Linear-Time Algorithms for Rectilinear Hole-free Proportional Contact Representations

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Abstract. A proportional contact representation of a planar graph is one where each vertex is represented by a simple polygon with area proportional to a given weight and adjacencies between polygons represent edges between the corresponding pairs of vertices. In this paper we study proportional contact representations that use only rectilinear polygons and contain no unused area or hole. There is an algorithm that gives a hole-free proportional contact representation of a maximal planar graph with 12-sided rectilinear polygons in $O(n \log n)$ time. We improve this result by giving a linear-time algorithm that produces a hole-free proportional contact representation of a maximal planar graph with a 10-sided rectilinear polygons. For a planar 3-tree we give a linear-time algorithm for a hole-free proportional contact representation with 8-sided rectilinear polygons. Furthermore, there exist a planar 3-tree that requires 8-sided polygons in any hole-free contact representation with rectilinear polygons. A maximal outerplanar graph admits a hole-free proportional contact representation with rectangles.

1 Introduction

Representing planar graphs as *contact graphs* has been an extensive field of research due to its theoretical appeal as well as practical interesets. In such a representation, vertices are represented by geometrical objects like curves, line-segments or polygons with edges corresponding to two objects touching in some specified fashion.

In this paper, we consider *contact representations* of planar graphs, with vertices represented by simple polygons and adjacencies represented by a non-trivial contact between the corresponding polygons. In the weighted version of the problem, an weight function on the vertices of the planar graph G is also given and the goal is to find a contact representation of G where the area of the polygon for each vertex is proportional to the weight assigned to it. We call such a representation a *proportional contact representation* of G. Such representations often lead to a more compelling visualization of a planar graph than usual node-link representations [4] and have practical applications in cartography, VLSI Layout, and floor-planning. Aesthetic, practical and cognitive requirements often prefer the use of rectilinear polygons and desire to limit the complexity of the polygons in such a representation. In practical areas like VLSI and architectural floor-planning, it is also desirable to minimize the unused area in the representation, also known as "holes". We thus address the problem of constructing a proportional contact representation of a planar graph with rectilinear polygons so that the the representation contains no hole and the maximum complexity of a polygon is as few as possible.

1.1 Related Work

Contact representations of planar graphs involves early result that dates back to 1936 when Koebe showed that a planar graph can be represented by touching circles. More recent results give contact representations of planar graphs with triangles [6] and with cubes in 3D [7]. Badent *et al.* [1] show that planar partial 3-trees and some series-parallel graphs have contact representations with homothetic triangles. Recently, Gonçalves *et al.* [10] proved that any 3-connected planar graph and its dual can be simultaneously represented by touching triangles.

While the above results deal with representations with point-contacts between polygons, representations with side-contact, though demanded more in practice, are less studied. Furthermore point-contacts tend to induce holes in the representation, which is extremely undesirable in some application areas. Gansner *et al.* [8] addressed this issue and showed that six sided polygons are sometimes necessary and always sufficient for hole-free contact representation of any planar graph with convex polygons. Outer-planar graphs have contact representations of triangles with side-contacts [9].

Application in VLSI or architectural layout design encourages the use if rectilinear polygons in a contact representation and a related research area deals with contact representations using rectilinear polygons that fits inside a rectangle. it is known that 8 sides are sometimes necessary and always sufficient for such a representation (see e.g. [11, 15, 25]). The characterization of graphs admitting such a representation with rectangles were obtained by Kozḿiński and Kinnen [14] and in the dual setting by Ungar [23]. A similar characterization of graphs having representations with 6-sided rectilinear polygons was given by Sun and Sarrafzadeh [21]. Buchsbaum *et al.* [4] give an overview on the state of the art concerning rectangle contact graphs.

Note that in all these contact representation results mentioned above, the areas of the polygons is not considered; these results deal with the unweighted version of the problem. We will now look into algorithms that take both area of shapes and existence of holes into account. Algorithms by van Kreveld and Speckman [24] and Heilmann *et al.* [12] yield representation with rectangular polygons but in the representations obtained by these algorithms, the adjacencies may be disturbed and there can also be a small distortions on the weight. De Berg *et al.* describe an algorithm for hole-free proportional contact representation with rectilinear polygons of at most 40 sides for an internally triangulated plane graph G (and of only 20 sides when G has four vertices on the exterior face and contains no separating triangles [5]. This was later improved to 34 sides [13].

The problem has also been studied in the dual settings, where the weights are assigned to the internal faces of a plane graph G instead of the vertices and a drawing of G with the area of the faces proportional to the prescribed weights is desired. All planar cubic graphs admit such a drawing [22] and so do all planar partial 3-trees [2] but not all planar graphs [18]. Rectilinear drawings with prescribed face areas were studied in [3, 17]. Beidl and Velázquez gave an algorithm to draw a cubic triconnected graph with at most 12 corners per face keeping the prescribed face areas [3] in $O(n \log n)$ time.

Since the dual of a maximal planar graph is a cubic triconnected graph, this result immediately gives a proportional contact representation of a maximal planar graph with 12-sided rectilinear polygons. Rahman *et al.* gave an algorithm for drawing of a special class of plane graphs [17] with prescribed face areas using only rectilinear 8-sided faces. However, generalizing this to general planar graphs remains open.

1.2 Our Contribution

In this paper, we improve the result of [3] by giving a linear-time algorithm for a hole-free proportional contact representation of a maximal planar graph with a 10-sided rectilinear polygons. We also give a linear-time algorithm for a hole-free proportional contact representation of a planar 3-tree with 8-sided rectilinear polygons. In both the cases 8 is the lower bound on the complexity of the polygons since there exist a planar 3-tree that requires 8-sided polygons in any hole-free contact representation with rectilinear polygons. We also showed that a hole-free proportional contact representation of a maximal outerplanar graph with rectangles can be constructed in linear time.

The following table summarizes our results in this paper.

Class of Graphs	Complexity Lower Bound	Complexity Upper Bound	Complexity of Outer Boundary	Time Complexity
Maximal Planar Graphs	8 [25]	10	4	$O(n)$
Planar 3-Trees	8	8	4	$O(n)$
Maximal Outer-planar	4	4	$O(n)$	$O(n)$

2 Representations for Maximal Planar Graphs

In this section, we give an algorithm for a hole-free proportional contact representation of a maximal planar graph with 10-sided rectilinear polygons. We construct the drawing using the concept of a "Schnyder realizer" [20], which we review here briefly. A *Schnyder realizer* of a fully triangulated graph G is a partition of the interior edges of G into three sets T_1, T_2 and T_3 of directed edges such that for each interior vertex v, the following conditions hold:

- v has out-degree exactly one in each of T_1, T_2 and T_3,
- the counterclockwise order of the edges incident to v is: entering T_1, leaving T_2, entering T_3, leaving T_1, entering T_2, leaving T_3.

The first condition implies that each T_i, $i = 1, 2, 3$ defines a tree rooted at exactly one exterior vertex and containing all the interior vertices such that the edges are directed towards the root. Schnyder proved that any triangulated planar graph has a Schnyder realizer and it can be computed in $O(n)$ time [20].

We now have the following theorem that states the main result of this section.

Theorem 1. *Let $G = (V, E)$ be a maximal planar graph and let $w : V \rightarrow R^+$ be a weight function. Then a hole-free proportional contact representation Γ with respect to w can constructed in linear time such that each vertex of G is represented by a 10-sided rectilinear polygon in Γ.*

For the rest of this section, we will prove Theorem 1 by giving a linear-time algorithm for constructing a hole-free proportional contact representation Γ of G where each vertex of G is represented by a 10-sided rectilinear polygon of a fixed shape, as illustrated in Figure 1(a). (Note that some sides of the polygon may be degenerate for a vertex.) This polygon can be decomposed by vertical lines into four rectangles and from left to right, these four rectangles are called *foot*, *leg*, *bridge* and *body* of the polygon, respectively. The two parallel horizontal lines containing the top and the bottom of the bridge is called the *bridge-strip* of the polygon. Similarly we define the *foot-strip* of the polygon.

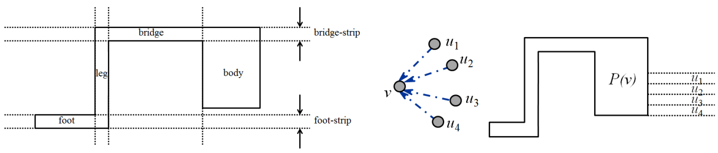


Fig. 1. (a) A 10-sided rectilinear polygon with decomposition into foot, leg, bridge and body, (b) Setting the foot-strip of of a polygon $P(v)$ representing a vertex v.

Let $G = (V, E)$ be a maximal plane graph with the three outer vertices v_1, v_2 and v_3 in the counterclockwise order. We first find a Schnyder realizer of G that partitions the interior edges into three rooted trees T_1, T_2 and T_3 rooted at v_1, v_2 and v_3, respectively, such that all the edges are oriented towards the outer vertex in each of these trees. We add the edges (v_1, v_2), (v_1, v_3) to T_1 and (v_2, v_3) to T_2 so that all the edges of G are partitioned into the three trees. For each vertex v of G, we denote by $f_i(v)$, $i = 1, 2, 3$, the parent of v in T_i. Let R be a rectangle with area equal to $\sum_{v \in V} w(v)$, where $w(v)$ denotes the weight assigned to v. We will construct a proportional contact representation of G inside R. We start by cutting a rectangle $P(v_1)$ with area $w(v_1)$ for v_1 form the top of R and cutting a rectangle $P(v_2)$ with area $w(v_2)$ for v_2 from the left side of $R - P(v_1)$. In the remaining part $R' = R - P(v_1) - P(v_2)$ of the rectangle, we draw the polygons for the other vertices. Let W and H be the width and the height of R', respectively. For each polygon $P(v)$ representing a vertex $v \in V - \{v_1, v_2\}$, we set a parameter $\lambda(v)$ to be the height of the foot and the bridge of $P(v)$ and the width of the leg of $P(v)$. A well chosen value of $\lambda(v)$ can lead to a 'balanced' appearance of $P(v)$, while a too large value of $\lambda(v)$ may cause the algorithm to fail. We fix the value such that $\lambda(v) \leq \frac{w(v)}{2H+W}$. We draw the polygons such that for each vertex v of G, the top of the bridge and the body of $P(v)$ is adjacent to the bottom of the bridge of $P(f_1(v))$, the left of the foot of $P(v)$ is adjacent to the right of the body of $P(f_2(v))$ and the bottom of the body of $P(v)$ is adjacent to the top of the foot of $P(f_3(v))$.

The embedding of G gives a natural left-to-right order for the children of each vertex in the rooted tree T_1. We now place the polygons $P(v)$ for each vertex $v \in V - \{v_1, v_2\}$ inside R' by a ordered depth-first traversal of T_1. We start the traversal from the the child of v_1 immediately to the right of v_2 and for each vertex $v \in V - \{v_1, v_2\}$ we do the followings:

– fix the bridge-strip, foot and leg of $P(v)$,
– recursively traverse each child of v in T_1 and fix $P(u)$ for each descendant u of v in T_1,
– fix the bridge and the body of $P(v)$.

We thus traverse each vertex $v \in V - \{v_1, v_2\}$ twice, first to fix the foot and leg and then to fix the bridge and body of $P(v)$. We have the following lemma whose proof is immediate from the order of the traversal of the vertices.

Lemma 1. *Let u and v be two vertices of G. Then the following conditions hold.*
(a) The first traversal of u comes before the first traversal of v if and only if u comes before v in the pre-order traversal of T_1.
(b) The second traversal of u comes before the second traversal of v if and only if u comes before v in the post-order traversal of T_1.
(c) The second traversal of u comes before the first traversal of v if and only if u comes before v in both the pre-order and post-order traversals of T_1.

As we traverse the tree we construct the layout inside R' from left to right. For each vertex v, when we fix the body of $P(v)$, we also fix the foot-strip of $P(u)$ for each vertex u of G such that $f_2(u) = v$. Let U be the set of vertices u of G such that $f_2(u) = v$. For each $u \in U$, we set the height of the foot-strip of $P(u)$ to be $\lambda(u)$. We fix the foot-strips for all the vertices u of U such that the top-to-bottom order of these strips are the clockwise ordering of the the corresponding vertices around v in G, the bottom of the strip for the rightmost vertex contains the bottom of the body of $P(v)$ and every pair of consecutive strips touches each other, as illustrated in Figure 1(b). We call this operation *to set the foot-strips* at v. As an initialization, we set the foot-strips at v_2. We now describe how we fix the foot, leg, bridge and body for each of the polygons.

When we traverse a vertex v for the first time, the bridge-strip of $P(f_1(v))$ has already been fixed according to the order of the traversal. We fix the bridge strip of $P(v)$ with height $\lambda(v)$ just under the bridge-strip of $P(f_1(v))$ so that the two bridge strip touches each other. Again since $f_2(v)$ precedes v in the both the pre-order and the post-order traversals of T_1, by Lemma 1 the foot-strip of $P(v)$ has already been fixed. Then we set the leg of $P(v)$ with width $\lambda(v)$ touching the rightmost side of the already drawn part of R' and extending up to the foot-strip of $P(v)$. From the leg, we also fix the foot of $P(v)$ up to the right of the body of $P(f_2(v))$. When we traverse v for the second time, we fix the bridge of $P(v)$ so that it extends from the leg of $P(v)$ to the rightmost side of the already drawn part of R'. We also fix the body of $P(v)$ so that it touches bottom of the bridge-strip of $P(f_1(v))$, the top of the foot-strip of $P(f_3(v))$ and has sufficient width so that $P(v)$ has area $w(v)$. This is possible because $f_2(f_3(v))$ always precedes in both the pre-order and the post-order traversals of T_1 and by Lemma 1 the foot-strip of $P(f_3(v))$ has already been fixed. We then set the foot-strips at v. We call this Algorithm **Draw_Contact**. Figure 2(b) illustrates a hole-free proportional contact representation of the maximal planar graph in Figure 2(a) computed by Algorithm **Draw_Contact**.

We now have the following lemma whose proof is omitted here and included in the appendix.

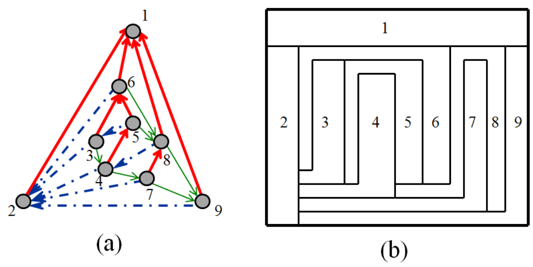


Fig. 2. (a) A maximal planar graph G, and (b) a hole-free proportional contact representation of G.

Lemma 2. *Let G be a maximal plane graph and let Γ be the representation of G obtained by Algorithm **Draw_Contact**. Then for any two vertices u and v in G, the polygons representing u and v do not cross each other in Γ.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let v_1, v_2, v_3 be the three outer vertices of G and let T_1, T_2, T_3 be the three Schnyder trees rooted at v_1, v_2 and v_3, respectively. Let Γ be the representation of G obtained by Algorithm **Draw_Contact**. For each vertex v of G, Let $P(v)$ be the polygon representing v in Γ. Clearly $P(v)$ has at most 10 sides for each vertex v of G. Then by Lemma 2, there is no crossing in Γ. One can see from the construction that the top of the bridge and the body of $P(v)$ is adjacent to the bottom of the bridge of $P(f_1(v))$, the left of the foot of $P(v)$ is adjacent to the right of the body of $P(f_2(v))$ and the bottom of the body of $P(v)$ is adjacent to the top of the foot of $P(f_3(v))$. Since each edge of G is in one of the three trees, this implies that Γ contains the necessary adjacencies. Furthermore since G is maximal planar and there is no crossing in Γ, Γ cannot contain any adjacency other than the ones defined by the edges in G. Also, the polygons are drawn with the correct areas. Thus Γ is a desired hole-free proportional contact representation of G. Computing a Schnyder realizer of G only takes linear time if we use the algorithm in [20]. It is not very hard to see from the description of the algorithm that we need a constant time to construct the polygon representing each vertex of G. The time complexity of the algorithm is thus linear. □

Theorem 1 gives an algorithmic upper bound of 10 on the complexity of the polygons in a hole-free proportional contact representation of a maximal planar graph with rectilinear polygons. Yeap and Sarrafzadeh gave a lower bound of 8 on this complexity in [25] where they found an example of a maximal planar graph (which is also a planar 3-tree) for which at least 8-sided polygons are necessary in a hole-free proportional contact representation with rectilinear polygons. We thus have the following theorem.

Theorem 2. *Ten-sided polygons are always sufficient and eight-sided polygons are sometimes necessary for a hole-free proportional Rectilinear contact representation of a maximal planar graph.*

3 Representations of Planar 3-trees

In this section we address the problem of hole-free proportional contact representations of planar 3-trees. A *3-tree* is either a 3-cycle or a graph G with a vertex v of degree three in G such that $G - v$ is a 3-tree and the neighbors of v are adjacent. If G is planar, then it is called a *planar 3-tree*. A *plane 3-tree* is a planar embedding of a planar 3-tree. It is trivial to see that starting with a 3-cycle, any planar 3-tree can be formed by recursively inserting a vertex inside a face and adding an edge between the newly added vertex and each of the three vertex on the face. It is known that the first 3-cycle to start with in this process can be presumed to form the outer face [2]. Let G be plane 3-tree. For a cycle C in G, we denote by $G(C)$ the plane subgraph of G inside C (including C). The following lemma follows from [2, 16].

Lemma 3. *Let G be a plane 3-tree with the outer vertices a, b and c and with at least one internal vertex. Then G contains exactly one internal vertex u that is adjacent to each of a, b and c. Furthermore, each of the three plane graphs inside and including the three cycles abu, bcu and cau is a planar 3-trees.*

Lemma 3 implies that for any planar 3-tree G, we can construct the "representation tree" of G, which is an ordered rooted ternary tree T_G spanning all the internal vertices of G as follows. Let u be the unique internal vertex of G that is adjacent to its three outer vertices a b and c in this counterclockwise order. We then construct T_G with the root u by making the roots of the trees corresponding to the three plane 3-trees G_{abu}, G_{bcu} and G_{cau}, the children of u in this order. We call T_G the *representation tree* of G. For any vertex v of T_G, we denote by U_v, the set of the descendants of v in T_G including v. The *predecessors* of v are the neighbors of v in G that are not in U_v. Clearly each vertex of T_G has exactly three predecessors. We now have the following lemma.

Lemma 4. *Let $G = (V, E)$ be a plane 3-tree and let $w : V \rightarrow R^+$ be a weight function. Then a hole-free proportional contact representation of G with respect to w can be obtained in linear time where each vertex of G is represented by an 8-sided rectilinear polygon.*

Proof. Let T_G be the representation tree of G. For any vertex v of T_G, let $W(v)$ denote the summation of the weights assigned to each of the vertices in U_v. Lemma 3 implies that T_G can be computed in linear time, for details see [16]. A linear-time bottom-up traversal of T_G is then sufficient for computing $W(v)$ for each vertex v of G. In the followings, we construct a hole-free proportional contact representation of G inside any rectangle R with area equal to the summation of the weights for all the vertices of G.

Let a, b, c be the three outer vertices of G in the counterclockwise order. We first draw the polygons for a, b and c. We cut a rectangle $P(a)$ with area $w(a)$ for a form the top of R, cut a rectangle $P(b)$ with area $w(b)$ from the left side of $R - P(a)$ and cut an L-shaped strip $P(c)$ of area $w(c)$ for c from the right side and the bottom of $R - P(a) - P(b)$, as illustrated in Figure 3(a). We now draw the polygons for the vertices in T_G inside the rectangle $R - P(a) - P(b) - P(c)$ by a top-down traversal of T_G. While we traverse a vertex v of T_G, we recursively draw the polygons for the vertices of U_v inside a rectangle R_v with area $W(v)$ such that R_v shares two of its side with the polygon for one of the predecessors of v and the other two sides with the polygons for the other two predecessors. Note that this condition holds for the rectangle $R - P(a) - P(b) - P(c)$ for the root of T_G. Let v be a vertex of T_G with predecessors p_1, p_2, p_3 and let $pqrs$ be the rectangle with area $W(v)$ where ps, pq and qrs are part of the boundary of the polygons for p_1, p_2 and p_3, respectively. From $pqrs$, we then cut three rectangles $R_1 = t_1t_2rs$, $R_2 = pt_3t_4t_5$ and $R_3 = qt_6t_7t_8$ with areas $W(u_1)$, $W(u_2)$ and $W(u_3)$, respectively, as illustrated in Figure 3(b), where u_1, u_2 and u_3 are the three children of v in T_G (some of them might be empty). Then the 8-sided polygon obtained by $pqrs - R_1 - R_2 - R_3$ has area $w(v)$ and has common boundary with all of the polygons representing its predecessors. Finally, we recursively fill out the rectangles R_1, R_2, R_3 by polygons representing the vertices in U_{u_1}, U_{u_2} and U_{u_3}, respectively. Clearly the polygon representing each vertex v of G can be computed in constant time. Thus the time complexity for constructing the representation of G is then linear. □

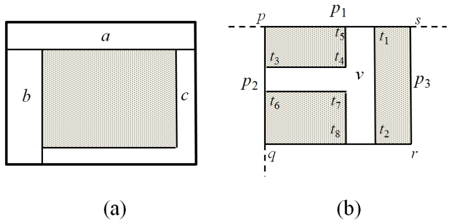


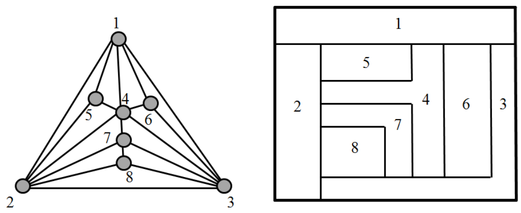
Fig. 3. Illustration for the proof of Lemma 4

Figure 4(b) illustrates a hole-free proportional contact representation of the planar 3-tree in Figure 4(a) computed by the algorithm described above. Lemma 4 implies that any planar 3-tree admits a hole-free proportional contact representation with 8-sided rectilinear polygons for any weight function. The corresponding lower bound on the complexity of the polygons is also 8 due to the planar 3-tree constructed by Yeap and Sarrafzadeh in [25] for which at least 8-sided polygons are necessary in a hole-free proportional contact representation with rectilinear polygons. We thus have the following Theorem.

Theorem 3. *Polygons with 8 sides are always sufficient and sometimes necessary for hole-free proportional contact representations of planar 3-trees with rectilinear polygons.*

4 Representations for Maximal Outer-planar Graphs

In this section we address the problem of hole-free proportional contact representations of maximal outer-planar graphs. An *outer-planar graph* is a graph that has a planar embedding where each of its vertices are in the outer face. Such an embedding is called an *outer-planar embedding* of the graph. A *maximal outer-planar graph* is an outer-planar graph to which no edges can be added without violating outer-planarity. It is



(a) (b)

Fig. 4. (a) A planar 3-tree G, and (b) a hole-free proportional contact representation of G.

trivial to see that each internal face in an outer-planar embedding of a maximal outer-planar graph is a triangle, and for $n \geq 3$ the outer-face is a simple cycle containing all vertices. We will give a linear-time algorithm to construct a hole-free proportional contact representation of a maximal outer-planar graph with rectangles. Before that, we need the following definitions.

Let Γ be a contact representation using rectangles for vertices (but with the outside not necessarily a rectangle). Let B be the bounding box of Γ. We say that a vertex v *occupies the top* of a representation Γ if there exists a horizontal line ℓ such that the rectangle representing v is exactly the intersection of B with the upper half-space of ℓ. In other words, the rectangle of v contains all of the top end of the bounding box of Γ. Similarly we define that a vertex v *occupies the right* of Γ.

Lemma 5. *Let G be a maximal outer-planar graph, and let (s, t) be an edge on the outer-face, with s before t in clockwise order. Then there exists a proportional hole-free contact-representation Γ using rectangles for vertices such that s occupies the top of Γ, and t occupies the right of $\Gamma - s$.*

Proof. The proof is by induction on the number of vertices. The claim is trivial for a single edge (s, t), see Figure 5. Now let G have at least 3 vertices, and let x be the (unique) third vertex on the inner face that is adjacent to (s, t). Then graph G can be split into two graphs at vertex x and edge (s, t): $G[s, x]$ consists of the graph induced by all vertices between s and x in counter-clockwise order around the outer-face, and $G[x, t]$ consists of the graph induced by the vertices between t and x.

Recursively draw $G[s, x]$ and remove s from it; call the result Γ_s. Recursively draw $G[x, t]$ and remove x and t from it; call the result Γ_t. Then scale the width of Γ_t until the bounding box of Γ_t is less wide than the rectangle of x in Γ_s. To maintain a proportional contact representation, scale the height of Γ_t by the inverse of the scale-factor for the width. Now Γ_t can be attached at the bottom right end of the representation of x in Γ_s. Add a rectangle for t on the right that spans the whole height (and extends below it at the bottom), and make its width such that its area is as prescribed for t. Add a rectangle for s such that it spans the whole width (and extends below it at the left), and make its height such that its area is as prescribed for s. This gives the desired representation. □

Since a rectangles is a rectilinear polygon with smallest complexity, we have the following theorem.

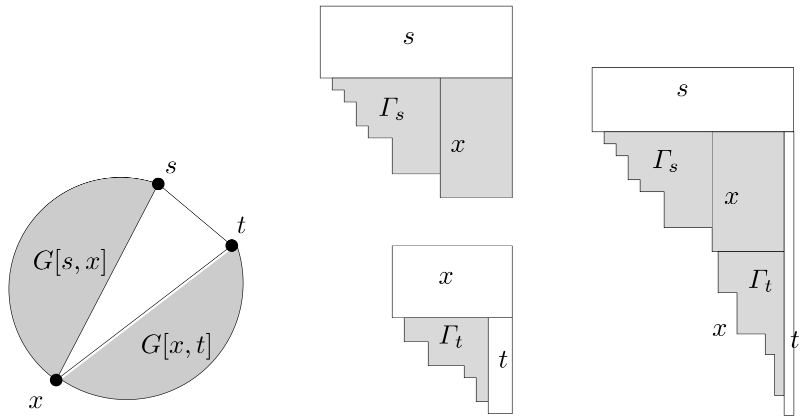


Fig. 5. Combining the drawings of two subgraphs.

Theorem 4. *For rectilinear hole-free proportional contact representation of a maximal outer-planar graph, rectangles are always necessary and sufficient.*

In our construction, the outer boundary of the representation will have size $\Theta(n)$ (it is a rectangle minus a stair-case with $n-2$ steps.) We can show that this is required. It was already known that the outer-face cannot be a rectangle if vertices are rectangles [19], but we improve this to a stronger result.

Lemma 6. *There exists a maximal outer-planar graph for which any contact representation with rectangles requires $\Omega(n)$ complexity of the outer-face.*

The proof of this lemma is omitted here and is included in the appendix.

5 Conclusion

We gave a linear-time algorithm for a hole-free proportional contact representation of a maximal planar graph with 10-sided rectilinear polygons, which improves the previously known upper bound of 12 for the complexity of the polygons in such a representation. We also gave algorithms that obtains hole-free proportional contact representations of planar 3-trees and maximal outerplanar graphs with optimal complexity of polygons. The corresponding lower bound on the complexity of the polygons in the representation of maximal planar graph is 8. Thus it would be interesting to find out whether 8-sided rectilinear polygons are always sufficient for such a representation for maximal planar graphs. The problem can also be investigated for other important subclasses of planar graphs.

References

1. M. Badent, C. Binucci, E. D. Giacomo, W. Didimo, S. Felsner, F. Giordano, J. Kratochvíl, P. Palladino, M. Patrignani, and F. Trotta. Homothetic triangle contact representations of planar graphs. In *CCCG 2007*, pages 233–236, 2007.

2. T. Biedl and L. E. Ruiz Velázquez. Drawing planar 3-trees with given face-areas. In *Proc. 17th International Symposium on Graph Drawing (GD'09)*, volume 5849 of *Lecture Notes in Computer Science*, pages 316–322. Springer, 2010.
3. T. Biedl and L. E. Ruiz Velázquez. Orthogonal cartograms with few corners per face. Technical Report CS 2010-21, School of Computer Science, University of Waterloo, 2010.
4. A. L. Buchsbaum, E. R. Gansner, C. M. Procopiuc, and S. Venkatasubramanian. Rectangular layouts and contact graphs. *ACM Transactions on Algorithms*, 4(1), 2008.
5. M. de Berg, E. Mumford, and B. Speckmann. On rectilinear duals for vertex-weighted plane graphs. *Discrete Mathematics*, 309(7):1794–1812, 2009.
6. H. de Fraysseix, P. O. de Mendez, and P. Rosenstiehl. On triangle contact graphs. *Combinatorics, Probability and Computing*, 3:233–246, 1994.
7. S. Felsner and M. C. Francis. Contact representations of planar graphs with cubes. In *Proc. ACM Symposium on Computational Geometry*, 2011.
8. E. R. Gansner, Y. Hu, M. Kaufmann, and S. G. Kobourov. Optimal polygonal representation of planar graphs. In *9th Latin Am. Symp. on Th. Informatics (LATIN)*, pages 417–432, 2010.
9. E. R. Gansner, Y. Hu, and S. G. Kobourov. On touching triangle graphs. In *Proceedings of the 18th Symposium on Graph Drawing (GD 2010)*, pages 250–261, 2010.
10. D. Gonçalves, B. Lévêque, and A. Pinlou. Triangle contact representations and duality. In *Proceedings of the 18th Symposium on Graph Drawing (GD 2010)*, pages 238–249, 2010.
11. X. He. On floor-plan of plane graphs. *SIAM Journal of Computing*, 28(6):2150–2167, 1999.
12. R. Heilmann, D. A. Keim, C. Panse, and M. Sips. Recmap: Rectangular map approximations. In *10th IEEE Symp. on Information Visualization (InfoVis 2004)*, pages 33–40, 2004.
13. A. Kawaguchi and H. Nagamochi. Orthogonal drawings for plane graphs with specified face areas. In *4th Conf. on Theory and Applications of Models of Comp.*, pages 584–594, 2007.
14. K. Koźmiński and E. Kinnen. Rectangular duals of planar graphs. *Networks*, 15:145–157, 1985.
15. C.-C. Liao, H.-I. Lu, and H.-C. Yen. Compact floor-planning via orderly spanning trees. *Journal of Algorithms*, 48:441–451, 2003.
16. D. Mondal, R. I. Nishat, M. S. Rahman, and M. J. Alam. Minimum-area drawings of plane 3-trees. In *Proceedings of the 22nd Annual Canadian Conference on Computational Geometry (CCCG)*, pages 191–194, 2010.
17. M. S. Rahman, K. Miura, and T. Nishizeki. Octagonal drawings of plane graphs with prescribed face areas. *Computational Geometry*, 42(3):214–230, 2009.
18. G. Ringel. Equiareal graphs. In R. Bodendiek, editor, *Contemporary Methods in Graph Theory*, pages 503–505. Wissenschaftsverlag, 1990.
19. I. Rinsma. Nonexistence of a certain rectangular floorplan with specified area and adjacency. *Environment and Planning B: Planning and Design*, 14:163–166, 1987.
20. W. Schnyder. Embedding planar graphs on the grid. In *Proceedings of the 1st ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 138–148, 1990.
21. Y. Sun and M. Sarrafzadeh. Floorplanning by graph dualization: l-shaped modules. *Algorithmica*, 10(6):429–456, 1993.
22. C. Thomassen. Plane cubic graphs with prescribed face areas. *Combinatorics, Probability & Computing*, 1:371–381, 1992.
23. P. Ungar. On diagrams representing graphs. *J. London Math. Soc.*, 28:336–342, 1953.
24. M. J. van Kreveld and B. Speckmann. On rectangular cartograms. *Computational Geometry*, 37(3):175–187, 2007.
25. K.-H. Yeap and M. Sarrafzadeh. Floor-planning by graph dualization: 2-concave rectilinear modules. *SIAM Journal on Computing*, 22:500–526, 1993.

Appendix

Proof of Lemma 2. Let v_1, v_2, v_3 be the three outer vertices of G and let T_1, T_2, T_3 be the three Schnyder trees rooted at v_1, v_2 and v_3, respectively. For each vertex v of G, Let $P(v)$ be the polygon representing v in Γ. By the choice of $\lambda(v)$, the bottommost bridge-strip is above the topmost foot-strip in Γ. Then by the construction, one can see that the only possible crossing might occur between a foot and a leg or between a foot and a body.

Let u and v be two vertices of G. We first assume that the foot of $P(u)$ crosses the leg of $P(v)$, as illustrated in Figure 6(a). Then by Lemma 2, u comes before v in the pre-order traversal of T_1; both u and v comes before both $f_2(u)$ and $f_2(v)$ in both the pre-order and the post-order traversals of T_1; and either $f_2(u) = f_2(v)$ or $f_2(u)$ comes before $f_2(v)$ in the post-order traversals of T_1. Let p_1 be the unique path from v to v_1, p_2 the leftmost path from v to one of its descendant leaf and p_3 the unique path from $f_2(v)$ to p_1 in T_1. Then u is to the right of the path $p_1 \cup p_2$ and $f_2(u)$ is inside the region enclosed by p_1, p_3 and the edge $(v, f_2(v))$, not on the path p_1. Then by the properties of Schnyder trees and by planarity, there cannot be any edge $(u, f_2(u))$, a contradiction.

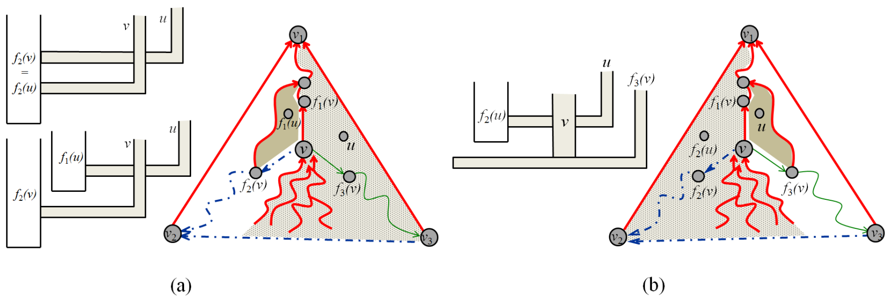


Fig. 6. Illustrations for the proof of Lemma 2.

We now assume that the foot of $P(u)$ crosses the body of $P(v)$. Then by Lemma 1, u precedes v in both the pre-order and the post-order traversals of T_1 and v precedes $f_2(u)$ in the post-order traversal of T_1, as illustrated in Figure 6(b). Had u been before $f_3(v)$ in the pre-order traversal of T_1, the the foot of u would cross the leg of $f_3(v)$, which is not possible according to the previous paragraph. We thus assume that u follows $f_3(v)$ in the pre-order traversal of T_1. Let p_1 be the unique path from v to v_1, p_2 the rightmost path from v to one of its descendant leaf and p_3 the unique path from $f_3(v)$ to p_1 in T_1. Then $f_2(u)$ is to the left of the path $p_1 \cup p_2$ and $f_2(u)$ is inside the region enclosed by p_1, p_3 and the edge $(v, f_3(v))$, not on the path p_1 and the edge $(v, f_3(v))$. Then by planarity, there cannot be any edge $(u, f_2(u))$, a contradiction. □

Proof of Lemma 6. Consider any maximal outer-planar graph G such that $\lfloor n/2 \rfloor$ vertices have degree two (any maximal outer-planar graph whose inner dual is a full binary tree suffices). Suppose Γ is a hole-free proportional contact representation of G with rectangles. Since rectangles are convex, no two of them can share two sides. Therefore any vertex v of degree 2 shares at most two of its sides with other vertices, and so at least two of its sides with the outer boundary of Γ. Furthermore, these two sides must be consecutive on $P(v)$, since otherwise v would be a cut vertex in G. The common endpoint of these two sides is then a corner of the outer boundary of Γ, so the outer-face has at least $\lfloor n/2 \rfloor$ sides. □