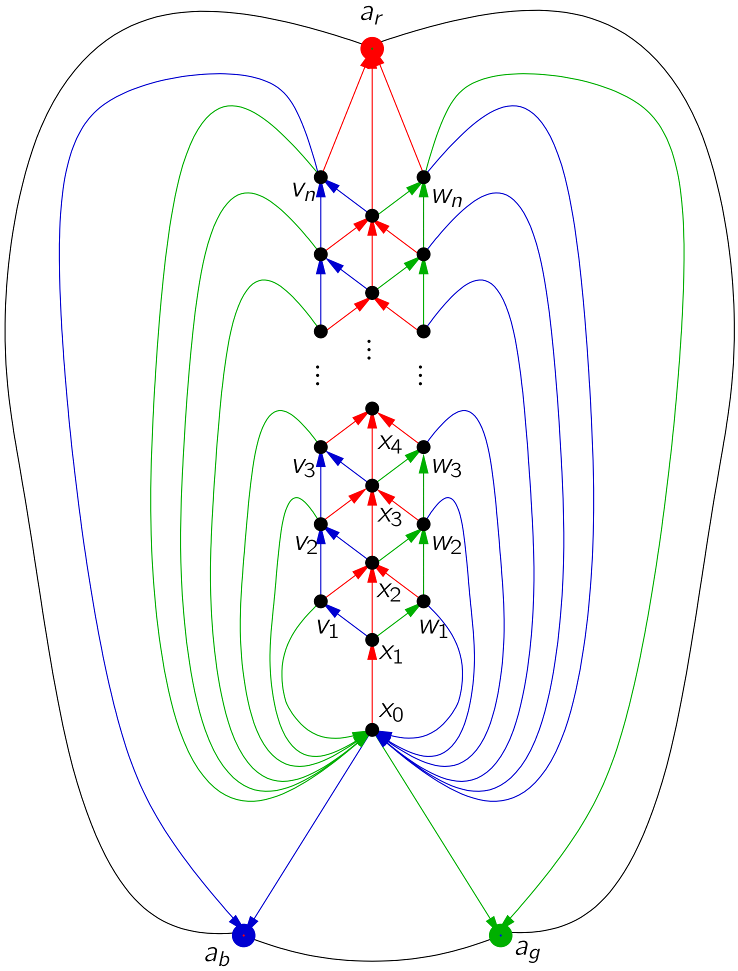

Figure 7: The triangulation T_n with a Schnyder wood.

Claim 2. $|\Omega_c| = 1$ and Figure 7 shows the unique Schnyder wood of this set.

Consider $S \in \Omega_c$. From $(x_0, x_1) \in S$ we conclude that $\{v_1, x_2, w_1\}$ are the out-neighbors of x_1. From the degrees it follows that all the edges between $\{v_1, x_2, w_1\}$ and $\{v_2, x_3, w_2\}$ are oriented upward in S. Inductively we find that all the edges between $\{v_{i-1}, x_i, w_{i-1}\}$ and $\{v_i, x_{i+1}, w_i\}$ are oriented upward in S. Since the edges (v_i, x_0) and (w_i, x_0) are in S anyway it follows that the orientation of all edges is fixed when (x_0, x_1) is fixed. The bijection between 3-orientations and Schnyder woods then yields that the Schnyder wood shown in Figure 7 is the unique element of Ω_c. △

Claim 3. $|\Omega_L| = |\Omega_R| > (2 + \sqrt{3})^{n-1}$.

From the symmetry of T_n we easily get that $|\Omega_L| = |\Omega_R|$. Now let P_k be the set of directed path from x_0 to v_k in the orientation S from Figure 7. If $p \in P_k$ then (v_k, x_0) together with p forms a directed cycle in S. Reverting this cycle yields a 3-orientation that contains the edge (x_0, v_k). This 3-orientation belongs to Ω_L. Different paths in P_k yield different orientations. Therefore, $|\Omega_L| \geq \sum_k |P_k|$ (in fact equality holds).

It remains to evaluate $g_k = |P_k|$. To do so let h_k be the number of directed paths from x_0 to x_k. Clearly, $h_{k+1} = h_k + 2g_k$ and $g_{k+1} = h_{k+1} + g_k$ with initial conditions $h_1 = g_1 = 1$. Standard techniques for solving linear recurrences yield

$$g_k = \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})^k - (2-\sqrt{3})^k\right) > (2+\sqrt{3})^{k-1}.$$

The claim now follows from $|\Omega_L| > |P_k| = g_n > (2+\sqrt{3})^{n-1}$. △

The three claims together with Lemma 1 yield $\Phi_{M_3(T_n)} \leq 1/(2+\sqrt{3})^{n-1}$. Which implies the theorem via Fact T. □

4.1 Slow mixing for 3-orientations with sub-linear maximum degree

As announced we now modify T_n to prove slow mixing for Schnyder woods of triangulations with a sub-linear maximum degree. For a given $m \in \mathbb{N}$ the triangulation $T_n(m)$ is constructed by replacing each edge $\{x_i, x_{i+1}\}$ with $i \geq 1$ by a path $x_i, y_{i,1}, \ldots, y_{i,m}, x_{i+1}$. Each vertex $y_{i,j}$ is also made adjacent to to v_i and w_i, see Figure 8. The resulting triangulation $T_n(m)$ has $3n+4+(n-1)m$ vertices and its maximum degree is $\max\{2n+3, m+5\}$.

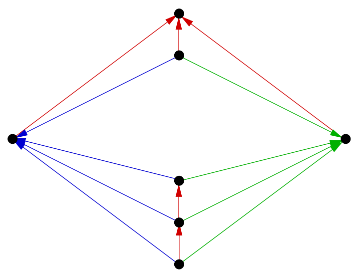

Figure 8: The gadget replacing $\{x_i, x_{i+1}\}$ in $T_n(m)$.

The definition of $\Omega_L, \Omega_c, \Omega_R$ for $T_n(m)$ is the same as for T_n. This is again a hour glass partition, i.e., there is no direct transition between Ω_L and Ω_R. Replacing the red edges (x_i, x_{i+1}) in Figure 7 by the colored gadget of Figure 8 yields the unique Schnyder wood S of Ω_c. To estimate $|\Omega_L|$ we again look at the set P_n of directed paths from x_0 to x_n in S. Since there are $2m+3$ directed path from x_i to x_{i+1} we get $|P_n| > (2m+3)^{n-1}$. From $|\Omega_L| > (2m)^{n-1}$ we obtain:

Proposition 2 *Let $T_n(2n)$ be the above triangulation on $2n^2+n+4$ vertices with maximum degree $2n+5$. The triangle flip Markov chain M_3 on 3-orientations of the triangulation $T_n(n)$ has $\tau_{mix} > 2^{(n-1)\log(4n)-2}$.*

4.2 Slow mixing for α-orientations with constant degree

In [20] and in this paper there are proofs for rapid mixing of the face flip Markov chain for α-orientations on graphs with small constant maximum degree and negative results in the sense of slow mixing of these Markov chains for graphs with large maximum degree. Could it be that the face flip Markov chain for α-orientations is rapidly mixing for all graphs of small maximum degree? In this subsection we show that this is not the case.

Our example family G_k is obtained from T_{3k-2}. In T_{3k-2} remove all edges incident to x_0 except those connecting to a_g and a_b. Let H_k be a patch taken from the triangular grid whose boundary is a regular hexagon with side length k, i.e., each side has $k+1$ vertices, and in total H_k has $3(k^2+k)+1$ vertices. Now identify two opposite corners of H_k with the vertices x_0 and x_1 of T_{3k-2}. Label the vertices on the left boundary of H_k as $v'_0 = x_1, v'_1, \ldots, v'_{3k-4}, v'_{3k-3} = x_0$ and on the right boundary as $w'_0 = x_1, w'_1, \ldots, w'_{3k-4}, w'_{3k-3} = x_0$. Add the missing edges to make v_i, v'_i, v'_{i+1} and w_i, w'_i, w'_{i+1} triangles for $i = 1, \ldots 3k-2$. Finally add the edges from v'_{3k-1} to a_b and from w'_{3k-1} to a_g. Figure 9 shows the result of the construction for $k=3$.

The graph G_k has $3(k^2+4k-1)$ vertices, the degrees are between 4 and 6. Let α be the function shown on the right part of Figure 9, the values taken by α range from 0 to 5. A key property of α is that except from the rigid edges which connect to a_g and a_b there is exactly one out-edge of H_k, i.e., one edge directed from a vertex of H_k to a vertex outside of H_k.

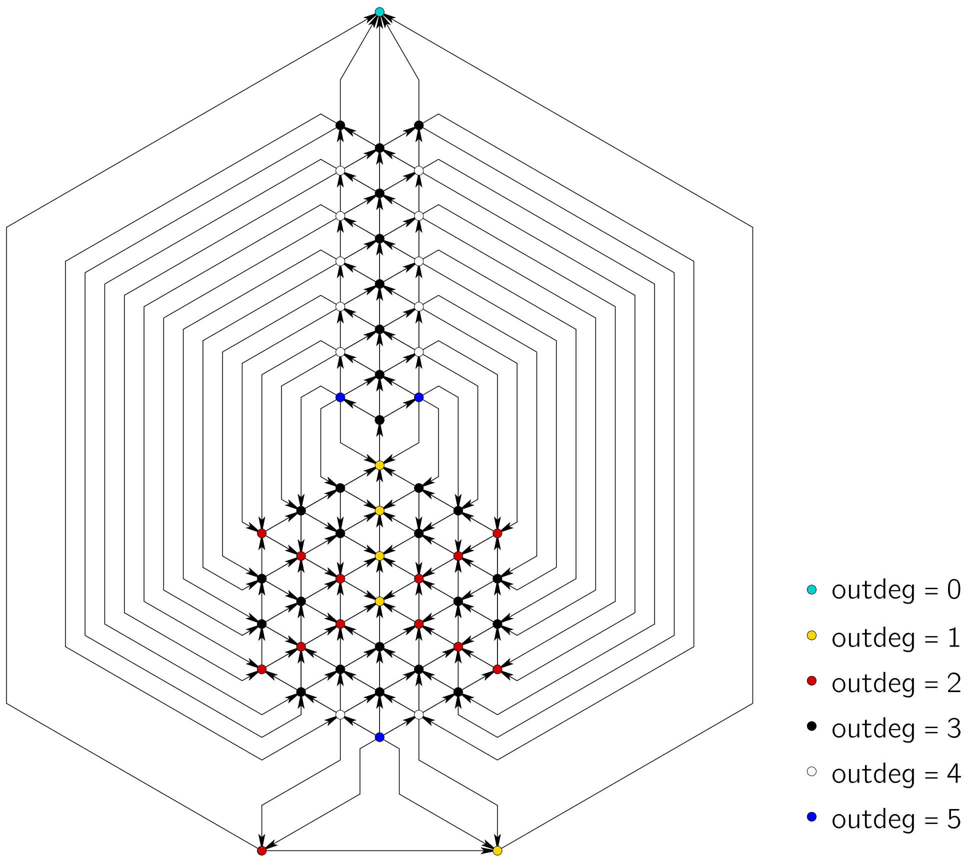

Figure 9: The graph G_3, the degrees prescribed by α, and the central α-orientation.

Let Ω denote the set of α-orientation of G_k We define a hour glass partition $\Omega_L, \Omega_c, \Omega_R$ of this set. Let (y, z) be the unique non-rigid out-edge of H_k. The edge (y, z) is called *left* if $z \in \{v_1, \dots, v_n\}$, it is *right* if $z \in \{w_1, \dots, w_n\}$ and it is *central* if $z = x_1$. Now $\Omega_L, \Omega_c, \Omega_R$ are the α-orientations where the edge (y, z) is left, central, and right respectively.

It is clear that there is no transition between elements of Ω_L and Ω_R, i.e., the partition has the hour glass property. The set Ω_c has precisely one element, this is the orientation shown in Figure 9. The size of Ω_L is at least as large as the size of the set for T_{3k-2} which has been shown to be $> 3.73^{3k-4}$. With the implied bound on the conductance and Fact T we obtain our last theorem.

Theorem 8 *The triangulation G_k on $3(k^2 + 4k - 1)$ vertices with maximum degree 6 and the function α shown in Figure 9 have a slow mixing face flip Markov chain M_α, more precisely $\tau_{mix}(M_\alpha) > 3.73^{3(k-2)}$.*

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