Bounds for the Jump Number of Partially Ordered Sets

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A *linear extension* of a partial order P is a linear order $L = x_1, x_2..., x_n$ respecting the order relations of P, i.e. $x_i < x_j$ implies $i < j$ for all $x_i, x_j \in P$. In other words, L is the linear sum $L = C_0 \oplus C_1 \oplus \ldots \oplus C_m$ of disjoint chains $C_0, C_1, \ldots, C_m$ in P, whose union is all of P, such that $x \in C_i$, $x' \in C_j$ and $x < x'$ implies $i \leq j$. We may assume the chains to be maximal, i.e. the last element of C_i, $\max C_i$, is noncomparable with the first element of C_{i+1}, $\min C_{i+1}$. The pairs $(\max C_i, \min C_{i+1})$ then are the *jumps* of P. The number of jumps of L is denoted by $s_P(L)$ and the *jump number* of P is

$$s(P) = \min\{s_P(L) : L \text{ is a linear extension of } P\}.$$

The problem of determining the jump number has been shown to be *NP*-hard by Pulleyblank [Pu], his result motivates the study of lower bounds for $s(P)$.

Width bounds and defect bounds

According to Dilworth's theorem, the minimum number of chains which form a partition of P is equal to the *width* $w(P)$ of P – the size of a maximum antichain. Therefore

$$s(P) \geq w(P) - 1. \tag{1}$$

A partial order P is called a *Dilworth* poset, if the above bound is exact, i.e. if $s(P) = w(P) - 1$. Duffus, Rival and Winkler [DRW] proved that cycle-free posets are Dilworth, Sysło, Koh and Chia [SKC] proposed a polynomial algorithm to recognize Dilworth posets of heigth one and Bouchitté, Habib [BH] proved that the recognition of Dilworth posets is *NP*-complete. In this paper we will propose some strengthenings of the width bound.

A different approach to bound $s(P)$ is due to Gierz, Poguntke [GP] they proved $rank(M_P) \geq |P| - s(P) - 1$, where M_P denotes the incidence matrix of P. Define the

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defect of a matrix M to be $def(M) = size(M) - rank(M)$ to obtain the more conveniant formulation

$$s(P) \geq def(M_P) - 1. \tag{2}$$

We now give a new, short and simple proof for this bound: Let L be a jump optimal linear extension of P, i.e. $s_L(P) = s(P)$. Order the rows and columns of M_P according to the order of L, M_P then is an upper diagonal matrix with zero diagonal. On the super-diagonal of M_P we find a 1 for each *bump* i.e. for each pair (x_i, x_{i+1}) with $x_i < x_{i+1}$ and a 0 for each jump (x_i, x_{i+1}) of L. Delete the first row and the last column of M_P (both have all entries 0) as well as all the rows and columns corresponding to a 0 on the super-diagonal. The matrix M_P^* thus obtained is an upper diagonal matrix with all 1 diagonal, hence $rank(M_P^*) = size(M_P^*) = |P| - s(P) - 1$. With $rank(M_P) \geq rank(M_P^*)$ we have finished the proof. □

We now show that the defect bound is superior to the width bound, i.e.

$$def(M_P) = |P| - rank(M_P) \geq w(P). \tag{3}$$

Let A be an antichain of size $w(P)$. The elements of P are partitioned into A, $A^{\downarrow} = \{x : x < a \text{ for some } a \in A\}$ and $A^{\uparrow} = \{x : x > a \text{ for some } a \in A\}$. Note that all the nonzero entries of M_P are in the union of the set of rows corresponding to elements of $A^{\downarrow}$ with the set of columns of the elements of $A^{\uparrow}$. Hence $rank(M_P) \leq |A^{\downarrow}| + |A^{\uparrow}| = |P| - |A|$. □

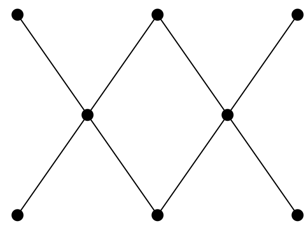
An observation first made by Chein and Habib [CH] leads to an improvement of the width bound. Let (x_i, x_{i+1}) be a bump in a linear extension L of P, then (x_i, x_{i+1}) is an edge in the diagram D_P, i.e. in the transitive reduction, of P. Therefore, the chains into which L partitions P even give a path partition of D_P. Given a digraph D, let the *width* $w(D)$ of D be the minimal size of a path partition of D. Then

$$s(P) \geq w(D_P) - 1 \geq w(P) - 1. \tag{4}$$

The following example shows that the diagram-width bound may be superior to the defect bound.

Example A

P:



$w(P) = 3$

$def(M_P) = 3$

$w(D_P) = 4$

Now let M_{D_P} be the incidence matrix of the diagram. Since our proof of (2) did only refer to the 1's induced by bumps of L we get the inequality $rank(M_{D_P}) \geq |P| - s(P) - 1$

and hence

$$s(P) \geq def(M_{D_P}) - 1. \tag{5}$$

Between this diagram-defect bound and the diagram-width bound a relation similar to (3) holds, namely

$$def(M_{D_P}) \geq w(D_P). \tag{6}$$

Proof: [Fe] exhibited a generalization of Dilworth's theorem to directed acyclic graphs. The duals to the path partitions of a dag $D = (V, E)$ are the so called 1–*weightings*. A function $\chi : V \longrightarrow \{-1, 0, 1\}$ is a 1–weighting if for all paths W in D we have $\chi(W) = \sum_{x \in W} \chi(x) \leq 1$. The *value* of a 1–weighting is $\chi(V) = \sum_{x \in V} \chi(x)$. The duality theorem is: The maximal value of an 1–weighting of D equals the minimum number of paths in a path partition of D.

Let χ be a 1–weighting of value $w(D_P)$ and $X = \chi^{-1}(1)$, $Z = \chi^{-1}(-1)$, then $w(D_P) = |X| - |Z|$. To get (6) we have to show $|P| - rank(M_{D_P}) \geq |X| - |Z|$ or equivalently

$$rank(M_{D_P}) \leq |P| - |X| + |Z|. \tag{$*$}$$

Now, delete in D_P all the elements of Z and all the edges incident with elements of Z to obtain D_P^0. The incidence matrix $M_{D_P^0}$ is obtained from M_{D_P} by deleting the $|Z|$ rows and the $|Z|$ columns corresponding to elements of Z. Therefore $rank(M_{D_P^0}) \geq rank(M_{D_P}) - 2|Z|$. The remaining elements, $P \setminus Z$, are partitioned by the 'antichain' X into the classes X, $X^{\downarrow}$ and $X^{\uparrow}$. An argument as in the proof of (3) gives $rank(M_{D_P^0}) \leq (|P| - |Z|) - |X|$. The two inequalities for $rank(M_{D_P^0})$ together give $(*)$. □

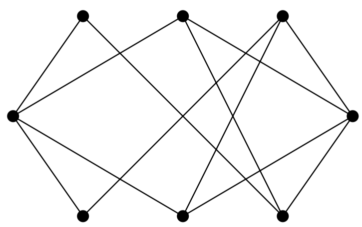
In fact, from each digraph F with intermediate edgeset, i.e. $P \geq F \geq D_P$, we obtain

$$s(P) \geq def(M_F) - 1 \geq w(F) - 1.$$

The quality of the width bound is monotone, i.e. $P \geq F_1 \geq F_2 \geq D_P$ implies $w(F_2) \geq w(F_1)$, however, the defect of M_F can be strictly larger then both, $def(M_P)$ and $def(M_{D_P})$.

Example B

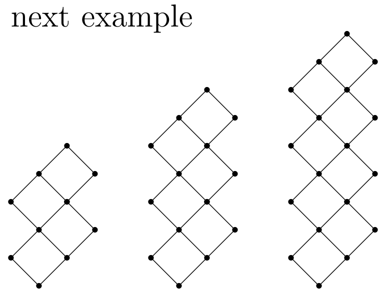
F:



$$def(M_P) = 3$$
$$def(M_F) = 4$$
$$def(M_{D_P}) = 3$$

The following problem stays open for further research: Given P, find a digraph F between P and D_P which minimizes $rank(M_F)$ (gives the best bound on $s(P)$). Anyway, even the best defect bound can be arbitrarily far from $s(P)$. This is shown by a family of orders described in the next example

Example C



The defect of all intermediate digraphs of these posets is 2, but the jump number can be arbitrarily large. We now exhibit a further bound related to path partitions.

On $\diamondsuit$-free path partitions

The crucial observation here is that every path in D_P induced by a linear extension of P is the unique connection between its minimum and ist maximum, in other words $(\min C)^{\uparrow} \cap (\max C)^{\downarrow} = C$ for all chains which are part of a linear extension. Call a path with this property a *$\diamondsuit$-free path* and denote the minimal size of a $\diamondsuit$-free path partition of D_P by $\diamondsuit(P)$. We obtain

$$s(P) \geq \diamondsuit(P) - 1 \geq w(D_P) - 1. \tag{7}$$

The following proposition proves this bound to be exact for the class of 2-dimensional lattices.

Proposition 1 *If P is a 2-dimensional lattice then $s(P) = \diamondsuit(P) - 1$.*

Proof: A partition $\mathcal{C} = \{C_1, \ldots, C_k\}$ of a diagram D_P into $\diamondsuit$-free paths can be arranged to form a linear extension only if the digraph $D_{\mathcal{C}}$ consisting of the vertices $C_1, \ldots, C_k$ and the arcs $C_i \to C_j$ if there are $x \in C_i$ and $y \in C_j$ with $x < y$ is acyclic – every 'linear extension' of $D_{\mathcal{C}}$ then gives a linear extension of P. A chordless cycle of length r in $D_{\mathcal{C}}$ would correspond to a subset (say $\{C_1, \ldots, C_r\}$) of $\mathcal{C}$, and elements $x_i, y_i \in C_i$, $i = 1 \ldots r$, satisfying

$$y_1 < x_1, x_1 > y_2, y_2 < x_2, x_2 > y_3 \ldots, x_{r-1} > y_r, y_r < x_r, x_r > y_1.$$

An ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r\}$ of size $2r$, $r \geq 2$, with these comparabilities and no others is called a *r-cycle*. A chordless cycle of length r in $D_{\mathcal{C}}$ would exhibit an induced r-cycle in P. However, r-cycles $r \geq 3$ are known to be 3-dimensional, hence they can not appear as induced suborders of a 2-dimensional lattice.

2-cycles $\{C_i, C_j\} \subseteq D_{\mathcal{C}}$ are excluded by the planarity of the diagram of a 2-dimensional lattice P (see[KR]): To elements $x_1, y_1 \in C_i$, $x_2, y_2 \in C_j$ forming a 2-cycle, i.e. $y_1 < x_1, x_1 > y_2, y_2 < x_2, x_2 > y_1$, there are paths W from y_1 to x_2 and W' from y_2

to x_1 in D_P. The paths W and W' must cross. By the planarity of D_P they can not contain crossing edges. Hence there is a point $p \in W \cap W'$. By disjointness $p \notin C_i$ or $p \notin C_j$, assume $p \notin C_i$, then the path from y_1 to x_1 through p contradicts the $\diamondsuit$-freeness of C_i. $\square$

Unfortunately I could not decide whether $\diamondsuit(P)$ is polynomially computable, bot I offer two strategies to approximate $\diamondsuit(P)$.

The first method is based on the network-flow technique used to compute $w(P)$ and $w(D_P)$ (see [Fu, Fe]). In the actual case we use a second commodity to avoid paths which are not $\diamondsuit$-free. Call a path W in D_P *minimaly dependent* if every subpath of W is $\diamondsuit$-free.

We first associate the network $N = (V, A)$ with D_P. Here

$$V = \{s_1, t_1\} \cup \{x' | x \in P\} \cup \{x'' | x \in P\},$$
$$A = \{(s, x') | \, x \in P\} \cup \{(x'', t) | \, x \in P\} \cup \{(x', y'') | \, (x, y) \in D_P\}$$

Now let $\mathcal{W}$ be the collection of all minimaly dependent paths in D_P. We extend N for the second commodity. To an arc $e \in D_P$ let $\mathcal{W}_e = (W_1^e, \ldots, W_{e_j}^e)$ be a list of all minimaly dependent paths containing e. If $e = (x, y)$ replace the edge $(x', y'') \in A$ by a path $x', w_1^{e,1}, w_1^{e,2}, \ldots, w_{e_j}^{e,1}, w_{e_j}^{e,2}, y''$, with new intermediate points. Add new points w', w'' for each $W \in \mathcal{W}$ and arcs $(w', w_i^{e,1})$ and $(w_i^{e,2}, w'')$ if W is the i-th member of $\mathcal{W}_e$. Finally insert a source s_2 and all edges (s_2, w') as well as a target t_2 and the edges (w'', t_2). This results in N^*. Note that the size of N^* is polynomially bounded in $|P|$.

Now let the capacity of all arcs in N^* be equal to one and maximize the flow F_1 from s_1 to t_1 under the requirement that the flow of the second commodity F_2 has flow value $|F_2| = |\mathcal{W}|$. Unfortunately we can not use the 2-commodity flow algorithm of Hu [Hu] here, since Hu's algorithm applies in the case of undirected networks only. But and we may solve the flow problem as a linear program.

If the solution is integral then by the usual argument the flow F_1 gives a path partition $\mathcal{C}$ of D_P into $|P| - |F_1|$ paths. The flow of the second commodity blocks one unit of flow 1 in every minimaly dependent path, hence flow 2 warrants that the paths in $\mathcal{C}$ are $\diamondsuit$-free. Vice versa every $\diamondsuit$-free path partition $\mathcal{C}$ of D_P leads to an integral flow with $|F_2| = |\mathcal{W}|$ and $|F_1| = |P| - |\mathcal{C}|$. Let $\diamondsuit_F(P) = |P| - \max |F_1|$. As we have seen $\diamondsuit_F(P)$ is a bound for $\diamondsuit(P)$.

$$\diamondsuit(P) \geq \diamondsuit_F(P) \geq w(D_P). \tag{8}$$

For the second inequality remember $w(D_P) = |P| - \max |F|$, the maximum being taken over all s_1, t_1 flows in N.

The second approximation of $\diamondsuit(P)$ is based on the *LP* formulation of partition problems. Note that the total number of $\diamondsuit$-free paths in D_P is bounded by $\binom{|P|}{2}$ since such a path is determined by its start and endpoint. Therefore the *LP*-relaxation of

$\diamondsuit$-free path partition can be solved in poynomial time. Let the objective value of an optimal solution be $\diamondsuit_{LP}(P)$. Obviously $\diamondsuit$-free path partition is a restriction of path partition; by the duality theorem cited above the *LP*-relaxation of path partition has an optimal integral solution of value $w(D_P)$, we thus get

$$\diamondsuit(P) \geq \diamondsuit_{LP}(P) \geq w(D_P). \tag{9}$$

Now we prove an inequality between $\diamondsuit_F(P)$ and $\diamondsuit_{LP}(P)$.

Proposition 2 *For every P we have $\diamondsuit_{LP}(P) \geq \diamondsuit_F(P)$.*

Proof: Let a feasible solution of the *LP*-relaxation of path partition be given, i.e. for every $\diamondsuit$-free path C in D_P we have a variable z_C, such that

$$\sum_{C:x\in C} z_C = 1 \text{ for all } x \in D_P.$$

We now construct flows F_1, F_2 in N^* with $|F_1| = |P| - \sum_C z_C$ and $|F_2| = |\mathcal{W}|$.

For edges $(x,y) \in D_P$ we push an ammount of $\sum_{x,y\in C} z_C$ trough the path $s_1, x', \ldots, y'', t_1$. This never violates the capacity constrains, since $\sum_{x,y\in C} z_C \leq \sum_{x\in C} z_C = 1$. To compute the total flow of the first commodity, $|F_1|$, we first observe

$$\sum_C z_C|C| = \sum_{x\in P}\sum_{C:x\in C} z_C = \sum_{x\in P} 1 = |P|.$$

Now

$$|F_1| = \sum_{(x,y)\in D_P}\sum_{C:x,y\in C} z_C = \sum_C z_C(|C|-1) = |P| - \sum_C z_C$$

as required.

It remains to show, that we still can realize a flow of commodity 2 and value one from w' to w'' for every $W \in \mathcal{W}$. The remaining capacity for a $w' \to w''$ flow is

$$\begin{aligned}\sum_{(x,y)\in W}(1 - F_1(x',y'')) &= \sum_{(x,y)\in W}\Big(1 - \sum_{C:x,y\in C} z_C\Big)\\ &= \Big(|W|-1\Big) - \Big(\sum_C z_C(|C\cap W|-1)\Big)\\ &= \Big(|W|-1\Big) - \Big(\sum_C z_C|C\cap W|\Big) + \Big(\sum_{C:C\cap W\neq\emptyset} z_C\Big).\end{aligned}$$

For the second term we obtain

$$\sum_C z_C|C\cap W| = \sum_{x\in W}\sum_{C:x\in C} z_C = |W|.$$

The third term can be estimated by

$$\sum_{C:C\cap W\neq\emptyset} z_C \geq \sum_{C:\min W\in C} z_C + \sum_{C:\max W\in C} z_C = 2,$$

the inequality holds since no $\diamondsuit$-free path can contain both of $\min W$ and $\max W$. Altogether we have established

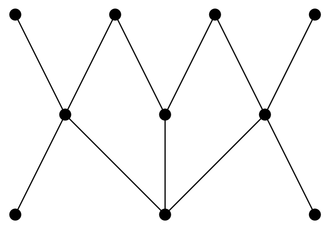
$$\big(|W|-1\big) - \Big(\sum_C z_C|C\cap W|\Big) + \Big(\sum_{C:C\cap W\neq\emptyset} z_C\Big) \quad \geq \quad (|W|-1) - |W| + 2 \quad = \quad 1$$

and hence the claim. □

The following example gives an order with $\diamondsuit_F(P) < \diamondsuit_{LP}(P)$.

Example D

P:



$$\diamondsuit_F(P) \quad = \quad 4.5$$
$$\diamondsuit_{LP}(P) = s(P)+1 \quad = \quad 5$$

For the comparison of the defect bounds and the $\diamondsuit$-free bounds note that

- For the family given in example C: $s(P) = \diamondsuit_F(P) - 1$
- For all crowns: $def(M_P) \geq \diamondsuit(P)$

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