Henneberg Steps for Triangle Representations

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Abstract. Which plane graphs admit a straight line representation such that all faces have the shape of a triangle? In previous work we have studied necessary and sufficient conditions based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. A flat angle assignment that fullfills these conditions is called good. The complexity for checking whether a graph has a good flat angle assignment remains unknown.

In this paper we deal with extensions of good flat angle assignments. We show that if G has a good flat angle assignment and G^+ is obtained via a planar Henneberg step of type 2, then G^+ also admits a good flat angle assignment. A similar result holds for certain combinations of Henneberg type 1 steps followed by a type 2 step. As a consequence we obtain a large class of pseudo-triangulations that admit drawings such that all faces have the shape of a triangle. In particular, every 3-connected, plane generic circuit admits a good flat angle assignment With several examples we show the limitations of our method.

1 Introduction

Planar graphs and drawings of planar graphs are widely studied. Highlights in the area are Tutte's rubber band representations of 3-connected graphs [Tut63] and Koebe's touching coins representations [Koe36]. More discrete but also very popular are triangle contact representations of de Fraysseix et al. [dFdMR94]. Graphs admitting a rectangle contact representation have also been widely studied [KK85, Ung53, BGPV08]. Planar graphs also have contact representations with convex hexagons (e.g. [DGH$^+$12]).

In this paper we study a representation of planar graphs in the classical setting, i.e., vertices are presented as points in the Euclidean plane and edges as straight line segments. We are interested in the class of planar graphs that admit a representation in which all faces are triangles. Note that in such a representation each face f has exactly $\deg(f) - 3$ incident vertices that have an angle of size π in f. Conversely each vertex has at most one angle of size π. In [AF] we have studied necessary and sufficient conditions based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. Flat angle assignments that fullfill these conditions are called *good*. The complexity for checking whether a graph has a good flat angle assignment remains unknown. In the second part of this introduction we give some details of the characterization of good flat angle assignments.

Graphs with only triangular regions have also been investigated in the dual setting, i.e., vertices are triangles and edges correspond to side contacts. Gansner, Hu and Kobourov [GHK11] show that outerplanar graphs, grid graphs and hexagonal grid graphs can be represented by Touching Triangle Graphs (TTG's). Alam, Fowler and Kobourov [AFK] consider proper TTG's, i.e., the union of all triangles of the TTG is a triangle and there are no holes. They present conditions for biconnected outerplanar graphs to have a TTG. Kobourov, Mondal and Nishat [KMN12] present construction algorithms for proper TTG's of 3-connected cubic graphs and some grid graphs.

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In [AF] we haven given necessary and sufficient conditions for a graph to have a Straight Line Triangle Representation (SLTR). The drawback of this characterization is that we are not aware of an efficient way of checking whether a given graph admits a flat angle assignment that fullfills the conditions.

In this paper we will investigate the relation between pseudo-triangulations and SLT representations. In Section 2 we start with some basic definitions and show that every SLTR is a pseudo-triangulation. In Section 3 we will consider two construction steps and define how to extend the GFAA along these steps, such that the resulting assignment is also a GFAA. However, there exist graphs that have a GFAA but can not be constructed using only the two steps we give in Section 3.

2 Preliminaries

A *pseudo-triangle* is a simple polygon with precisely three convex angles, all other vertices of the polygon admit a concave angle at the interior of the polygon. A *pseudo-triangulation* (PT) is a planar graph with a drawing such that all faces are pseudo-triangles. An example of a PT is given in Fig. 1 (a).

A pseudo-triangulation is *pointed* if each vertex has an angle of size $> \pi$. A pointed pseudotriangulation with n vertices must have exactly $2n-3$ edges. Indeed pointed pseudotriangulations have the Laman property: they have $2n-3$ edges, and subgraphs induced by k vertices have at most $2k-3$ edges. Laman graphs, and hence also pointed pseudotriangulations, are minimally rigid graphs. A detailed survey on pseudo-triangulations has been given by Rote et al. [RSS08].

Pointed pseudotriangulations are defined by an assignment of big angles to vertices. Fig. 1 (a) shows an example of a pointed pseudotriangulation whose big angles do not constitute a good flat angle assignment. Fig. 1 (c) shows a plane Laman graph, i.e., a graph admitting big angle assignment that yields a pointed pseudotriangulation but with the given outer face the graph has no good flat angle assignment.

A *Straight Line Triangle Representation* of a graph G is a plane drawing of G such that all edges are straight line segments and all faces are triangles. Throughout this paper $G = (V, E)$ will be a plane, internally 3-connected graph with three suspension vertices. A plane graph G with suspensions s_1, s_2, s_3 is said to be *internally 3-connected* when the addition of a new vertex v_∞ in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected graph. The three suspension vertices are the corners of the outer face. With little effort it can be shown that a graph that admits an SLTR but is not internally 3-connected is a subdivision of an internally 3-connected graph that admits an SLTR [AF].

A *flat angle assignment* (FAA) of a graph is a mapping from a subset U of the non-suspension vertices to faces such that

[C_v] Every vertex of U is assigned to at most one face,

[C_f] For every face f, precisely $|f|-3$ vertices are assigned to f.

An FAA is called *good* when it induces an SLTR. In [AF] we have shown that when an FAA that is good (GFAA), it induces a contact family of pseudosegments Σ which has the following property:

[C_P] Every subset S of Σ with $|S| \geq 2$ has at least three free points.

Definition 2.1 (Contact Family of Pseudosegments)**.** A contact family of pseudosegments is a family $\{c_i\}_i$ of simple curves $c_i : [0,1] \to \mathbb{R}^2$, with $c(0) \neq c(1)$, such that any two curves c_i and c_j $(i \neq j)$ have at most one point in common. If c_i and c_j have a common point, then this point is an endpoint of (at least) one of them.

Definition 2.2 (Free Point)**.** Let Σ be a family of pseudosegments and S a subset of Σ. A point p of a pseudosegment from S is a free point for S if

1. p is an endpoint of a pseudosegment in S, and
2. p is not interior to a pseudosegment in S, and
3. p is incident to the unbounded region of S, and
4. p is a suspension or p is incident to a pseudosegment that is not in S.

The drawback of the characterization given in [AF] is that we are not aware of an efficient way to test whether a given graph has an FAA that is good.

A *combinatorial pseudo-triangulation* (CPT) is an assignment of the labels *big* and *small* to the angles around each vertex. Each vertex has at most one angle labeled *big* and each inner face has precisely three incident angles labeled *small*, the outer face has precisely three *big* angles. For an interior angle labeled *big*, let the incident vertex be assigned to the incident face, and a vertex is not assigned if it has no angle labeled *big*. The vertices incident to the angles labeled *big* in the outer face are the suspensions. Hence a CPT satisfies C_v and C_f and therefore it is an FAA.

A CPT does not always induce a PT, Haas et al. have shown that the *generalized Laman condition* is necessary and sufficient for a CPT to induce a PT [HOR+03].

Definition 2.3 (Generalized Laman Condition)**.** Let G be the graph of a pseudo-triangulation of a planar point set in general position. Every subset of x not assigned vertices plus y assigned vertices of G, with $x + y \geq 2$ spans a subgraph with at most $3x + 2y - 3$ edges.

Laman graphs are minimally generically rigid graphs, a Laman graph $G = (V, E)$ satisfies $|E| = 2|V| - 3$ and for all subsets $H \subseteq V$ the induced graph $G[H]$ has at most $2|H| - 3$ edges. Haas et al. show that PT's such that every vertex has an angle labeled *big* are precisely planar Laman graphs. The Generalized Laman Condition is a property of the embedding and the vertices chosen to be not assigned, and not of the graph itself. However for a plane Laman graph, the Generalized Laman Condition always holds.

A plane, internally 3-connected graph admits a Flat Angle Assignment (FAA) only if there is a matching of vertices to faces such that C_v and C_f are satisfied. We will give an upper and a lower bound on the number of edges of a graph (and all induced subgraphs) that admits an FAA.

Proposition 2.4 (Lower Bound)**.** A plane graph $G = (V, E, F)$ has an FAA if and only if $|E| \geq 2|V| - 3$ and for every subset of the vertices H, with f_H as outer boundary of the induced graph $G[H]$,

$$|E(G[H])| \geq 2|H| - 3 - |f_H|.$$

Proof. Let $G = (V, E, F)$ a 3-connected plane graph, let X the set of not assigned vertices, $|X| \geq 3$ since there are three suspension vertices. By Hall's mariage theorem we need $|V| - |X| = \sum_{f \in F}(|f| - 3)$ and for every subset $H \subseteq V$ we have $|H \setminus X| \geq \sum_{f \in FG[H]}(|f| - 3)$. Further we use the identity $\sum_{f \in F} |f| = 2|E|$ and Euler's planarity condition.

$$|V| = \sum_{f \in F}(|f| - 3) + |X| = \sum_{f \in F}(|f|) - 3|F| + |X| = 2|E| - 3|F| + |X|$$

And then

$$3|E| - 3|V| + 6 = 3|F| = 2|E| - |V| + |X|, \qquad \text{hence,} \qquad |E| = 2|V| - 6 + |X|.$$

As $|X| \geq 3$ we have $|E| \geq 2|V| - 3$. By the same reasoning, for every subset $H \subseteq V$, such that f_H is the outer boundary of $G[H]$:

$$|EG[H]| \geq 2|H| - 3 - |f_H|.$$

□

Proposition 2.5 (Upper Bound)**.** Given a plane graph $G = (V, E, F)$ with a set $X \subseteq V$ of not assigned vertices, $|X| \geq 3$. Then G has an FAA if and only if $|E| = 2|V| + |X| - 6$ and for every subset $H \subseteq V$,

$$|EG[H]| \leq 3x + 2y - 3,$$

where $x = |X \cap H|$ and $y = |H \setminus X|$.

Proof. Let $G = (V, E, F)$ an internally 3-connected plane graph. By Proposition 2.4 we have $|E| = 2|V| + |X| - 6$.

Let $H \subseteq V$, suppose $G[H]$ is connected, let f_H the boundary of the outer face of $G[H]$ and assume it contains all the edges interior to $G[H]$ in G. Let p_H the number of vertices that need not be assigned inside $G[H]$, thus $p_H = |H| - \sum_{f \in FG[H]^{int}}(|f| - 3)$.

$$3|FG[H]| = 2|EG[H]| - |f_H| + 3 - |H| + p_H, \text{ and }, 3|FG[H]| = 6 + 3|EG[H]| - 3|H|.$$

Let $z = |X \cap f_H|$ the not assigned vertices that are incident to the outer face of $G[H]$, so $z \leq |f_H|$ and then we find

$$\begin{aligned} |EG[H]| &= 3|H| - 6 - |f_H| + 3 - |H| + p_H \\ &= 2|H| - 3 - |f_H| + p_H \\ &\leq 2|H| + p_H - z - 3 \\ &= (3p_H - 3z) + (2|H| - 2p_H + 2z) - 3 \\ &= 3x + 2y - 3 \end{aligned}$$

□

Theorem 2.6. Every FAA induces a PT.

Proof. By Prop. 2.5 every FAA satisfies the Generalized Laman Condition and by the result of Haas et al. this is sufficient. □

Figure 1 (a) shows a PT that is not an FAA. The induced FAA satisfies the generalized Laman condition but is not a GFAA, when changing the big angles to size π angles, all vertices but the top suspension will be between the bottom two suspensions. However this graph does have a GFAA.

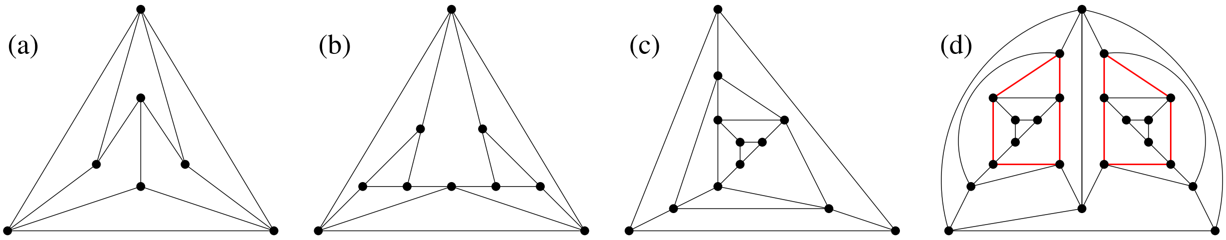

Figure 1: (a) A pseudo-triangulation that does not induce an SLTR, (b) a Laman graph that has an SLTR but can not be constructed using only the two steps we give in Section 3, (c) a plane Laman graph that has no SLTR for this embedding, (d) a Laman graph that has no SLTR.

Laman graphs admit a special construction, denoted with Henneberg Construction, and in the next section we will take a closer look at the steps of this construction. A Laman graph $G = (V, E)$ satisfies $|E| = 2|V| - 3$, by Prop. 2.4 this is the minimal number of edges an SLTR can have.

Not every plane Laman graph admits an SLTR, see Figure 3. To reason that a plane graph has no FAA that is good, it is most convenient to use the notion of outline cycles. An *outline cycle* of G is a closed walk that can be obtained as outline cycle of some connected subgraph of G. Outline cycles may have repeated edges and vertices, see Fig. 2. The interior $\int(\gamma)$ of an outline cycle $\gamma = \gamma(H)$ consists of H together with all vertices, edges and faces of G that are contained in the area enclosed by γ.

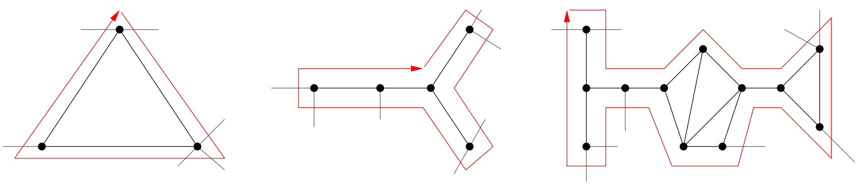

Figure 2: Three examples of outline cycles.

In [AF] we have shown that an FAA is good if and only if each outline cycle that is not the outline cycle of a path, has at least three combinatorially convex corner.

Definition 2.7. Given an FAA a vertex v of an outline cycle γ is a *combinatorial convex corner* for γ if

- v is a suspension vertex, or
- v is not assigned to a face and there is an edge e incident to v with $e \notin \int(\gamma)$, or
- v is assigned to a face f, $f \notin \int(\gamma)$ and there exists an edge e incident to v with $e \notin \int(\gamma)$.

Now to see that the plane graph in Figure 3 does not have an FAA that is good, consider the outline cycles following d, e, f, k, h and h, e, j, l, k, if both have three combinatorially convex corners, then both h and k must be assigned to the face (d, h, k, g) which contradicts C_f.

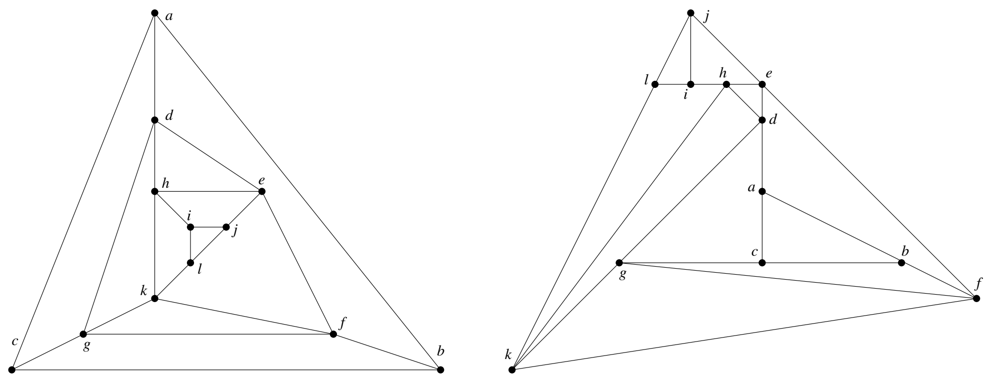

Figure 3: A plane Laman graph that does not admit an SLTR (left) and a different embedding of the same graph such that there does exist an SLTR (right).

The Generalized Laman Condition is a property of the embedding and the vertices chosen to be not assigned, and not of the graph itself. Whether a graph has an SLTR does depend on the embedding, even for Laman graphs, as Figure 3 shows. However, not every *planar* Laman graph admits an embedding such that there exists an SLTR. The graph in Fig. 1 (d) has no SLTR, when the red (thick) cycles are embedded as here, there must be four of the five vertices of the cycle that are assigned on the outside, but there is only “space” for three such assignments in the neighboring faces. There is no embedding such that both the red cycles are turned inside out.

By Prop. 2.4, Laman graphs are the graphs with the minimal number of edges such that there could exist an FAA. Every Laman graph can be constructed from an edge by Henneberg steps [Hen11, Whi97], therefore also a graph $G = (V, E)$, with $|E| = 2|V| - 3$ that admits an SLTR must have such a construction. In the next section we will investigate how to use the Henneberg construction such that a GFAA can be extended along the steps.

3 Construction Methods

Every graph $G = (V, E)$, with $|E| = 2|V| - 3$, that has an SLTR, must be a plane Laman graph. Every Laman graph can be constructed from an edge by Henneberg steps [Hen11, Whi97], therefore also a graph $G = (V, E)$, with $|E| = 2|V| - 3$ that admits an SLTR must have such a construction.

Henneberg Steps

- Henneberg Type 1 step ($\textsc{hen}_1$, Figure 4 (a)) adds a vertex and connects it to two disjoint vertices of the graph.
- Henneberg Type 2 step ($\textsc{hen}_2$, Figure 4 (b)) subdivides an edge and connects the new vertex to a third vertex of the graph.

It has been shown that planar Laman graphs admit a planar Henneberg construction [HOR+03]. Since we consider plane graphs (with a given set of suspension vertices), we consider the steps in a plane setting, that is, each of the steps takes place in a face of the plane graph.

Figure 4: A Henneberg Type 1 step (a) and a Henneberg Type 2 step (b).

A $\textsc{hen}_2$ step on a 3-connected graph, results in a 3-connected graph, a $\textsc{hen}_1$ step does not. In the following section we will proof that a GFAA can be extended after a $\textsc{hen}_2$ step such that the new assignment also is a GFAA. Hence graphs that can be constructed with $\textsc{hen}_2$ steps only from a graph that admits an SLTR, also admit an SLTR. The GFAA can be constructed along the $\textsc{hen}_2$ construction of the graph.

Not all graphs that admit an SLTR can be constructed with $\textsc{hen}_2$ steps only, in Figure 5 the vertices d, e, f can not be introduced with $\textsc{hen}_2$ steps only.

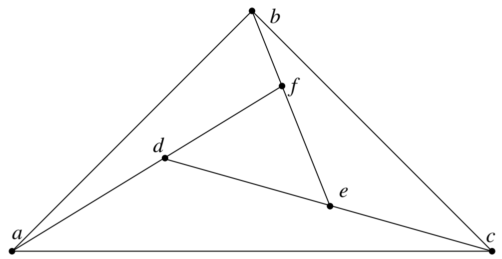

Figure 5: The vertices d, e, f can not be introduced with $\textsc{hen}_2$ steps only in the cycle a, b, c

A GFAA can also be extended along a particular combination of n $\textsc{hen}_1$ steps followed by a $\textsc{hen}_2$ step in a face. Consider the cycle (a, b, c) of Fig. 5:

1. ($\textsc{hen}_1$) add d and connect it to a and c,
2. ($\textsc{hen}_1$) add e and connect it to d and b,
3. ($\textsc{hen}_2$) subdivide the edge cd and connect the new vertex f to e.

In Section 3.2 we will discuss and proof when such a step can be extended.

3.1 Henneberg Type 2 steps are good

Given a graph G and a GFAA ψ of G. Let uv the edge that is subdivided, x the new vertex and w the third vertex to which x is connected (see Figure 6). The face f, incident to uv and w is splitted into f_u (the face incident to u) and f_v. The other face incident to uv is denoted with f_x. The resulting graph is denoted G^+. We will construct an assignment ψ^+ for G^+ and proof that ψ^+ is a GFAA.

There are three vertices not assigned to f under ψ, we will call them *corners* of f. We consider two cases, firstly f_u is incident to all corners of f, secondly, f_u is incident to precisely two corners of f. Note that if w is a corner of f it will be a corner for both f_u and f_v. The vertices different from u, v, w, x, that are assigned to f under ψ, will be assigned in the trivial way under ψ^+, i.e. such a vertex is assigned to f_u resp. f_v, if in G^+ it is incident to f_u resp. f_v.

Case 1: *f_u is incident to all corners of f.* If u or w is assigned to f under ψ, it is assigned to f_u under ψ^+. The vertex v is assigned to f_x and x to f_u under ψ^+.

Case 2: *f_u is incident to precisely two corners of f.* If u or w is assigned to f under ψ, it is assigned to f_u under ψ^+, if v was assigned to f it is assigned to f_v under ψ^+ and x is assigned to f_x.

This yields an assignment ψ^+ for G^+.

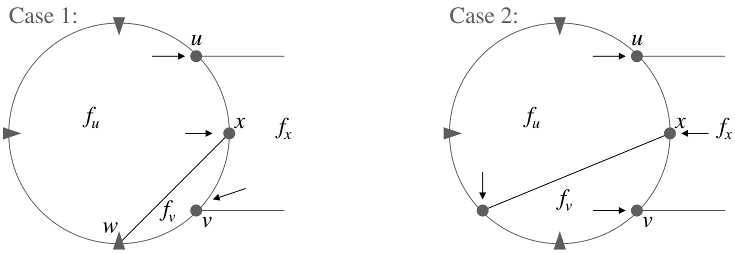

Figure 6: Updating the assignment after a HEN$_2$ step. The triangles denote the corners of f, dots denote non-corners, arrows denote assignments of a vertex to a face.

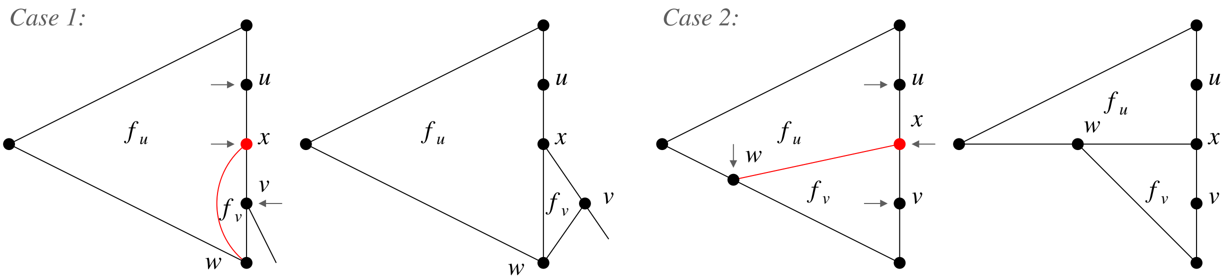

Figure 7: A stretched representation of the original face, and of the results of a HEN$_2$ step in Case 1 and Case 2.

Theorem 3.1. Given a 3-connected, plane graph G with a GFAA ψ. Let G^+ be the result of a HEN$_2$ step applied to G and let ψ^+ be the updated assignment. Then ψ^+ is a GFAA and G^+ admits an SLTR.

Proof. It is trivial that ψ^+ satisfies C_v and C_f and hence is an FAA.

We consider the induced families of pseudosegments, Σ and Σ^+ of ψ and ψ^+ respectively. Since ψ is a Good FAA, we know that every subset of Σ has at least three free points or cardinality at most one. Let $S \subseteq \Sigma^+$ have cardinality at least two.

Case 1: Let s_x resp. s_v be the pseudosegment that has x resp. v as interior point and let s_w the pseudosegment containing the edge uw. If S does not contain s_x, s_v or s_w then S is also a subset of Σ, hence it must have three free points. Suppose $S \subseteq \{s_x, s_v, s_w\}$, then S has three free points since no two pseudosegments of $\{s_x, s_v, s_w\}$ touch twice and if $S = \{s_x, s_v, s_w\}$ there are precisely three of the six endpoints covered. So suppose S contains at least one pseudosegment not of $\{s_x, s_v, s_w\}$. Consider the comparable set S' of Σ, that is

- If $s_x \in S$ then replace s_x by the pseudosegment s'_x of Σ that has u and v as interior points.
- If $s_v \in S$ then replace s_v by the pseudosegment s'_v of Σ that ends in v and contains all the edges of s_v but the edge vx.
- If $s_w \in S$ then delete s_w.

Now we have $S' \in \Sigma$, thus S' has three free points unless $|S'| = 1$.

- If $s_x \in S$ then s_x contributes the same free points to S as s'_x to S'.
- If $s_v \in S$ then if v was a free point for S', then x is for S. Hence s_v contributes the same number of free points to S as s'_v to S'.

- If $s_w \in S$ and $|S'| = 1$ and s_w contributes at least one free point to S and it covers no other points, thus S has three free points, when $|S'| > 1$ then S' has at least three free points, adding s_w does not cover any of them and therefore S has at least three free points.

We conclude that S has at least three free points.

Case 2: If w is a corner of f then there is one new pseudosegment s_{xw} consisting of only the edge xw. Let $S \subseteq \Sigma^+$, if $s_{xw} \notin S$ then S has at least three free points, if $|S| > 3$ then S has at least three free points since s_{xw} does not cover any free point, there is no pseudosegment that covers both endpoints of s_{xw} and hence if $|S| = 2$ and $s_{xw} \in S$ the set S also has at least three free points.

Suppose w is not a corner of f. Let s_x resp. s_w be the pseudosegment that has x resp. w as interior point and let s_c the pseudosegment containing the edge from w to the corner of f which is incident to f_v. If S does not contain s_x, s_w or s_c then S is also a subset of Σ, hence it must have three free points. Suppose $S \subseteq \{s_x, s_w, s_c\}$, then S has three free points since no two pseudosegments of $\{s_x, s_w, s_c\}$ touch twice and if $S = \{s_x, s_w, s_c\}$ there are precisely three of the six endpoints covered. So suppose S contains at least one pseudosegment not of $\{s_x, s_w, s_c\}$. Consider the comparable set S' of Σ, that is:

- If $s_x \in S$ then replace s_x by the pseudosegment s'_x of Σ that has u and v as interior points,
- If $s_w \in S$ then replace s_w by the pseudosegment s'_w of Σ that w as an interior point,
- If $s_c \in S$ then if $s_w \notin S$, replace s_c by the pseudosegment s'_w, otherwise, delete s_c.

Now we have $S' \in \Sigma$, thus S' has three free points unless $|S'| = 1$. If $|S'| = 1$ then $S = \{s_w, s_c\}$ which contradicts the assumption that S contains at least one pseudosegment not of $\{s_x, s_w, s_c\}$, thus $|S'| > 1$.

- If $s_x \in S$ then s_x contributes the same free points to S as s'_x to S'.
- If $s_w \in S$ then if x is covered in S, then c is covered or an endpoint of two pseudosegments in S'. The other endpoint of s_w is an endpoint of s'_w, hence replacing s'_w by s_w leaves the number of free points intact.
- If $s_c \in S$ and $s_w \notin S$ then s_c contributes at least as many free points to S as s'_w to S', so assume also $s_w \in S$. The free points that s'_w contributes to S' are then also free points of S' as the endpoints of s'_w are also endpoints for $\{s_w, s_c\}$. Hence S has at least three free points.

We conclude that S has at least three free points, hence ψ^+ is a GFAA. □

A graph $G = (V, E)$ is a *generic circuit* if $|E| = 2|V| - 2$ and for all subsets $H \subseteq V$ the induced graph $G[H]$ has at most $2|H| - 3$ edges. The generic circuit with the smallest number of vertices is the complete graph on four vertices (K_4).

Theorem 3.2. Every 3-connected, plane generic circuit admits an SLTR.

Proof. A 3-connected, generic circuit can be constructed with HEN_2 steps from K_4 (Berg and Tibór [BJ03]) and K_4 admits an SLTR. Every plane 3-connected generic circuit can be constructed with HEN_2 steps from K_4 such that all intermediate graphs are plane. By Thm. 3.1 we have that every 3-connected, plane generic circuit admits an SLTR. □

3.2 A combination step: n times a Henneberg 1 step followed by a Henneberg 2 step.

A plane Henneberg Type I step (HEN_1) adds a vertex, v_0, in a face, connecting it to two vertices incident to the face, it splits the face in two parts. The resulting graph is 2–connected as the new vertex v_0 has only two neighbors. In order to preserve 3–connectedness, the HEN_1 step needs to be followed by another step which assigns a third neighbor to v_0. This could be another HEN_1 step, in which case we find a new vertex v_1 with only two neighbors, or a HEN_2 step, which results in a 3–connected graph.

Not any such combination step will preserve the possibility to stretch the graph to an SLTR, e.g. the graph in Figure 3 can be constructed with a sequence of HEN_2 steps followed by one combination step. We will present rules for a combination step such that the GFAA ψ of the graph G can be extended to a GFAA ψ_n for the resulting graph G_n.

Remark 3.3. Note that if the HEN$_2$ step subdivides an edge of the original face, then the whole step can be replaced by a sequence of HEN$_2$ steps. As this has been proven to be extendible in the previous section, we will not consider this as an option in this section.

Throughout this section, we denote the face in which we are placing the HEN$_{1^n2}$ step with f, hence all vertices incident to f are vertices of G, the starting graph. The corners of a face are again the vertices incident to a face but not assigned to this face.

The rules The HEN$_{1^n2}$ step denotes a sequence of n HEN$_1$ steps followed by one HEN$_2$ step, such that the following *rules* are satisfied.

1. All the steps take place in a bounded $(n+1)$–face f.
2. The starting HEN$_1$ step, $[1_0]$, adds vertex v_0 between two neighbors (x_0 and y_0) of f.
3. The i–th HEN$_1$ step, $[1_i]$, $0 < i < n$, takes place in the *allowed face* of v_{i-1} and it adds v_i between v_{i-1} and z_i such that z_i is on f and a neighbor of x_{i-1} or of y_{i-1}. If z_i a neighbor of x_{i-1}, set $x_i = z_i$ and $y_i = y_{i-1}$, otherwise set $x_i = x_{i-1}$ and $y_i = z_i$. Note that this yields that after n HEN$_1$ steps all vertices of f have been assigned a new neighbor.
4. The HEN$_2$ step, $[2]$, takes place in the allowed face of v_{n-1}, denoted with $f_{a_{n-1}}$, such that both new faces are incident to at least one corner of $f_{a_{n-1}}$, not v_{n-1}.

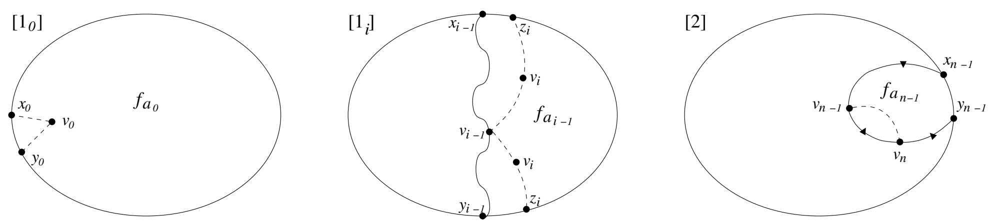

Figure 8: The three elements of a HEN$_{1^n2}$ step. The triangles in the rightmost figure denote the corners of $f_{a_{n_1}}$, note that also v_{n-1} is a corner as it is not yet assigned, but it will be assigned inside $f_{a_{n-1}}$.

Note that the last rule depends on the assignment after the HEN$_1$ steps not on the steps itself. This rule is introduced to simplify the proof that the new assignment is correct. Later we will proof that for any sequence that obeys the first three rules, the assignment until the HEN$_2$ step can be chosen so, that the last rule is obeyed (Lemma 3.5).

The Assignment Given a graph G with a GFAA ψ. Let G_n be the result of a HEN$_{1^n2}$ step applied to G and let ψ_n be the updated assignment, also we denote with G_i and ψ_i the resulting graph and updated assignment after the i-th part of the HEN$_{1^n2}$ step.

The vertices different from x_i, y_i, v_i, that are assigned to $f_{a_{i-1}}$ under ψ_{i-1}, will be assigned in the trivial way under ψ_i, i.e. the new face it is incident to in G^i.

In Figure 9 a visual representation of the assignment is given, in the first column the assignement after the first HEN$_1$ step $[1_0]$, the second column after the i-th HEN$_1$ step $[1_i]$ and the assignment after the HEN$_2$ step $[2]$ in the rightmost column. In a $[1_i]$ step such that the corners of the previous allowed face ($f_{a_{i-1}}$) are well distributed over the the allowed face and the not-allowed face[1], as in the bottom figure of the $[1_i]$ column of Figure 9 we consider two different methods for the assignment. Note that this only occurs when z_i is not a corner. We denote the methods with OLD-FIRST-method and NEW-FIRST-method. The OLD-FIRST-method prefers to assign vertices of the original face (f) to not-allowed faces, and the

[1]Faces in which we do not continue are denoted *not-allowed*

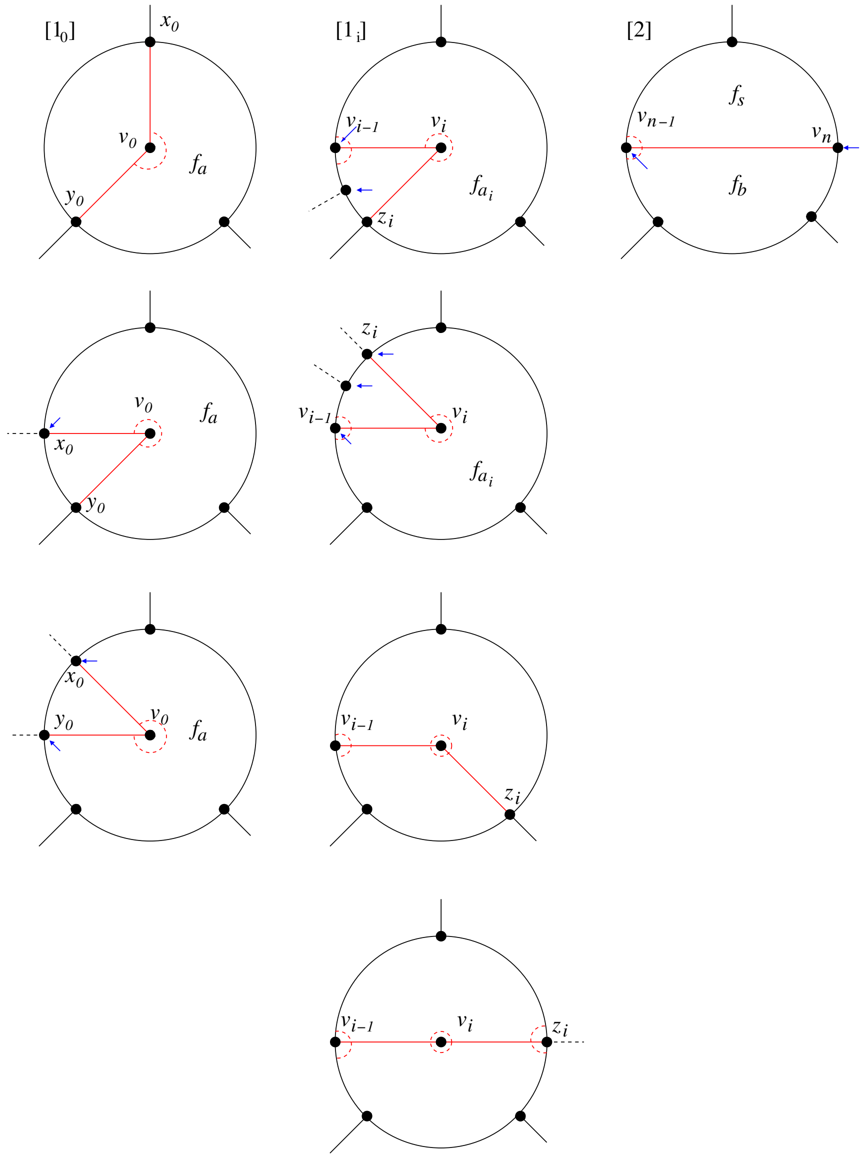

Figure 9: Updating the Assignment during a HEN_{1^n2} step.

NEW-FIRST-method, prefers to assign new vertices (i.e. v_{i-1} in step i) to not-allowed faces. Recall that the corners of a face are the vertices not assigned to the face[2] the are presented as triangles in Fig. 9.

[1_0] Let v_0 denote the new vertex, connected to x_0 and y_0, which are neighbors, splitting the face f in a 3-face f_0 and an $(n-2)$-face, f_{a_0}. If x_0 or y_0 was assigned to f, it will now be assigned to f_a. We call f_{a_0} the allowed face and in the next step this face will be splitted and v_0 will be assigned to either of the new faces.

[1_i] Let v_i denote the new vertex connected to v_{i-1} and z_i, z_i is a neighbor of x_{i-1} or y_{i-1} and z_i is incident to f. The current allowed face $f_{a_{i-1}}$ is splitted, the face in which we continue is called f_{a_i}, the new allowed face, and the other new face is called f_i. In the next step f_{a_i} will be splitted and v_i will be assigned to either of the new faces.

If f_{a_i} is incident to all corners of $f_{a_{i-1}}$ then v_{i-1} is assigned to f_{a_i}, if z_i was assigned to $f_{a_{i-1}}$ then it is now assigned to f_{a_i}.

[2] After each HEN_1 step, the allowed face has four corners, as the previously added vertex will be assigned one step after it has been added.

If f_{a_i} incident to at most three corners of $f_{a_{i-1}}$ and z_i was not assigned to $f_{a_{i-1}}$ we assign v_{i-1} to f_i.

Otherwise, we distinguish between the methods. When using the OLD-FIRST-method, z_i is assigned to f_i and v_{i-1} is assigned to the face incident to three of the four corners of $f_{a_{i-1}}$. When using the NEW-FIRST-method, v_{i-1} is assigned to f_i and z_i is assigned to the face incident to three of the four corners of $f_{a_{i-1}}$.

[2] Let v_n denote the new vertex introduced by the subdivision of an edge, v_n is connected to v_{n-1}. The face $f_{a_{n-1}}$ is splitted into f_b, the new face incident to at least three corners of $f_{a_{n-1}}$ and f_s. The face that is also incident to the subdivided edge is denoted f_n. Assign v_n to f_n and v_{n-1} to f_b.

The Correctness

Theorem 3.4. Given a 3-connected, plane graph G with a GFAA ψ. Let G_n be the result of a HEN_{1^n2} step applied to G and let ψ_n be the updated assignment. Then ψ_n is a GFAA and G_n admits an SLTR.

Proof. It is trivial that ψ_n satisfies C_v and C_f after the HEN_2 step and thus ψ_n is an FAA.

We consider the induced families of pseudosegments, Σ and Σ_i of ψ resp. ψ_i $(i = 0, \ldots, n)$, where ψ_i denotes the assignment after step i. Since ψ is a GFAA we know that every subset of Σ has at least three free points or cardinality at most one. Obviously ψ_i satisfies C_v and it also satisfies C_f in all faces but the allowed face f_{a_i}.

In every step we consider a subset S of Σ_i, with $|S| \geq 2$, and show that S has at least three free points. In step i, $(0 < i \leq n)$, we rely on the fact that we have already shown that every subset of Σ_{i-1} has at least three free points or cardinality at most one.

A *covering* denotes a vertex which is interior to one pseudosegment and an endpoint for another, we say that v *is a covering in* S if there exist pseudosegments $s, t \in S$ such that v is interior to s and an endpoint of t.

[1_0] Let s_x and s_y denote the two pseudosegments ending in v_0 incident to x_0 resp. y_0. Let s^+ the pseudosegment that is incident to both x_0 and y_0. If S does not contain s_x, s_y or s^+, the free points of the comparable set in Σ are the free points of S, hence S has three free points. Any subset of $\{s_x, s_y, s^+\}$ of cardinality at least two, has three free points. So let S contain at least one pseudosegment not of $\{s_x, s_y, s^+\}$. Now consider the comparable set S' of Σ, that is, delete s_x, s_y, s^+ from S and add the pseudosegment of Σ incident to both x_0 and y_0, denoted with s. Since $|S'| > 1$, S' has three free points.

- If $s_x \in S$ then v_0 is free for S and the other endpoint of s_x is also an endpoint of s and it is free for S only if it is free for S'.
- If $s_y \in S$ then v_0 is free for S and the other endpoint of s_x is also an endpoint of s and it is free for S only if it is free for S', if both $s_x, s_y \in S$ then together they contribute at least one free point more to S (namely v_0) than s contributes to S'.
- If $s^+ \in S$ and $s_x, s_y \notin S$ then s^+ contributes at least as many free points to S as s to S'. Suppose an endpoint of s^+ is not free for S, then either, it is also an enpoint of s and not free in S', or it is covered by s_x (or s_y) in which case the related endpoint of s is free for S' implies that this endpoint is contributed as a free point to S by s_x (or s_y).

It follows that if s contributed free points, then the deleted pseudosegment(s) of S contribute as many free points for S. Hence S has at least three free points.

Note that, when $|S| > 2$ and $s_x, s_y \in S$ then S has three free points different from v_0.

[1_i] Let s_{i-1} denote the pseudosegment that has v_{i-1} as an interior point and s_i the *other* pseudosegment with v_i as an endpoint. We have now named two pseudosegments bounding the not allowed face, let s^+ be the third. Consider the set S.

Any subset of $\{s_i, s_{i-1}, s^+\}$ of cardinality at least two, has three free points. So let S contain at least one pseudosegment *not* of $\{s_i, s_{i-1}, s^+\}$.

Suppose z_i is a corner of $f_{a_{i-1}}$, then v_{i-1} is the only possible new covering. Suppose v_{i-1} is a free point for the comparable set S', but not for S, then $s_{i-1} \in S$ and v_i is a "new" free point.

On the other hand, when z_i is not a corner of $f_{a_{i-1}}$, then z_i is a free point for the comparable set S' only if it is also a free point for S.

Hence as all $S' \in \Sigma_{i-1}, |S'| \geq 2$ have at least three free points and there is no new covering possible that does not induce a "new" free point, it must hold that all $S \in \Sigma_i, |S| \geq 2$ have at least three free points.

Note that, when $|S| > 2$ and $s_i, s_{i-1} \in S$ then S has three free points different from v_i, since the comparable set under Σ_{i-1} has three free points different from v_{i-1}.

[2] Note that $|\Sigma_{n-1}| = |\Sigma_n|$, i.e. no new pseudosegment is introduced. Since all vertices of the original face f have gotten precisely one new neighbor during the HEN_1 steps, we know that the face f_{a_n}, in which the HEN_2 step takes place has precisely one edge in common with f. This is not the subdivided edge. There is only one new incidence introduced, the pseudosegment for which v_{n-1} is an interior point (s'_{n-1}) and the pseudosegment for which v_n is an interior point (s'_n) have v_n as a common point. We will first show that these two pseudosegments touch only once in Σ_n.

Claim: every pseudosegment of Σ_{n-1} touches f at most once.

Proof. We want to show that every pseudosegment s shares at most one (connected) path with f. Suppose otherwise. If s is inside f, see Figure 10 (a), s touches f in two disjoint points. Let a, b the two points on f that are also in s and consider outline cycles, $\gamma(h_1), \gamma(h_2)$, of the two halves of f, h_1 and h_2, both bounded by a part of f and (part of) s. Let $\gamma(h_1)$ have three convex corners, say a, b and some point not incident to h_2. Now if a, b are also convex corners for $\gamma(h_2)$, all convex corners of f are used, hence $\gamma(h_2)$ has at most two, contradiction. Suppose a is not convex for f, then, if a is assigned inside h_2, at most one more convex corner of f can contribute to $\gamma(h_2)$, but again, $\gamma(h_2)$ has at most two. On the other hand, if a is assigned inside h_1 and we assumed $\gamma(h_1)$ has at least three convex corners, we may conclude that the third convex corner of f contributes only to $\gamma(h_1)$ and again, $\gamma(h_2)$ has at most two convex corners. Similarly if b is not convex for f we find that $\gamma(h_2)$ has at most two convex corners, hence there is no such pseudosegment s.

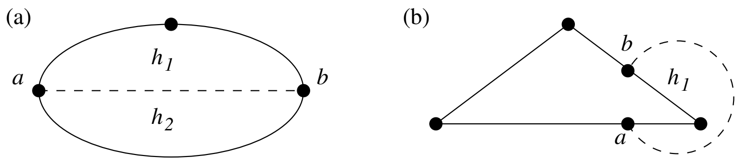

Figure 10: The dotted line represents a pseudosegment that touches f twice.

Secondly suppose s touches f twice and lies outside of f, see Figure 10 (b). Note again that a, b can not be vertices of the same pseudosegment, as then we find two pseudosegments that touch twice. Consider the outline cycle of $h_1 \cup f$, two convex corners of f will be convex for this outline cycle but the third convex corner of f lies inside. Hence this outline cycle has at most two convex corners. Therefore, every pseudosegment of Σ_{n-1} touches f at most once. □

Suppose that the pseudosegments s'_{n-1} and s'_n touch twice in Σ_n, consider the comparable pseudosegments s_{n-1}, s_n, of Σ_{n-1}, which contain v_{n-1} resp. the subdivided edge. For s'_{n-1} and s'_n touch twice in Σ_n, the comparable pseudosegments s_{n-1} and s_n must have a common point p. We distinguish three cases, p lies outside f, inside f or, on the boundary of f.

- *(p strictly outside f)* Then both s_{n-1} and s_n must continue outside f, hence they both have an edge in common with f (as two edges incident to a vertex but not neighboring edges in the cyclic order around the vertex, can not belong to the same pseudosegment). Then there are two comparable pseudosegments of Σ that are on f and touch in p strictly outside f. As f is incident to precisely three pseudosegments of Σ, which pairwise have a point on f in common, we conclude that we have two pseudosegments in Σ that touch twice, a contradiction. So p is not strictly outside f.

– *(p strictly inside f)* Consider Figure 11 (a). The allowed face of v_{n-1}, i.e. $f_{a_{n-1}}$, shares precisely one edge with f. Since p is strictly interior of f, p must have been introduced at some HEN_1 step and it has a neighbor on f. Let n_1, n_2 be the points where s_{n-1} resp. s_n touch $f_{a_{n-1}}$ for the first time after leaving p. There is no possibility for a vertex between n_1 and n_2 to have a neighbor on f unless s_{n-1} or s_n touches f. A pseudosegment shares at most two points with f, therefore there are at most two points q_1, q_2 between n_1, n_2, which have a neighbor on f incident to s_{n-1} or s_n. Both n_1 and n_2 also need a neighbor on f, this is not the neighbor q_1 resp. q_2, hence it must be either the first point of s_{n-1} resp. s_n towards p, or the third neighbor of n_1 resp. n_2. If both of them have their neighbor on f also in $f_{a_{n-1}}$ there can not be a vertex v_{n-1} in $f_{a_{n-1}}$ that is interior to s_{n-1}, contradiction. Since every vertex strictly inside f has degree three, we know that at least s_{n-1} touches f towards p. But since p has to be assigned as well, either s_n must touch f in the neighbor of p on f or n_1, n_2 are neighbors, the first implies that either s_n or s_{n-1} touches f twice, contradiction. So assume n_1, n_2 to be neighbors, s_{n-1} touches f towards p and the neighbor of p on f is interior to s_{n-1} as in Figure 11 (b).

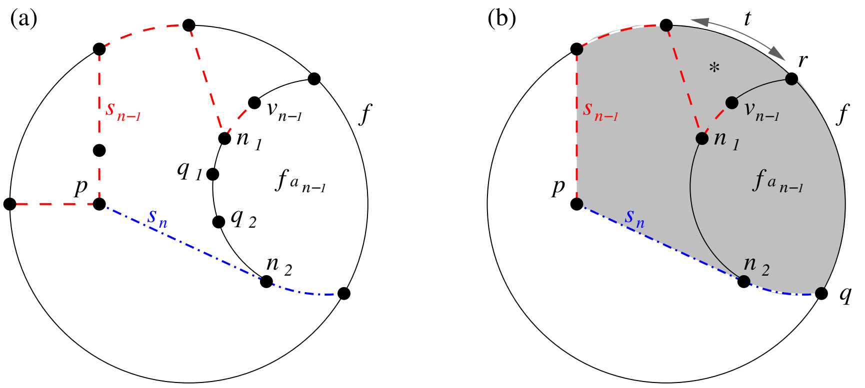

Figure 11: Point p is strictly inside f.

Consider the region R bounded by p and the parts of s_{n-1}, s_n from p to f, R contains n_1 and n_2 (grey area of Figure 11 (b)). There must be at least three convex corners on its boundary, say p, q and r. Trivially p is a convex corner. If q and r are both incident to $f_{a_{n-1}}$, then $f_{a_{n-1}}$ has five vertices (namely, q, r, n_1, n_2 and v_{n-1}) not assigned to it under ψ_{n-1}, contradiction. Suppose either q or r is not a convex corner for $f_{a_{n-1}}$, then there must be a convex corner t of R incident to the $*$ region in Figure 11 (b) and to f. Since all the interior vertices of f have degree 3 (except for v_{n-1}), any such t must have a neighbor strictly inside f. But then r must have been assigned to a face in the $*$ region as otherwise there is a face (containing at least r, v_{n-1}, a neighbor of v_{n-1} interior to s_{n-1} and a neighbor of r on the boundary of f) bounded by four different pseudosegments under ψ_{n-1}. As this is not the allowed face, we have a contradiction. Suppose there is another convex corner t', then similarly as above, we find that t is not a corner for f. It follows that there can be at most one convex corner of R incident to the $*$ region and not to $f_{a_{n-1}}$, and this is possible only if r is not a convex corner of R yet it is a convex corner of $f_{a_{n-1}}$.

Since r must be a convex corner at least for $f_{a_{n-1}}$ and also q must be a convex corner for R, $f_{a_{n-1}}$ must have five vertices not assigned to it under ψ_{n-1}, this is a contradiction. Therefore p is not strictly inside f.

– *(p on f)* Since p is on f, s_{n-1} and s_n can not touch f on the other side. There are four pseudosegments bounding $f_{a_{n-1}}$, one of them shares an edge with f, say s_4, see Figure 12. Now the meeting point, t, of s_n with s_4, must have a third neighbor in f, and since t is assigned, it will be an interior point of s_4 or s_n.

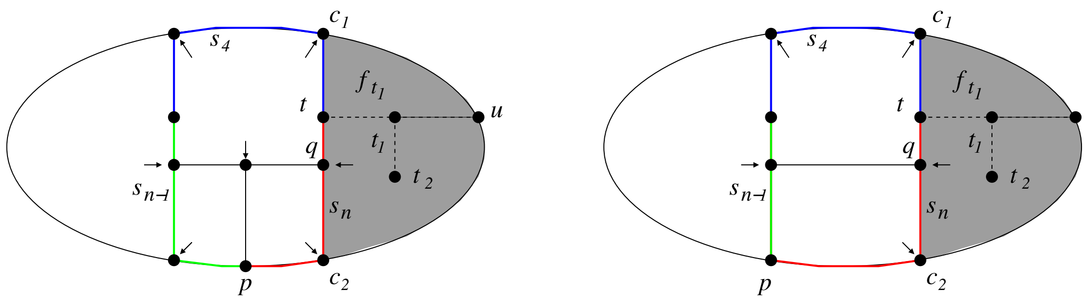

Figure 12: Point p is on f.

Consider the region R, the grey colored area in Figure 12, it must have at least three convex corners, c_1, c_2 are two of them and the third is not t. Vertex t has its third neighbor outside $f_{a_{n-1}}$ and t is assigned, therefore there must be a convex corner u on f. Since t is assigned, it will continue s_4 or s_n, hence cannot be the v_0 vertex[3] (otherwise s_4 or s_n touches f twice). Then there exists a t_1 neighbor of t which is introduced in a HEN$_1$ step. But then t_1 must be assigned and it cannot continue s_4, s_n to f hence it is assigned otherwise, but then it can not be v_0 and there must exist a t_2. Since this sequence $t, t_1, t_2, \ldots$ will not contain v_0, at some point there must be a vertex which has one neighbor on f and a neighbor w which is an interior point of s_n or s_4.

Since t can not be a neighbor of u, there is no vertex between t and f on s_4, hence w must be interior to s_n. Also w can not be q or t. If w lies between q and t, it has no neighbor on f, contradiction. If w lies between q and f then q has no neighbor on f, contradiction. Hence there is no point w, which implies that there is no vertex u and R does not have three convex corners under ψ_{n-1}, which is a contradiction and therefore p is not on f.

We conclude that there is no point p in which s_n and s_{n-1} touch.

Consider any set $S \subseteq \Sigma_n$. Suppose $s'_n, s'_{n-1} \notin S$ then the comparable set under Σ_n has three free points, none of which are covered in the HEN$_2$ step, hence S has the same set of free points under Σ_n. If $s'_n \in S$ and $s'_{n-1} \notin S$ also nothing has changed.

Let $s'_n, s'_{n-1} \in S$ and consider the comparable set S' of Σ_{n-1}, i.e. replace s'_{n-1} with s_{n-1} and $s'_n \in S$ with s_n. Let s^+ the pseudosegment that ends in v_{n-1}. If $s^+ \in S$ then the comparable set has three free points different from v_{n-1} and those must also be free points for S. So suppose $s^+ \notin S$. Consider $S \cup s^+$, this set has three free points and s^+ contributes at most one. But since s^+ and s'_n touch, either an endpoint of s^+ is covered by s'_n, or an endpoint of s'_n is covered by s^+ or they share an endpoint, in either case, removing s^+ from $S \cup s^+$ leaves the number of free points unchanged and hence S must have three free points.

□

Left to show is that there exists an assignment which obeys the last rule. That is, the HEN$_2$ step splits a face $f_{a_{n-1}}$ into two faces, which are both incident to at least one of the not-assigned vertices of $f_{a_{n-1}}$ different from v_{n-1}.

Lemma 3.5. Given a 3-connected, plane graph $G = (V, E)$ with a GFAA ψ. For every HEN$_{1^n2}$ step in a bounded $(n+1)$-face f of graph G, let $G' = (V', E')$ the resulting graph and let ψ_{n-1} the intermediate assignment after all HEN$_1$ steps that follows the OLD-FIRST-method and ψ^+_{n-1} the one that follows the NEW-FIRST-method. Then if neither ψ_{n-1} nor ψ^+_{n-1} is such that the last rule of the HEN$_{1^n2}$ step is obeyed, then ψ_{n-1} or ψ^+_{n-1} is such that v_{n-2} and z_{n-1} are not assigned to the same face in step $n-1$. In the latter the assignment of v_{n-2} and z_{n-1} can be swopped, hence we have a sequence of OLD-FIRST-steps followed by one NEW-FIRST-step (or the other way around), such that the last rule is obeyed.

[3]The vertex v_0 is the first introduced vertex, recall that it is connected to two neighbors on f, x_0 and y_0.

Proof. We consider the complete HEN$_{1^n2}$ step to be known when the assignment is updated. Consider ψ_{n-1}, if now the last rule of the HEN$_{1^n2}$ step is obeyed, we are done. So suppose not. Let f_b the face which is incident to all corners of $f_{a_{n-1}}$ and f_s, the face incident to none under ψ_{n-1}.

First suppose the NEW-FIRST-method assigns the vertices in f_b to f_b, i.e. now f_b is the face that is not incident to a corner of $f_{a_{n-1}}$. Note that this implies that $|f_{a_{n-1}}| \geq 7$ after the HEN$_1$ steps. Before the HEN$_2$ step both ψ_{n-1} and ψ^+_{n-1} satisfy C^+_o (Part of the proof of Theorem 3.4).

Claim 1.: If $f_{a_{n-1}}$ is interior to outline cycle γ, then γ has three convex corners in $V'\backslash\{v_{n-1}, v_n\}$ under both ψ_{n-1} and ψ^+_{n-1}. Suppose not, γ has v_{n-1} as third convex corner. Consider a Henneberg type 2 step that does satisfy the last rule of the HEN$_{1^n2}$ step, then v_{n-1} is assigned inside of γ, hence no longer a convex corner, but the assignment is an SLTR by Theorem 3.4. Hence γ must have at least three convex corners in $V'\backslash\{v_{n-1}, v_n\}$.

We consider the outline cycles of h_s and h_b as in Figure 13 (a) resp. (b), both have at least three convex corners, under both assignments.

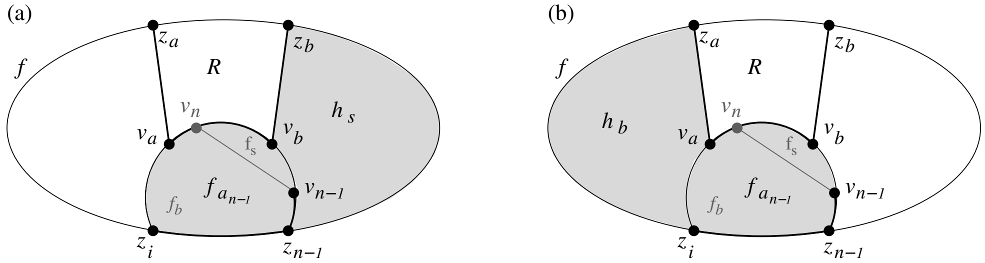

Figure 13: Definition of h_s and h_b, thick lines represent edges, thin lines denote that there may be more vertices on this path.

Consider h_b under the OLD-FIRST-method assignment, then v_{n-1} is a convex corner, but by Claim 1. there are at least three convex corners in $V'\backslash\{v_{n-1}, v_n\}$. One of which may be z_a, another z_{n_1} and the third one lies between z_a and z_i and is also a convex corner for f. Secondly consider h_b under the NEW-FIRST-method assignment, then there must be three convex corners on f, one of which may be z_b, the other two must also be convex corners for f. Since f only has three convex corners, both z_a and z_b must be convex for h_s and h_b respectively, but not for f, therefore they are assigned inside the region R that is not in h_b nor in h_s. Looking at the region R seperately, it is clear that z_a assigned inside under OLD-FIRST-method implies that z_b must be assigned outside in this case (since R only has four possible corners, including z_a, z_b). But then we consider both $h_b \cup R$ and $h_s \cup R$ under their respective assignments (see Figure 14) to see that there must be at least one more convex corner on f, hence f must have had four corners before this step, which is a contradiction.

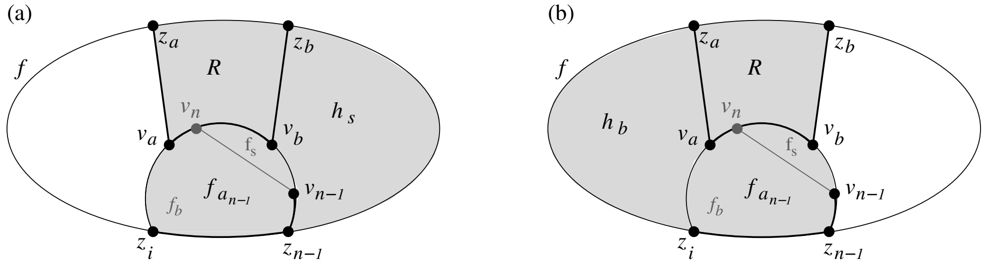

Figure 14: The possible corners of h_b and h_s united with the region R.

Assume both methods assign the vertices in f_s to f_s. First note that the face f_{n-1}, which is closed when introducing v_{n-1}, must be a 4-face consisting of vertices $z_{n-1}, z_{n-2}, v_{n-1}, v_{n-2}$ (in every step the current allowed face is divided into a 4-face and a "rest", since $f_{a_{n-1}}$ is not a 4-face, f_{n-1} must be). Secondly, v_{n-1} is not yet assigned, hence precisely one of $z_{n-1}, z_{n-2}, v_{n-2}$ must be assigned to f_{n-1}. If in both methods

z_{n-2} is assigned to f_{n-1}, we must have three convex corners of f incident to $f_{a_{n-1}}$, but $f_{a_{n-1}}$ is incident to precisely two vertices of f. Hence z_{n-2} is assigned to f_{n-1} in at most one of the two methods. If z_{n-1} is a convex corner of f we may conclude that one of ψ_{n-1} and ψ^+_{n-1} must obey the last rule, hence assume z_{n-1} is not a convex corner. Now one of ψ_{n-1} and ψ^+_{n-1} does not assign z_{n-2} to f_{n-2}, hence either v_{n-2} or z_{n-1} is assigned to f_{n-2} (and the other one is assigned to f_s under both methods). But since z_{n-2} now is a convex corner of $f_{a_{n-2}}$ the assignment of a vertex to f_{n-2} is one where we follow one of the methods. Hence we now choose the different method for the last step and assign the other one of v_{n-2} or z_{n-1} to f_{n-2}. We end with an assignment that does obey the last rule. □

Since Theorem 3.4 considers any assignment and not necessarily one that follows either the OLD-FIRST or NEW-FIRST method only, we conclude that Theorem 3.4 together with Lemma 3.5 proof that this particular combination step obeys.

There are plane Laman graphs that admit an SLTR but can not be constructed with the two steps. For example the graph in Figure 1 (b). This graph requires a $\text{HEN}_{1^l 2}$ step in an n-face with $l < n$. But if for such a construction step the assignment could be extended along this step, then the graph in Figure 1 (d) would have an SLTR.

4 Conclusion and Open Problems

We have given two construction steps such that a GFAA can be extended along these steps and the extended assignment is also a GFAA. However, this does not define the class of Laman graphs that have an SLTR. Therefore the problem: Is the recognition of graphs that have an SLTR (GFAA) in P? is still open, even for graphs in which all non-suspension vertices have to be assigned.

It would be interesting to be able to decide whether a graph has a Henneberg type 2 construction.

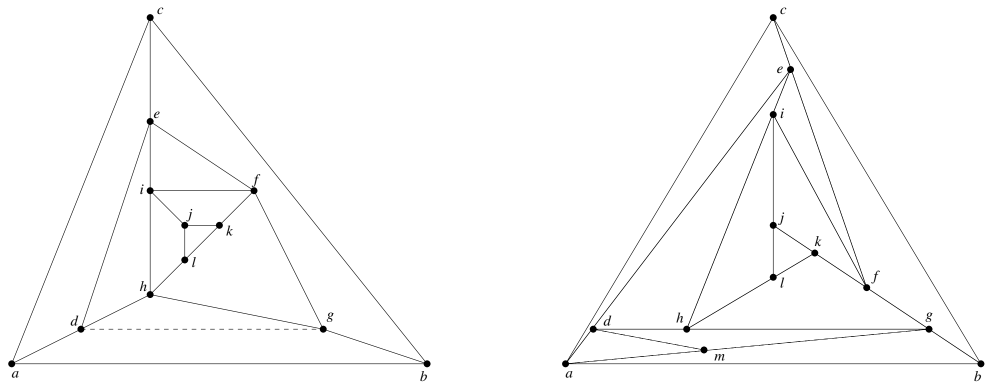

Figure 15: A graph that does not admit an SLTR but a Henneberg 2 extension of the graph does.

In Figure 15 two graphs are shown, the left graph does not admit an SLTR (consider the outline cycles following g, f, e, i, h and h, l, k, f, i, if both have three combinatorially convex corners, then both h and i must be assigned to the face (d, h, i, e) which contradicts C_f). Then apply a Henneberg type 2 step, subdivide (d, g), add m and connect m to a, to find the right graph which admits an SLTR and a Henneberg type 2 construction (starting with the triangular prism graph). A reverse Henneberg 2 step may result in a graph that does not have a Henneberg type 2 construction while the graph before the reverse step does.

The class of 3-connected quadrangulations is well-defined, e.g. Brinkmann et al. give a characterization using two expansion steps [BGG+05]. Adding a diagonal edge in the outer face of a plane, 3-connected quadrangulation yields a Laman graph. One of the expansion steps (denoted P_3 in [BGG+05]) is a Henneberg Combination step, hence a GFAA can be extended along this step. It would be interesting to know if a GFAA could also be extended along the other expansion step (denoted P_1 in [BGG+05]). If so, can all Laman graphs that admit an SLTR be constructed with the three steps P_1, $\text{HEN}_{1^n 2}$ and HEN_2?

Adding an edge in a plane graph that has a GFAA requires only minor changes to the GFAA of the original graph to obtain a GFAA for the resulting graph. An interesting question arises: Does every graph that admits an SLTR in which not every non-suspension vertex admits a straight angle, have a spanning Laman subgraph that admits an SLTR?

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