Approximating hitting sets of axis-parallel rectangles with opposite corners separated by a monotone curve

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Abstract. In this note, we present a simple combinatorial factor 8 algorithm for approximating the minimum hitting set of a family $\mathcal{R} = \{R_1, \ldots, R_n\}$ of axis-parallel rectangles in the plane such that there exists an axis-monotone curve γ separating the same two opposite corners of each rectangle R_i (say, the lower left and upper right corners).

1 Introduction

Let $\mathcal{R} = \{R_1, \ldots, R_n\}$ be a family of axis-parallel rectangles of $\mathbb{R}^2$. A set of points $T \in \mathbb{R}^2$ is said to be a *transversal* or a *hitting or piercing set* of $\mathcal{R}$ if $T \cap R_i \neq \varnothing$ for any $R_i \in \mathcal{R}$. The *transversal number* $\tau(\mathcal{R})$ is the minimum size of a hitting set of $\mathcal{R}$. The *packing number* $\nu(\mathcal{R})$ is the maximum number of pairwise disjoint rectangles of $\mathcal{R}$. In terms of the intersection graph of $G_{\mathcal{R}}$ of the family of rectangles the packing number is the independence number $\alpha(G_{\mathcal{R}})$ and due to the Helly property of rectangles the transversal number equals the clique covering number $\theta(G_{\mathcal{R}})$.

Computing, approximating, and relating $\tau(\mathcal{R})$ and $\nu(\mathcal{R})$ is both an algorithmic and combinatorial question with numerous applications. In 1965, Wegner [17] asked if it is always true that $\tau(\mathcal{R}) \leq 2\nu(\mathcal{R}) - 1$ and Gyárfás and Lehel [12] relaxed this question by asking if $\tau(\mathcal{R}) \leq c\nu(\mathcal{R})$ for a universal constant c not depending of $\mathcal{R}$. Gyárfás and Lehel [12] noticed that $\tau(\mathcal{R}) \leq \nu^2(\mathcal{R})$. Károlyi [14] proved that $\tau(\mathcal{R}) \leq \nu(\mathcal{R})\lfloor \log \tau(\mathcal{R}) \rfloor + 2$ and Fon-Der-Flaass and Kostochka [10] gave a simpler proof. Gyárfás and Lehel [12] also proved that $\tau(\mathcal{R}) \geq \frac{3}{2}\nu(\mathcal{R})$ and Fon-Der-Flaass and Kostochka [10] improved this lower bound by showing that $\tau(\mathcal{R}) \geq \frac{5}{3}\nu(\mathcal{R})$ for a set of 23 rectangles. Ahlswede and Karapetyan [3] announced that $\tau(\mathcal{R}) \leq 4\nu(\mathcal{R})$ if $\mathcal{R}$ is a family of squares and that $\tau(\mathcal{R}) \leq 2\nu(\mathcal{R})$ if $\mathcal{R}$ consists of unit squares.

Let P_b and P_r be two sets of points in the plane and let $\mathcal{R}$ be the family of all rectangles with lower left corner in P_b (blue) and upper right corner in P_r (red). Soto and Telha [16] show that in this case $\tau(\mathcal{R}) = \nu^2(\mathcal{R})$, morover optimal transversals and packings can be computed efficiently. In general he problems of computing the transversal and packing numbers of a family of axis-parallel rectangles are NP-hard. Hardness has been proven even for the case when all rectangles are unit squares [9]. Hochbaum and Maass [13] presented a PTAS for approximating $\tau(\mathcal{R})$ for unit squares and Chan [6] provided a PTAS for arbitrary axis-parallel squares. Chan and Mahmood [8] described a PTAS for families of unit-height rectangles. Agarwal and Mustafa [1] presented a constant factor approximation of $\nu(\mathcal{R})$ when the rectangles of $\mathcal{R}$ are pseudodiscs (alias pairwise non-piercing), i.e., the intersection of the boundaries of any two rectangles consists of at most two points. More recently, Chan and Har-Peled [7] and Mustafa and Ray [15] extended the approach of [1] to arbitrary pseudodiscs and presented a PTAS for approximating $\nu(\mathcal{R})$ (Chan and Har-Peled [7] also noticed that in this case $\nu(\mathcal{R}) = O(\tau(\mathcal{R}))$ holds). Aronov, Ezra, and Sharir [2] proved the existence of $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$-nets for families of axis-parallel rectangles, which, combined with a result of Brönnimann and Goodrich [4], leads to a factor $O(\log\log\tau(\mathcal{R}))$ approximation algorithm for the transversal number $\tau(\mathcal{R})$ (notice also that Chalermsook and Chuzhoy [5] described an $O(\log\log n)$ algorithm for approximating $\nu(\mathcal{R})$ for a set $\mathcal{R}$ of n rectangles).

In this note, we present a factor 8 approximating algorithm for $\tau(\mathcal{R})$ and establish that $\tau(\mathcal{R}) \leq 8\nu(\mathcal{R})$ for families $\mathcal{R}$ of axis-parallel rectangles whose opposite corners can be separated by an axis-monotone curve γ. More exactly, let $A = A(\mathcal{R})$ denote the set of all upper right corners and $B = B(\mathcal{R})$ denote the set of all lower left corners of rectangles of $\mathcal{R}$ (we will suppose that the points of A are colored red and the points of B are colored blue). An *axis-monotone curve* is an unbounded Jordan curve γ such that the intersection of γ with each horizontal or vertical line is a segment. Each axis-monotone curve γ separates the plane into two halves H'_γ and H''_γ. We say that a family of axis-parallel rectangles $\mathcal{R}$ is *separable* if there exists an axis-monotone curve γ which separates the red corners of $\mathcal{R}$ from the blue corners, i.e., such that $A(\mathcal{R}) \subset H'_\gamma$ and $B(\mathcal{R}) \subset H''_\gamma$. Here is the main result of this note:

Theorem 1. *If a family $\mathcal{R}$ of axis-parallel rectangles is separable, then $\tau(\mathcal{R}) \leq 8\nu(\mathcal{R})$ and it is possible to construct in polynomial time a hitting set T of $\mathcal{R}$ of size at most $8\tau(\mathcal{R})$ and a packing P of $\mathcal{R}$ of size at least $\nu(\mathcal{R})/8$.*

2 Preliminary results

We begin with a simple lemma that allows to decompose packing and hitting problems.

Lemma 1. *Suppose that a family of sets $\mathcal{F}$ is partitioned into m subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and that for each $\mathcal{F}_i$ there exists a polynomial algorithm that computes a hitting set T_i and a packing P_i of $\mathcal{F}_i$ such that $|T_i| \leq k_i|P_i|$. Then*

a. $\bigcup_{i=1}^m T_i$ *is a hitting set of size at most* $(k_1 + \ldots + k_m)\tau(\mathcal{F})$.

b. *The largest of the sets* P_i *is a packing of size at least* $\nu(\mathcal{F})/(k_1 + \ldots + k_m)$.

This leads to a factor $k_1 + \ldots + k_m$ *approximation algorithms for the minimum hitting set and the maximum packing problem for* $\mathcal{F}$. *Moreover,*

c. $\tau(\mathcal{F}) \leq (k_1 + \ldots + k_m)\nu(\mathcal{F})$.

Proof. An optimal hitting set for $\mathcal{F}_i$ has size at least $|P_i|$, i.e., $|P_i| \leq \tau(\mathcal{F}_i)$. Therefore, $|T_i| \leq k_i\tau(\mathcal{F}_i) \leq k_i\tau(\mathcal{F})$ and $\left|\bigcup_{i=1}^m T_i\right| \leq \sum_{i=1}^m |T_i| \leq \sum_{i=1}^m k_i\tau(\mathcal{F}) = (k_1 + \ldots + k_m)\tau(\mathcal{F})$.

Since $\bigcup_{i=1}^m T_i$ is a hitting set we obtain $\sum_{i=1}^m k_i|P_i| \geq \sum_{i=1}^m |T_i| \geq \left|\bigcup_{i=1}^m T_i\right| \geq \nu(\mathcal{F})$. It follows that if P_{i_0} is the largest of the sets P_i, then $(k_1 + \ldots + k_m)|P_{i_0}| \geq \nu(\mathcal{F})$.

For the final part **c.** note that $\tau(\mathcal{F}) \leq \left|\bigcup_{i=1}^m T_i\right| \leq \sum_{i=1}^m |T_i| \leq \sum_{i=1}^m k_i|P_i| \leq (k_1 + \ldots + k_m)|P_{i_0}| \leq (k_1 + \ldots + k_m)\nu(\mathcal{F})$. □

A family of axis-parallel rectangles is said to be *linearly separable* if there exists an axis-monotone Jordan curve γ such that for each rectangle $R \in \mathcal{R}$ the intersection $R \cap \gamma$ is a non-empty subcurve of γ and for any $R', R'' \in \mathcal{R}$ we have $R' \cap R'' \neq \varnothing$ if and only if $R' \cap R'' \cap \gamma \neq \varnothing$.

Lemma 2. *If* $\mathcal{R}$ *is linearly separable, then* $\tau(\mathcal{R}) = \nu(\mathcal{R})$.

Proof. Let $\mathcal{I}_\gamma := \{R \cap \gamma : R \in \mathcal{R}\}$. First notice that since γ is homeomorphic to the real line $\mathbb{R}$, up to this homeomorphism, $\mathcal{I}_\gamma$ can be viewed as a family of intervals in $\mathbb{R}$. Consider the interval graph G defined by $\mathcal{I}_\gamma$ and note that $\nu(\mathcal{I}_\gamma) = \alpha(G)$ and due to the Helly prperty of intervals $\tau(\mathcal{I}_\gamma) = \theta(G)$. Since interval graphs are perfect (c.f. [11]) we obtain $\tau(\mathcal{I}_\gamma) = \nu(\mathcal{I}_\gamma)$. Thus, it suffices to show that $\tau(\mathcal{R}) = \tau(\mathcal{I}_\gamma)$ and $\nu(\mathcal{R}) = \nu(\mathcal{I}_\gamma)$. The second equality is obvious because γ is a linear separating curve, i.e., two rectangles of $\mathcal{R}$ are disjoint if and only if their intersections with γ are disjoint. From the definition of $\mathcal{I}_\gamma$ it follows that any hitting of $\mathcal{I}_\gamma$ is also a hitting set of $\mathcal{R}$, hence, $\tau(\mathcal{R}) \leq \tau(\mathcal{I}_\gamma)$. Together with $\tau(\mathcal{I}_\gamma) = \nu(\mathcal{I}_\gamma) = \nu(\mathcal{R}) \leq \tau(\mathcal{R})$ this yields $\tau(\mathcal{R}) = \tau(\mathcal{I}_\gamma)$. □

A family $\mathcal{R}$of axis-parallel rectangles is *cross separable* if there exists an axis-monotone Jordan curve γ such that either γ intersects the left and the right side of all rectangles R of $\mathcal{R}$ or γ intersects the top and the bottom side of all rectangles R of $\mathcal{R}$. In the first case we say that $\mathcal{R}$ is $\|$-*cross separable* while in the second case γ is $=$-cross separable.

Lemma 3. *If* $\mathcal{R}$ *is cross separable, then* $\mathcal{R}$ *is linearly separable.*

Proof. Suppose without loss of generality that $\mathcal{R}$ is $\|$-cross separated by γ. Consider the vertical projection π from $\mathbb{R}^2$ to $\mathbb{R}$. Since γ intersects the left and the right side of each rectangle R we have $\pi(R) = \pi(R \cap \gamma)$ for all R in $\mathcal{R}$. Now, if $R' \cap R'' \neq \varnothing$, then there is a point p in $\pi(R') \cap \pi(R'') = \pi(R' \cap R'') \neq \varnothing$. Axis-monotonicity of γ implies that $\pi^{-1}(p) \cap \gamma$ is a possibly degenerate segment s. Since $p \in \pi(R' \cap \gamma)$ the intersection of s and R' is non-empty and since γ only intersects the left and the right side of R' this implies that $s \subset R'$. Similarly, $s \subset R''$, hence, $R' \cap R'' \cap \gamma \neq \varnothing$ as required. □

3 The algorithm and its analysis

Let $\mathcal{R}$ be a separable family of axis-parallel rectangles and let γ be an axis-monotone curve separating the red corners A from the blue corners B of rectangles of $\mathcal{R}$. We can suppose without loss of generality that no rectangle R of the family is properly contained in another rectangle $R' \in \mathcal{R}$: if this happen, we simply remove R' from $\mathcal{R}$. Each rectangle R of $\mathcal{R}$ is assigned to one of four subfamilies $\mathcal{R}_\neg, \mathcal{R}_\|, \mathcal{R}_=, \mathcal{R}_\llcorner$ depending of how γ intersects R. Namely, R belongs to $\mathcal{R}_\|$ (respectively, to $\mathcal{R}_=$) if γ intersects both vertical sides of R (respectively, both horizontal sides of R). Otherwise, R belongs to $\mathcal{R}_\neg$ if γ intersects the top and the right side of R and, finally, R belongs to ,$\mathcal{R}_\llcorner$ if γ intersects the left and the bottom side of R. Clearly, $\mathcal{R}$ is the disjoint union of the four subfamilies $\mathcal{R}_\neg, \mathcal{R}_\|, \mathcal{R}_=$, and $\mathcal{R}_\llcorner$. Since the two families $\mathcal{R}_\|$ and $\mathcal{R}_=$ are cross-separable (as certified by the curve γ), from Lemma 2 and 3 we immediately obtain:

Lemma 4. *$\tau(\mathcal{R}_\|) = \nu(\mathcal{R}_\|)$ and $\tau(\mathcal{R}_=) = \nu(\mathcal{R}_=)$.*

Since computing a minimum hitting set and a maximum set of pairwise disjoint intervals of a family of intervals of a real line can be done in linear time (if the ends of the intervals are sorted) [11], we deduce that the hitting and packing numbers of the families $\mathcal{R}_\|$ and $\mathcal{R}_=$ can be efficiently computed.

Next we will show how to approximate in polynomial time the transversal numbers of the two remaining families $\mathcal{R}_\neg$ and $\mathcal{R}_\llcorner$. We analyse a simple algorithm which construct a hitting set for the family $\mathcal{R}_\neg$ (a similar algorithm works also for $\mathcal{R}_\llcorner$). The idea is to partition the rectangles of $\mathcal{R}_\neg$ into two subfamilies $\mathcal{R}'_\neg$ and $\mathcal{R}''_\neg$. For the first family $\mathcal{R}'_\neg$, we construct a hitting set $T'_\neg \cup T^0_\neg$ and a packing $\mathcal{P}'_\neg \subset \mathcal{R}'_\neg$ such that $|T'_\neg| = |\mathcal{P}'_\neg|$ and $|T^0_\neg| \leq |T'_\neg|$. For the second family $\mathcal{R}''_\neg$ in the partition we can prove that it is $\|$-cross separable by the axis-monotone curve μ which is the upper envelope of the points of $T'_\neg$, thus by Lemmata 2 and 3 we conclude that $\tau(\mathcal{R}''_\neg) = \nu(\mathcal{R}''_\neg)$ and that an optimal hitting set and an optimal packing can be computed efficiently.

Recall that a point $q = (q_x, q_y)$ is said to *dominate* a point $p = (p_x, p_y)$ if $p_x \leq q_x$ and $p_y \leq q_y$. For a finite set $S \subset \mathbb{R}^2$ let S_0 be the set of all of S that are not dominated by any other point in S. The set S_0 is just the set of maxima of the dominance order on S. The *upper-envelope* $\mu(S)$ of S is the axis-monotone staircase passing through all points of S_0. Equivalently, the upper-envelope $\mu(S)$ is the boundary ∂U of the union $U = \bigcup_{p \in S} Q_p$ of the closed quadrants $Q_p = \{q = (q_x, q_y) \in \mathbb{R}^2 : q_x \leq p_x \text{ and } q_y \leq p_y\}$ consisting of all points of $\mathbb{R}^2$ dominated by $p = (p_x, p_y)$. Notice that $\mu(S)$ is an axis-monotone polygonal line whose convex corners are the points of S_0.

We continue with the description of the algorithm:

Algorithm HittingSet($\mathcal{R}_\neg$)

Input: The family $\mathcal{R}_\neg$.
Output: A partition of $\mathcal{R}_\neg$ into two families $\mathcal{R}'_\neg$, $\mathcal{R}''_\neg$ together with
a hitting set $T'_\neg \cup T^0_\neg$ and a packing $\mathcal{P}'_\neg$ of $\mathcal{R}'_\neg$ and
a hitting set $T''_\neg$ and a packing $\mathcal{P}''_\neg$ of $\mathcal{R}''_\neg$.
Initialization: $T'_\neg \leftarrow \varnothing, T^0_\neg \leftarrow \varnothing$, and $\mathcal{P}'_\neg \leftarrow \varnothing$

1. **while** $\mathcal{R}_\neg \neq \varnothing$ **do**
2. Pick a rectangle R of $\mathcal{R}_\neg$ with a highest lower left (blue) corner (if there are several such rectangles, then select one with the largest height) and let c_R be the lower right corner of R.
3. Set $\mathcal{P}'_\neg \leftarrow \mathcal{P}'_\neg \cup \{R\}$ and $T'_\neg \leftarrow T \cup \{c_R\}$.
4. Remove from $\mathcal{R}_\neg$ all rectangles R' containing the point c_R and insert them in $\mathcal{R}'_\neg$.
5. Remove from $\mathcal{R}_\neg$ all rectangles R'' that intersect the rectangle R (but do not contain c_R) and insert them in $\mathcal{R}''_\neg$.
6. **endwhile**
7. Let $\mu(T'_\neg)$ be the upper envelope of $T'_\neg$ and let $T^0_\neg$ be the set of all concave corners of $\mu(T'_\neg)$.
8. Remove from $\mathcal{R}''_\neg$ all rectangles R'' such that $R'' \cap T^0_\neg \neq \varnothing$ and insert them in $\mathcal{R}'_\neg$.
9. Compute a hitting set $T''_\neg$ and a packing $\mathcal{P}''_\neg$ of the cross-separable family $\mathcal{R}''_\neg$.
10. Return the subfamilies $\mathcal{R}'_\neg$, $\mathcal{R}''_\neg$, $\mathcal{P}'_\neg$, and $\mathcal{P}''_\neg$ of $\mathcal{R}_\neg$ and the point sets $T'_\neg \cup T^0_\neg$ and $T''_\neg$.

We begin the analysis of the algorithm by looking at the family $\mathcal{R}'_\neg$.

Lemma 5. *The set $T'_\neg \cup T^0_\neg$ is a hitting set and $\mathcal{P}'_\neg$ is a packing of $\mathcal{R}'_\neg$. The sizes of the sets are related by $|T'_\neg \cup T^0_\neg| = 2 \cdot |\mathcal{P}'_\neg| - 1$.*

Proof. From the description of the algorithm, we conclude that $\mathcal{P}'_\neg$ consists of pairwise disjoint rectangles, i.e., it is a packing, and that $|T'_\neg| = |\mathcal{P}'_\neg|$. Since $T^0_\neg$ is the set of concave corners of the staircase $\mu(T'_\neg)$ whose convex corners are the points of $T'_\neg$ we obtain that $|T^0_\neg| = |T'_\neg| - 1$. The definition of $\mathcal{R}'_\neg$ implies that $T'_\neg \cup T^0_\neg$ is a hitting set. □

Now, we continue with the basic property of the family $\mathcal{R}''_\neg$.

Proposition 1. *The family $\mathcal{R}''_\neg$ is $\|$-cross separable with respect to $\mu := \mu(T'_\neg)$.*

Proof. From the definition of $\mathcal{R}_\neg$ and the construction of the set $T'_\neg$ we conclude that each point of $T'_\neg$ is below the axis-monotone curve γ separating the sets A and B. Thus the upper envelope μ of $T'_\neg$ is also below γ. Let $R'' \in \mathcal{R}''_\neg$ and let R be the rectangle of $\mathcal{P}'_\neg$ because of which R'' was inserted in $\mathcal{R}''_\neg$, i.e., $R'' \cap R \neq \varnothing$ and $c_R \notin R''$. Moreover, since R'' was not removed from $\mathcal{R}''_\neg$ at Step 8 of the algorithm, $R'' \cap T^0_\neg = \varnothing$ holds. The remaining part of the proof is split into three claims.

Claim 1: *The right side of R is to the right of the right side of R'' and the bottom side of R is at least as high as the bottom side of R''.*

Proof of Claim 1: By the algorithm, at the moment when R was inserted in $\mathcal{P}'_\neg$, the rectangle R'' is still in $\mathcal{R}_\neg$, therefore the blue corner of R is at least as high as the blue corner of R''. Since $R'' \cap R \neq \emptyset$ and $c_R \notin R''$, necessarily we are done. This concludes the proof of Claim 1.

Claim 2: *If μ intersects a rectangle $R'' \in \mathcal{R}''_{\urcorner}$, then μ necessarily $\|$-cross R''.*

Proof of Claim 2: Since R'' is not removed from $\mathcal{R}''_{\urcorner}$ at Step 8, R'' contains no corner of μ. Therefore, μ either $\|$-cross or $=$-cross R''. Suppose by way of contradiction that μ $=$-cross R'' and let s be the vertical segment of μ traversing R''. Let $c = (c_x, c_y)$ be the lower extremity of s. Since c is a concave corner of μ it belongs to $T^0_{\urcorner}$. Let R be the rectangle intersecting R'' because of which R'' was inserted in $\mathcal{R}''_{\urcorner}$. From Claim 1 we conclude that the right side of R is to the right of the right side of R'' and the bottom side of R'' is at most as high side of R. This implies that that the corner c_R of R inserted in $T'_{\urcorner}$ either belongs to the open quadrant $\{q = (q_x, q_y) : c_x < q_x \text{ and } c_y < q_y\}$ or to the horizontal segment s' of μ incident to c. In the first case, the point c is dominated by $c_R \in T'_{\urcorner}$, contrary to the fact that μ is the upper envelope of $T'_{\urcorner}$. Otherwise, if c_R belongs to s', then from the choice of R by the algorithm we conclude that the bottom side of R'' also belongs to the horizontal line passing through c. Since s intersects R'' we find that $c \in R''$ and from $c \in T^0_{\urcorner}$ we conclude that R'' is removed from $\mathcal{R}''_{\urcorner}$ at Step 8. This contradiction establishes Claim 2.

Claim 3: *μ intersects all rectangles of $\mathcal{R}''_{\urcorner}$.*

Proof of Claim 3: Suppose by way of contradiction that $R'' \cap \mu = \emptyset$ for some $R'' \in \mathcal{R}''_{\urcorner}$. If R'' is above μ, from Claim 1 we conclude that the loweer right corner $c_R \in T'_{\urcorner}$ of R is also above μ. This is in contradiction to the definition of μ as the upper envelope of $T'_{\urcorner}$. Therefore, R'' is below μ but since μ is below γ we find by transitivity that R'' is below γ. This is in contradiction to the fact that γ is the curve certifying that the family $\mathcal{R}$ is separable. This contradiction establishes Claim 3 and concludes the proof of the proposition. □

We can now conclude the proof of Theorem 1. We have partitioned the set $\mathcal{R}$ of rectangles as $\mathcal{R} = \mathcal{R}_{\urcorner} \uplus \mathcal{R}_{\|} \uplus \mathcal{R}_{=} \uplus \mathcal{R}_{\llcorner}$ and refined the partition with $\mathcal{R}_{\urcorner} = \mathcal{R}'_{\urcorner} \uplus \mathcal{R}''_{\urcorner}$ and $\mathcal{R}_{\llcorner} = \mathcal{R}'_{\llcorner} \uplus \mathcal{R}''_{\llcorner}$. Each of the families $\mathcal{R}_{\|}$, $\mathcal{R}_{=}$, $\mathcal{R}''_{\urcorner}$, and $\mathcal{R}''_{\llcorner}$ is cross-separable (Lemma 4 and Proposition 1). Therefore, hitting sets $T_{\|}$, $T_{=}$, $T''_{\urcorner}$, and $T''_{\llcorner}$ and packings $\mathcal{P}_{\|}$, $\mathcal{P}_{=}$, $\mathcal{P}''_{\urcorner}$, and $\mathcal{P}''_{\llcorner}$ of pairwise equal size can easily be computed. Moreover, the algorithm returns a hitting set $T'_{\urcorner} \cup T^0_{\urcorner}$ and a packing $\mathcal{P}'_{\urcorner}$ for $\mathcal{R}'_{\urcorner}$ and a hitting set $T'_{\ulcorner} \cup T^0_{\ulcorner}$ and a packing $\mathcal{P}'_{\ulcorner}$ for $\mathcal{R}'_{\ulcorner}$. Since $|T'_{\urcorner}| = |\mathcal{P}'_{\urcorner}| \leq \tau(\mathcal{R}'_{\urcorner})$ and $|T^0_{\urcorner}| = |T'_{\urcorner}| - 1$, we conclude that $|T'_{\urcorner} \cup T^0_{\urcorner}| \leq 2\tau(\mathcal{R}'_{\urcorner})$. Analogously, $|T'_{\ulcorner} \cup T^0_{\ulcorner}| \leq 2\tau(\mathcal{R}'_{\ulcorner})$.

From Lemma 1 we obtain that $T := T_{\|} \cup T_{=} \cup T'_{\urcorner} \cup T^0_{\urcorner} \cup T''_{\urcorner} \cup T'_{\ulcorner} \cup T^0_{\ulcorner} \cup T''_{\ulcorner}$ is a hitting set of $\mathcal{R}$ of size at most $8\tau(\mathcal{R})$ and that the largest of the six packings is a packing of $\mathcal{R}$ of size at least $\nu(\mathcal{R})/8$. Part **c.** of the lemma implies the inequality $\tau(\mathcal{R}) \leq 8\nu(\mathcal{R})$.

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