

Rectangle and Square Representations of Planar Graphs

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Abstract

In the first part of this survey we consider planar graphs that can be represented by a dissections of a rectangle into rectangles. In rectangular drawings the corners of the rectangles represent the vertices. The graph obtained by taking the rectangles as vertices and contacts as edges is the rectangular dual. In visibility graphs and segment contact graphs the vertices correspond to horizontal or to horizontal and vertical segments of the dissection. Special orientations of graphs turn out to be helpful when dealing with characterization and representation questions. Therefore, we look at orientations with prescribed degrees, bipolar orientations, separating decompositions, and transversal structures.

In the second part we ask for representations by a dissections of a rectangle into squares. We review results by Brooks et al. (1940), Kenyon (1998) and Schramm (1993) and discuss a technique of computing squarings via solutions of systems of linear equations.

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1 Introduction

Questions around representations of graphs by geometric objects are intensively studied. Motivation comes from practical applications and the fascinating exchange between geometry and graph theory and other mathematical areas.

One of the nicest results about representations of graphs by geometric objects is Koebe's "Coin Graph Theorem" [Koe36], [Sac94], [BS93]. It asserts that every planar graph can be represented by a set of disjoint discs, one for each vertex, such that two discs touch exactly if there is an edge between the corresponding vertices. Such a representation is called a *circle contact representation*. Figure 1 shows an example.

In the 1980's Thurston observed a connection between circle packings and the Riemann Mapping Theorem. From there the theory has developed into a discrete analog of complex analysis, Stephenson's book [Ste05] gives a comprehensive introduction.

In this paper we focus on representations of planar graphs based on rectangles. We look at rectangular dissections as shown in Figure 2 and graphs that can be derived from it.

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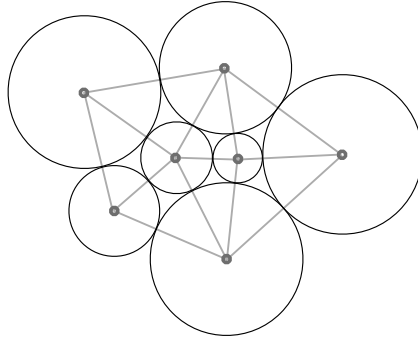


Figure 1: A circle contact representation of a planar graph.

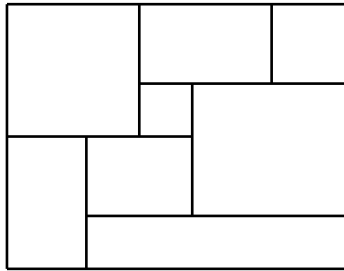


Figure 2: A rectangular dissection.

Suppose that $\phi : R \rightarrow G_\phi(R)$ corresponds to a specific mapping that associates a graph $G_\phi(R)$ with a rectangular dissection R and that $G_\phi(R)$ belongs to a class \mathfrak{G}_ϕ of graphs. Then we can ask whether a given graph G from class \mathfrak{G}_ϕ is *representable*, i.e., whether G is in the image of ϕ . The representability question can be treated as a characterization problem or as an algorithmic problem. If G is representable we can also ask for a *representation*, i.e., for a dissection R such that $G = G_\phi(R)$. In this survey we consider several graphs associated to a dissection and the corresponding representability and representation problems.

In Section 2 we look at rectangular drawings and rectangular duals. Taking the corners of a rectangular dissection as vertices and the connecting line segments as edges yields a rectangular drawing. In 2.1 we review the theory of rectangular drawings and in 2.3 we present an algorithm to decide whether a graph admits such a drawing and if so generate it. The algorithm is based on an orientation of the angular graph with prescribed out-degrees. In some variants of the problem this approach yields the fastest known algorithm. In 2.2 we consider rectangular duals, in this model the vertices of the planar graph are represented by rectangles and edges by contacts.

In Section 3 we take the horizontal and vertical segments or just the horizontal segments of a rectangular dissection as vertices. Based on horizontal and vertical segments we define the segment contact graph of a dissection and then proceed to consider more general segment contact graphs. With Theorem 3.2 we prove an unpublished characterization of segment 2-contact graphs due to Thomassen. In 3.1.1 segment contact graphs of rectangular dissections are shown to be closely related to separating decompositions. In 3.2 we consider the visibility graph of a dissection. This leads to the study of bipolar orientations. At the end of

the section, in 3.3, we look at the relation between bipolar orientations and separating decompositions.

In the second part of the survey we focus on square dissections, i.e., rectangular dissections where all rectangles are squares. Section 4 deals with the square analogs of visibility and segment contact graphs. We begin in 4.1 with the classical connection between squarings and electricity. In 4.2 we study a system of linear equation obtained from a separating decomposition and show that a solution yields a squaring. Kenyon [Ken98] developed a more general theory relating trapezoidal dissections and Markov chains, this is the subject of 4.3.

Section 5 is based on Schramm [Sch93]. The result is a characterization of graphs admitting a square dual. Finally, in 5.1 we relate square duals and transversal structures and propose an alternative method for computing square duals. The method is simple but comes with the drawback that its correctness still depends on a conjecture.

2 Rectangular drawing and rectangular duals

2.1 Rectangular drawing

Think of R as a union of interiorly disjoint rectangles. The union of the boundaries of the rectangles is the *skeleton* $\text{skel}(R)$ of the dissection R . Let $C(R)$ be the set of corners of the rectangles of R . The skeleton of R can be viewed as a graph $G_{\text{skel}}(R)$. The vertices of $G_{\text{skel}}(R)$ are the points in $C(R)$ and the edges of $G_{\text{skel}}(R)$ are the connecting line segments. More formally, the edges correspond to the connected components of $\text{skel}(R) \setminus C(R)$. The *skeleton graph* $G_{\text{skel}}(R)$ has four vertices of degree two incident to the outer face. All the other vertices are of degree three or four. The edges are drawn as horizontal or vertical line segments.

If a graph G is represented by R , i.e., $G = G_{\text{skel}}(R)$, then we call the representation a *rectangular drawing* of G . A characterization of graphs with $\Delta \leq 3$ that admit a rectangular drawing was obtained by Thomassen [Tho84]. Thomassen's result is based on an earlier result of Ungar [Ung53] who gave a characterization in the model where the corners of the outer face are regarded to be bends in an edge instead of vertices of degree two. Ungar's characterization is dual to Theorem 2.2 below.

Algorithms for the construction of rectangular drawings have been considered by various authors. One of the key results is the following:

Theorem 2.1. *Let G be a plane graph with four distinguished corner vertices of degree two and vertices of degrees at most three otherwise. There is an algorithm that decides whether G is a skeleton graph and if so computes a rectangular drawing in linear time.*

A survey of algorithms and many references can be found in Chapter 6 of the book of Nishiseki and Rhaman [NR04]. In Subsection 2.3 we present an approach to rectangular drawings and a proof of Theorem 2.1 that is not covered there. This leads to improvements in the running times of some variants of the problem.

In the graph drawing literature rectangular drawings have been extended and generalized in various directions.

- Edges are allowed to bend but remain constrained to the orthogonal drawing mode, i.e., are composed of horizontal and vertical segments. A highlight of the theory is

the application of min-cost-flow algorithms for bend minimization pioneered by Tamassia [Tam87].

- To overcome the degree restriction some authors allow that in the drawing vertices are represented by boxes. With boxes and bends every planar graph can be represented. If bends are forbidden the problem can be reduced to finding a rectangular drawing of a derived graph [RNN00].

For more on the topic we refer to the books on graph drawing [DETT99], [NR04] and the survey about orthogonal graph drawing [EFK01].

2.2 Rectangular dual

Let $F(R)$ be the set of rectangles of a rectangular dissection R . It is convenient to include the enclosing rectangle in $F(R)$. The *dual* of R is the graph $G_*(R)$ with vertex set $F(R)$ and edges joining pairs of rectangles that share a boundary segment, Figure 3 shows an example. If a graph G allows a representation as dual of a rectangular dissection R , i.e., $G = G_*(R)$, then G is called a *rectangular dual* of R . It is tempting to think that the graph $G_*(R)$ is the dual graph of $G_{\text{skel}}(R)$, this is almost true but there are some issues about the multiplicity of edges incident to the outer face of $G_{\text{skel}}(R)$, i.e., to the enclosing rectangle.

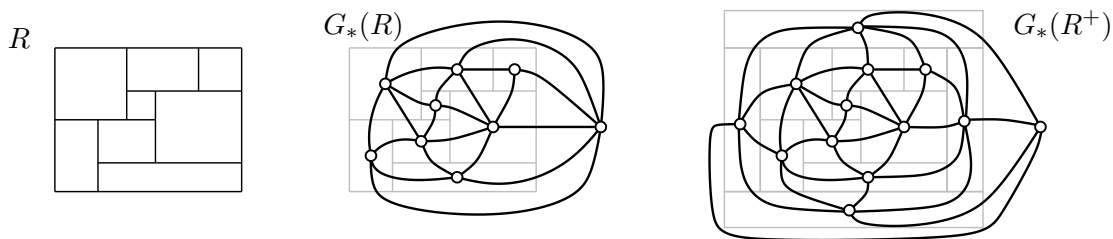


Figure 3: A rectangular dissection R , its dual $G_*(R)$, and the dual of the framed dissection R^+ .

Indeed graphs admitting a rectangular dual may have unwanted features, e.g., a vertex represented by a corner rectangle may have degree three and there may be a double-edge between an inner vertex w and the outer vertex v_∞ . A clean characterization is obtained if we assume that the degree of v_∞ is four. In terms of a rectangular dissection this can be achieved by adding a frame of four rectangles, one for each side, see Figure 3.

Theorem 2.2. *A planar triangulation with designated outer vertex v_∞ of degree four admits a rectangular dual exactly if it has no separating triangle, i.e., if it is 4-connected.*

There are many related characterizations, e.g., Kozmiński and Kinnen [KK85] or the earlier result of Ungar [Ung53] in the dual setting. Buchsbaum et al. [BGPV08] have many pointers to the literature. An elegant approach to proving the theorem is to split the task into two: In the first step it is shown that the graph can be enriched with some combinatorial structure, in the second step this structure is used to construct the geometric representation. Such an approach was taken by Bhasker and Sahni [BS88] and later refined by He [He93], Kant and He [KH97] and Fusy [Fus07]. The later of these papers use transversal structures

(c.f., Subsection 5.1) as the intermediate combinatorial structure. The approach results in the linear time construction of rectangular duals with integer coordinates bounded by n .

Problems where some region is to be partitioned into subregions subject to restrictions on the shapes of the subregions and some adjacency constraints are denoted as *floor-planing* problems. They arise in applications in VLSI chip design and cartography. In view of these applications specific optimization tasks are of interest. We mention two directions:

- Find a floorplan of a general planar graph such that the shapes of the modules representing the vertices are simple (e.g., orthogonal with ≤ 8 corners) and the total area of the floorplan is small. This problem is studied in [LLY03].
- A rectilinear cartogram is a diagram in which geographic regions have been replaced by orthogonal polygons. The neighbor relation on polygons and on their corresponding regions has to be the same, in addition the areas of the polygons corresponds to some numerical data associated to the regions. Eppstein et al. [EMSV09] have studied cartograms where all polygons are rectangles and with the flexibility that they can accommodate arbitrary area assignments. Alam et al. [ABF⁺11] show that a planar triangulation with area assignments can be represented by a cartogram using polygons with 8 corners, this is best possible.

2.3 An algorithm for rectangular drawings.

Let G be the input graph. We assume that G is given with a planar embedding and with four distinguished corner vertices of degree two, all the other degrees are three or four. With $G = (V, E)$ we consider its *trimmed angle graph* $\check{A}(G)$. The vertex set of this graph consists of the primal vertex set V together with the dual vertex set save the dual of the unbounded face, i.e., $V^* \setminus \{f_\infty\}$. Edges of $\check{A}(G)$ correspond to incidences between vertices and bounded faces of G or equivalently to the internal angles of G . To emphasize the bipartition of $\check{A}(G)$ think of vertices from V as white and of vertices corresponding to faces of G as black vertices, see Figure 4.

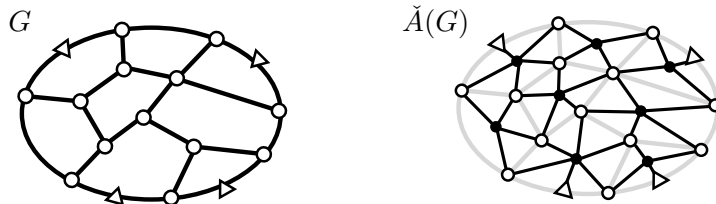


Figure 4: A planar graph G and its trimmed angle graph $\check{A}(G)$.

If G has a rectangular drawing, i.e., $G = G_{\text{skel}}(R)$ for some rectangular dissection R , then we can classify the angles of G as either being a corner or being flat with respect to R . We note:

- Each inner face is represented as a rectangle and thus has exactly four corner angles.
- Each inner vertex of degree three has exactly one flat angle.

Orient the edges of $\check{A}(G)$ such that $\{v, f\}$ is oriented as $f \rightarrow v$ when v is a corner of the rectangle corresponding to f and as $v \rightarrow f$ when the angle is flat. Now consider the out-degrees of this orientation and note that:

- $\text{out-deg}(f) = 4$ for all black vertices f . For a white vertex v we have $\text{out-deg}(v) = 1$ if v is an inner vertex of G with $\text{deg}(v) = 3$ and $\text{out-deg}(v) = 0$ if $\text{deg}(v) = 4$ or if v is a vertex of the outer face of G .

An orientation of $\check{A}(G)$ obeying the above rules for the out-degrees is denoted an α_{skel} -orientation.

Theorem 2.3. *Let G be a plane graph with four distinguished corner vertices of degree two at the outer face and vertices of degrees three or four otherwise. There is a rectangular drawing of G if and only if $\check{A}(G)$ has an α_{skel} -orientation.*

Proof. From the above we know that if $G = G_{\text{skel}}(R)$ for some R , then there is an α_{skel} -orientation of $\check{A}(G)$.

For the converse suppose that $\check{A}(G)$ has an α_{skel} -orientation. Identify the four corners of a rectangular frame F with the degree two vertices of G in clockwise order. From α_{skel} we can read off which vertices of G are corners of a given face f , they are the out-neighbors of f in the orientation. For faces f sharing an edge with the outer face f_∞ of G which is represented by F we know which corner is top-left, top-right, bottom-right and bottom-left, i.e., the *alignment* of the rectangles R_f . If we know the alignment of R_f and if faces f and f' share an edge in G , then we know the alignment of $R_{f'}$. Thus the alignment can be passed through along dual paths to all faces. The claim is that the alignment of R_f is independent of the dual path from f_∞ to f that has been used. This can be established with a homotopy type argument, the key to the argument is to check that if v is a vertex and f, f' are faces incident to v in G , then passing the alignment information from R_f to $R_{f'}$ on either of the two paths on the dual cycle around v yields the same result. This amounts to looking at the pictures of Figure 5 with all possible choices for f and f' .

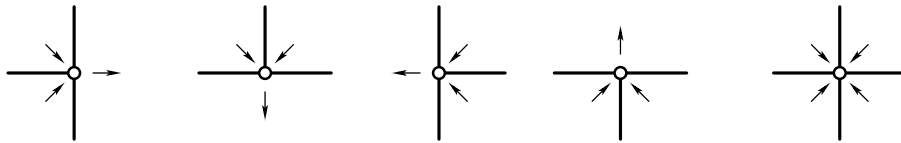


Figure 5: Consistent alignment of rectangles sharing a vertex v , the arrows indicate the underlying α_{skel} orientation of $\check{A}(G)$.

The alignment of the rectangles is equivalent to a red-blue coloring and orientation of edges of G such that red edges are horizontal with orientation from left to right and blue edges are vertical and downwards oriented. The boundary of each face consists of two directed paths in this orientation. The coloring of one of the two paths has a sequence of red edges followed by a sequence of blue edges, this is the *upper path*, the other path has blue edges followed by red edges and is called the *lower path*.

We use the red-blue coloring in the following description of how to construct the rectangular dissection R for G . Let p_0 be the lower path of the outer face f_∞ . Match the blue part of p_0 to the left side of the frame F and the red part of p_0 to the bottom side of F . This requires an arbitrary specification of positions for the vertices of degree three contained in p_0 . The third edge of each such vertex will have to be extended into the interior of F , from the coloring we know whether it is horizontal or vertical but we do not yet know its length. Such an initial piece of an edge will be called a *stump*. Starting from $p = p_0$ we add

rectangles one by one always keeping the invariant that the boundary of the set of already placed rectangles is a directed path p from the top-left corner to the bottom-right corner of F . We now focus on the placement of a new rectangle. Figure 6 shows an α_{skel} -orientation and an intermediate stage of the algorithm.

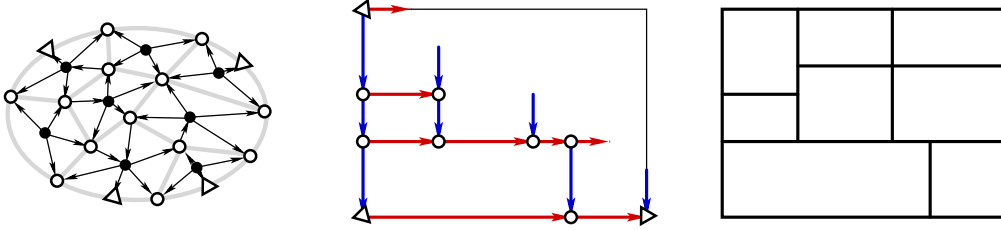


Figure 6: An α_{skel} -orientation and a corresponding rectangular dissection.

Along the path p we see some stumps, the first is a red stump on the top side of F and the last is a blue stump on the right side of F . Therefore, somewhere along p there is a red stump at a vertex v_1 followed by a blue stump at v_2 . Let p' be obtained by restricting p to the part between v_1 and v_2 . We claim that p' consists of a sequence of blue edges followed by a sequence of red edges. This can be verified as follows: The stump at v_1 is outgoing and red, the edge of p' incident to v_1 is also outgoing and hence must be blue. Similarly the edge of p' incident to v_2 has to be red. A vertex of p' where a red edge is ingoing and a blue is outgoing would have a stump which is impossible since the stumps at v_1 and v_2 are consecutive. This completes the proof of the claim. It follows that p' is the lower path of some rectangle R_f . We place R_f in the unique consistent way inside F and replace p' in p by the upper path of R_f . We also have to choose positions for the non-corner vertices contained in the upper path of R_f . Unless the top-right corner of R_f is the top-right corner of F and the dissection is complete there is at least one new stump at the top-right corner of R_f and possibly some more along the upper path. Now the status of the directed path p and its stumps is as before so that the next rectangle can be placed. \square

Algorithmically the construction of $\check{A}(G)$ from a given G and the construction of the rectangle dissection R from a given α_{skel} -orientation of $\check{A}(G)$ are both easy, they can be done in linear time. The computation of an α_{skel} -orientation can be modeled as a flow problem [Fel04b] in $\check{A}(G)$ and with methods from [MN95] be solved in $O(n^{1.5})$. In [Uec11] it is shown that the computation of an α_{skel} -orientation can be reduced to a shortest path problem. Using the currently fastest algorithm for planar shortest paths [MWN10] yields an overall running time of $O(n \log^2 n / \log \log n)$. If the input graph has no vertices of degree four we can do even better: We construct a suitably modified dual G_q^* of G having four vertices corresponding to the outer face of G , one for each segment between degree two vertices on the outer face. The α_{skel} -orientations of $\check{A}(G)$ are in bijection with transversal structures (a.k.a. regular edge labelings) of G_q^* as defined by Fusy [Fus09], [Fus07]. In his thesis [Fus07] Fusy has shown that if G_q^* has no separating triangle, then a transversal structure exists and can be computed in $O(n)$ time. This gives an α_{skel} -orientation of $\check{A}(G)$ in linear time. The result is summarized in the following:

Theorem 2.4. *Let G be a plane graph with four distinguished corner vertices of degree two and vertices of degrees three or four otherwise. There is an algorithm based on α_{skel} -*

orientations that decides whether G is a skeleton graph and if so computes a rectangular drawing in $O(n)$ time if $\Delta \leq 3$ and in $O(n \log^2 n / \log \log n)$ time if there are vertices of degree four.

Rahman et al. [RNG04] have a linear-time algorithm for rectangular drawings of graphs with maximum degree $\Delta \leq 3$ even in the case where no plane embedding of the graph is prescribed.

Miura et al. [MHN06] have an $O(n^{2.5} / \log n)$ algorithm to recognize whether a plane graph G with $\Delta = 4$ has a rectangular drawing. Their result is stated in a more general form: they allow an outer face of more complex shape. Since they prescribe which outer vertices are convex resp. concave corners the shape of the outer face can be modeled by adapting the out-degrees of α_{skel} . Therefore, this generalization is also covered by our approach. This yields a solution with $O(n \log^2 n / \log \log n)$ running time.

3 Segment contact and visibility graphs

3.1 Segment contact graphs

Think of a rectangular dissection R as a set of segments, some horizontal and some vertical. If R contains no point where four rectangles meet, intersections between segments only occur between horizontal and vertical segments and they involve an endpoint of one of the segments, i.e., they are contacts. Otherwise, we break one of the two segments of each crossing point into two to get a system of interiorly disjoint segments. The *segment contact graph* $G_{\text{seg}}(R)$ of a rectangulation R is the bipartite planar graph whose vertices are the segments of R and edges correspond to contacts between segments. From Figure 7 we see that $G_{\text{seg}}(R)$ is indeed planar and that the faces of $G_{\text{seg}}(R)$ are in bijection with the rectangles of R and are uniformly of degree 4. Hence, $G_{\text{seg}}(R)$ is a maximal bipartite planar graph, i.e., a quadrangulation.

If H is some subgraph of $G_{\text{seg}}(R)$, then H can also be represented as segment contact graph of some set of interiorly disjoint horizontal and vertical segments in the plane, i.e., H is a segment contact graph. A segment contact representation for H is obtained from R by removing some segments (vertex deletion) and slightly pulling back the ends of some segments to get rid of contacts (edge deletion).

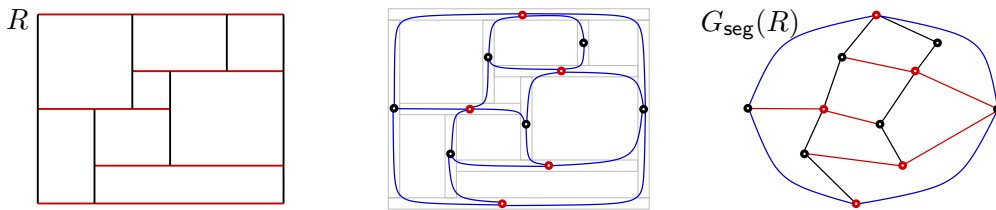


Figure 7: A rectangular dissection R and two drawings of its segment contact graph $G_{\text{seg}}(R)$.

Theorem 3.1. *Every planar bipartite graph H admits a contact representation with interiorly disjoint horizontal and vertical segments.*

This was shown by Hartman et al. [HNZ91] and by de Fraysseix et al. [dFOdMP95]. In the next subsection we sketch a proof of the theorem based on the concept of separating decompositions of quadrangulations.

A contact representation with interiorly disjoint segments such that the intersection of any $k + 1$ segments is empty is called a k -contact representation. Thomassen characterized the class of graphs admitting a 2-contact representation:

Theorem 3.2. *A planar graph $G = (V, E)$ has a 2-contact representation if and only if $|E[W]| \leq 2|W| - 3$ for every subset W of the vertices. As usual $E[W]$ denotes the set of edges with both ends in W .*

Thomassen presented the result on the Graph Drawing conference 1993 but never published his proof. Below we provide a simple proof based on rigidity theory. The condition stated in the theorem can be efficiently checked, see e.g. Lee and Streinu [LS08]. Hliněný [Hli01] showed that the recognition of general contact graphs of segments is NP-complete, actually he shows that even the recognition of graphs admitting a 3-contact representation is NP-complete.

A related class of graphs are the intersection graphs of segments. A long standing conjecture dating back to Scheinerman's Ph.D. thesis was that every planar graph is a segment intersection graph. The conjecture was finally resolved by Chalopin and Gonçalves [CG09]. Kratochvíl and Kuběna [KK98] asked whether all complements of planar graphs are segment intersection graphs.

Proof of Theorem 3.2. The necessity of the condition is easily seen: Let \mathcal{S} be the set of segments of a 2-contact representation of G . For $W \subset V$ let X_W be the set of end-points of segments in \mathcal{S} corresponding to vertices of W . We have $|X_W| = 2|W|$. There is an injection ϕ from edges in $E[W]$ to points in X_W . Corners of the convex hull of X_W , however, cannot be in the image of ϕ . Since the convex hull contains at least three corners we get: $|E[W]| \leq |X_W| - 3 = 2|W| - 3$.

For the converse we need some prerequisites. A *Laman graph* is a graph $G = (V, E)$ with $|E| = 2|V| - 3$ and $|E[W]| \leq 2|W| - 3$ for all $W \subset V$. Laman graphs are of interest in rigidity-theory, see e.g. [GSS93], [Fel04a]. Laman graphs admit a *Henneberg construction*, i.e., an ordering v_1, \dots, v_n of the vertices such that if G_i is the graph induced by the vertices v_1, \dots, v_i , then it holds that G_3 is a triangle and G_i is obtained from G_{i-1} by one of the following two operations:

- (H₁) Choose vertices $x \neq y$ from G_{i-1} and add v_i with the two edges (v_i, x) and (v_i, y) .
- (H₂) Choose an edge (x, y) and a third vertex z from G_{i-1} , remove (x, y) and add v_i together with the three edges (v_i, x) , (v_i, y) , and (v_i, z) .

In [HOR⁺05] it is shown that planar Laman graphs admit a planar Henneberg construction in the sense that the graph is constructed together with a plane straight line embedding and vertices stay at their position once they have been inserted. Moreover, the Henneberg construction can start with an outer triangle which remains unchanged throughout the construction.

Now let G be a planar graph fulfilling the condition of the theorem. We may assume that G is Laman since we can easily get rid of edges in a segment contact representation by retracting ends of segments. Consider a planar Henneberg construction $G_3, \dots, G_n = G$.

Starting from three pairwise touching segments representing G_3 we add segments one by one. For the induction we need the following invariant:

- After adding the i th segment s_i we have a 2-contact representation of G_i and there is a correspondence between the cells of the segment representation and the faces of G_i which preserves edges, i.e., if (x, y) is an edge of the face, then one of the corners of the corresponding cell is a contact between s_x and s_y .

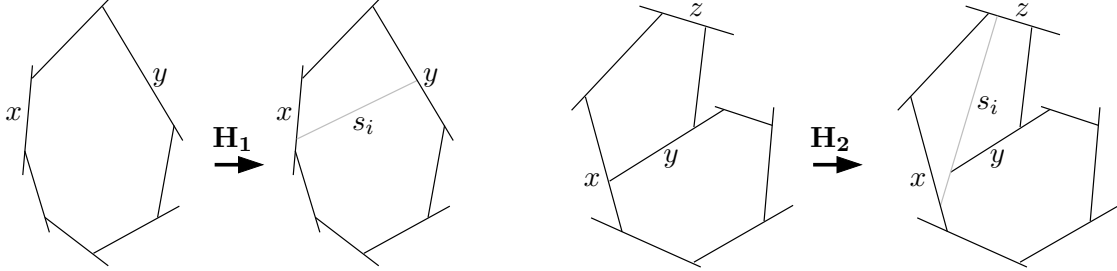


Figure 8: The addition of segment s_i .

From the count of edges it follows that the endpoints of all the segments except the three outer segments are used in contacts, therefore all faces in the segment contact representation are convex. Figure 8 shows how to add segment s_i in the cases where v_i is added by \mathbf{H}_1 , resp. \mathbf{H}_2 . It is evident that the invariant for the induction is maintained. \square

3.1.1 Separating decompositions and segment contact representations

Let Q be a quadrangulation, we call the color classes of the bipartition white and black and name the two black vertices on the outer face s and t . A *separating decomposition* of Q is an orientation and coloring of the edges of Q with colors red and blue such that:

- (1) All edges incident to s are ingoing red and all edges incident to t are ingoing blue.
- (2) Every vertex $v \neq s, t$ is incident to a non-empty interval of red edges and a non-empty interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval of a color is outgoing and all the other edges of the interval are incoming. If v is black, the outgoing edge is the last one in its color in clockwise order (see Figure 9).

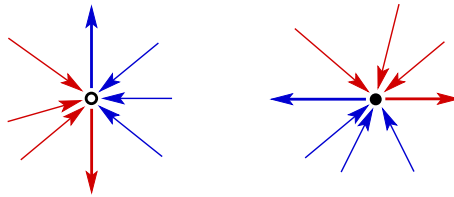


Figure 9: Edge orientations and colors at white and black vertices.

Separating decompositions have been studied in [dFOdM01], [FHKO10], and [FFNO11]. To us they are of interest because of the following lemma:

Lemma 3.3. *A segment contact representation of Q with horizontal and vertical segments induces a separating decomposition of Q .*

Proof. We assume w.l.o.g. that the segment contact representation and the plane embedding of Q are compatible in the sense that for every vertex v the clockwise order of the edges around v corresponds to the clockwise order of the contacts around the segment s_v of v . We also assume that the segments representing s and t are horizontal, s is the bottom and t is the top segment and their endpoints have no contact with another segment.

An edge of Q is represented by a contact where an endpoint of one segment is touching the interior of another segment. Orient the edge such that the vertex contributing the endpoint is the tail of the oriented edge, this yields a 2-orientation of Q , i.e., an orientation where every vertex except s and t has out-degree two. Since s is horizontal all neighbors of s have to be vertical. Tracing this kind of argument through the graph we conclude that all black vertices are represented by horizontal segments and all white vertices by vertical segments. Color the edge corresponding to the left contact of a horizontal segment blue and the edge of other right contact red. Similarly the edge induced by the top contact of a vertical segment is blue and the edge of the bottom contact is red. This construction yields a separating decomposition of Q . For an example see Figure 10. \square

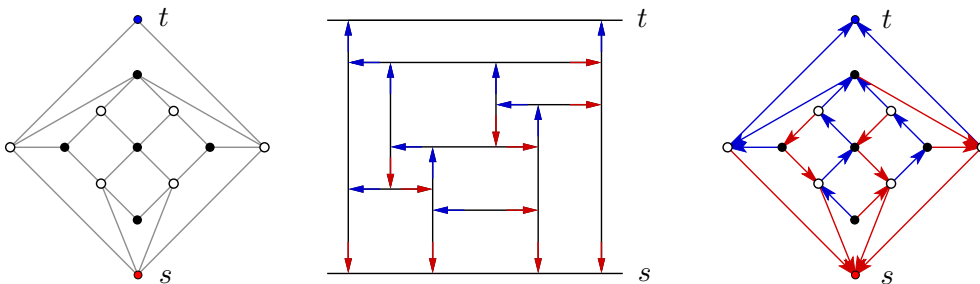


Figure 10: A quadrangulation Q , a segment contact representation of Q and the induced separating decomposition of Q .

In the following we sketch a construction for the converse. We start with a separating decomposition of Q and construct a segment contact representation. The algorithm behind the construction may not be the fastest and the construction itself not the most flexible tool for further applications. I have decided to include it because it nicely and unexpectedly combines some combinatorial structures. Details can be found in [FFNO11]. To begin with we need some facts.

- Every quadrangulation admits a separating decomposition.
- The red edges of a separating decomposition form a tree rooted at s that spans all vertices except for t . Symmetrically, the blue edges form a tree rooted at t that spans all vertices except for s .
- There exists a simple closed curve that contains all vertices of Q except s and t and avoids all edges of Q such that all red edges are in the interior of the curve and all blue edges are in the exterior. Deleting the piece of this curve that runs in the outer face we obtain the *equatorial line* of the separating decomposition.
- By straightening the equatorial line we obtain a 2-book embedding of Q , see Figure 11.

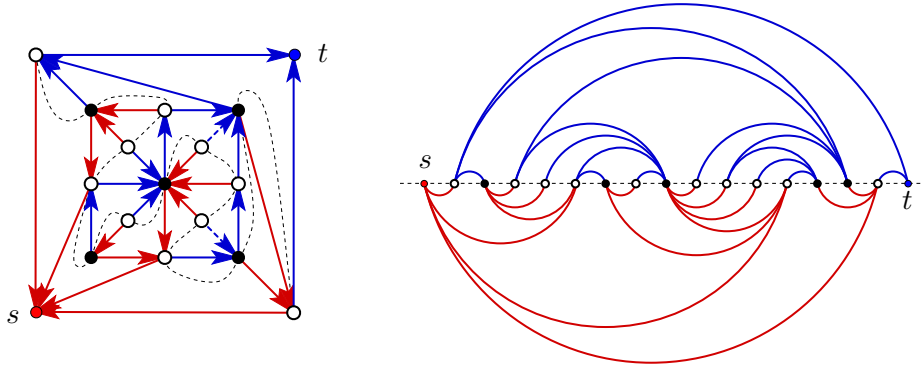


Figure 11: A quadrangulation Q with a separating decomposition S , and the 2-book embedding induced by the equatorial line of S .

An *alternating layout* of a plane tree T is a non-crossing drawing of T such that the vertices are placed on the x -axis and all edges are embedded in the halfplane above the x -axis (or all below). Moreover, for every vertex v it holds that all its neighbors are on one side, either they are all left of v or all right of v .

It can be shown that the 2-book embedding of Q obtained from S yields alternating layouts of the two trees of S . The roots of the two trees are the extreme vertices and in addition black vertices have all their blue neighbors on the left and all their red neighbors on the right while for white vertices the converse holds.

Figure 12 indicates a bijection between alternating trees and binary trees such that left/right vertices of the alternating tree correspond to left/right leaves of the binary tree.

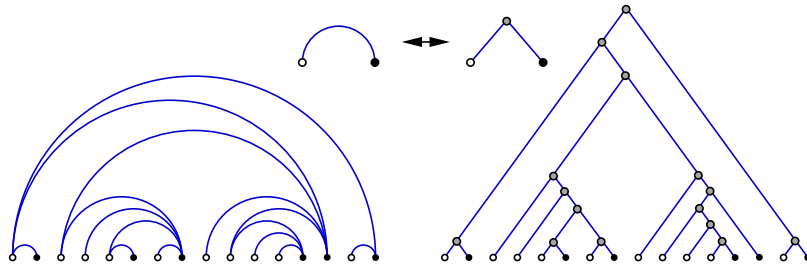


Figure 12: A bijection between alternating trees and the binary trees.

Modify the 2-book embedding by merging s with its successor into a new vertex s^+ and symmetrically t with its predecessor into t^+ . Then apply the bijection of Figure 12 to each of the two trees on its side and tilt the picture by 45° . The result is a segment contact representation of Q , see Figure 13.

3.2 Visibility graphs

A family of disjoint horizontal segments in the plane defines a *visibility graph*. The vertices of the graph are the segments and edges are based on vertical visibility: a segment s' is *visible* from segment s if there is a vertical ray r leaving s such that s' is the first segment in the family reached by r . The visibility graph is undirected since if s' is visible from s

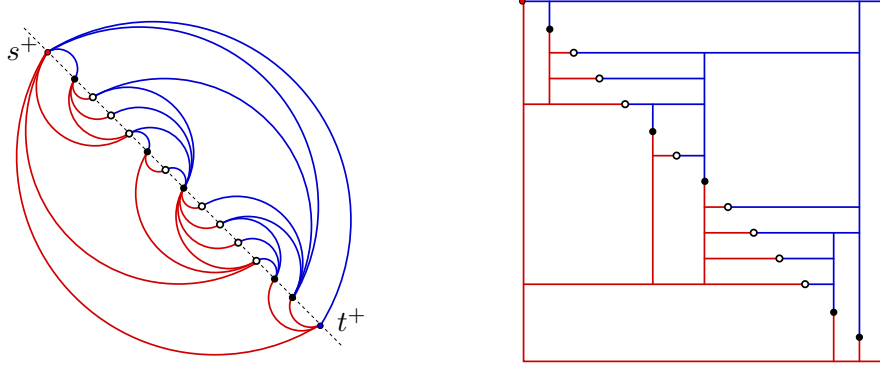


Figure 13: The 2-book embedding from Figure 11 after merging s and t with their respective neighbors and the associated rectangulation.

via an up-ray, then there is a down-ray proving visibility of s from s' . However if we only care of the up-rays we get a natural orientation of the visibility graph, this orientation is a bipolar orientation. Indeed bipolar orientations and visibility representations of planar graphs are intimately related. The study of this connection and its use for the construction of visibility representations has been pioneered by Rosenstiehl-Tarjan [RT86] and Tamassia-Tollis [TT86].

3.2.1 Bipolar orientations

Let G be a graph with distinguished adjacent vertices s and t . A *bipolar orientation* of G is an acyclic orientation of G such that s is the unique source and t is the unique sink, the oriented edge (s, t) is the *root* of the orientation. A graph G admits a bipolar orientation exactly if G is 2-connected. Actually the root edge of a 2-connected graph can be chosen arbitrarily. In the literature these facts are frequently treated in the disguise of st -numberings. Sometimes it is convenient to omit the root edge, in slight abuse of notation we also speak of a bipolar orientation in this case. This is e.g. done in the next paragraph and in Figure 14.

Bipolar orientations of a plane 2-connected (multi)graph G with designated s and t on the outer face are characterized by two facts:

Fact F. Every face f of G has exactly two angles where the orientation of the edges coincide. (The vertices incident to these angles are $v_{\text{source}}(f)$ and $v_{\text{sink}}(f)$).

Fact V. Every vertex $v \neq s, t$ of G has exactly two angles where the orientation of the edges differ. (The faces incident to these angles are $f_{\text{left}}(v)$ and $f_{\text{right}}(v)$).

An orientation of a plane graph G induces an orientation on the dual graph G^* : define the orientation of a dual edge e^* as left-to-right relative to the orientation of e . That is, when looking in the direction of an oriented edge e the dual edge e^* is oriented from the left face to the right face of e .

The *bipolar dual* of a plane bipolar orientation is this dual graph endowed with this dual orientation rooted at (s^*, t^*) where s^* is the face on the right and t^* is the face on the left of (s, t) , i.e, the orientation of (s^*, t^*) is not dual to (s, t) . Essentially Facts **V** and **F** are dual and we have set up the orientation of the root edge (s^*, t^*) such that: The bipolar dual of a plane bipolar orientation is again a bipolar orientation.

A more comprehensive treatment including proofs of the material in this subsection can e.g. be found in [dFOdMR95].

3.2.2 From a bipolar orientation to a rectangular dissection

The input of the following algorithm is a 2-connected plane graph G with an oriented root edge (s, t) on the outer face. The output is a rectangular dissection R such that the visibility graph of the horizontal segments of R is G .

ALGORITHM RECTANGULAR DISSECTION

- Compute a bipolar orientation B of G with root edge (s, t) .
- Compute the bipolar dual B^* with root edge (s^*, t^*) of B .
- For each primal vertex v let $y(v)$ be the length of the longest directed $s \rightarrow v$ path in B . For each dual vertex f let $x(f)$ be the length of the longest directed $s^* \rightarrow f$ path in B^* .
- With a primal vertex $v \neq s, t$ associate the horizontal segment with left-end at the point $(x(f_{\text{left}}(v)), y(v))$ and right-end at $(x(f_{\text{right}}(v)), y(v))$. With a dual vertex $f \neq s^*, t^*$ associate the vertical segment with lower-end $(x(f), y(v_{\text{source}}(f)))$ and upper-end $(x(f), y(v_{\text{sink}}(f)))$.

The special vertices s, t, s^* and t^* need special treatment, as endpoints of these four segments we can choose the four points $(0, 0), (0, y(t)), (x(t^*), y(t))$, and $(x(t^*), 0)$. If the visibility between s and t is required the right endpoint of these two segments can be shifted to the right by one unit.

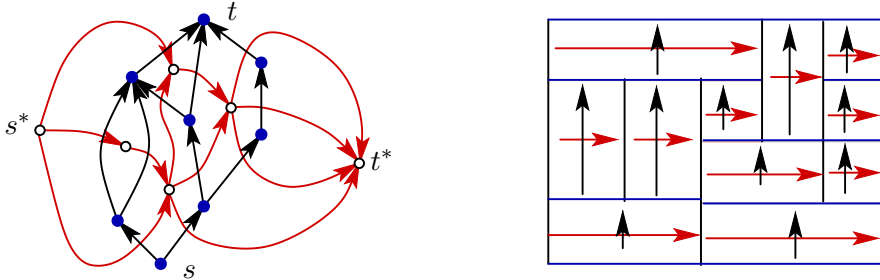


Figure 14: A dual pair B, B^* of bipolar orientations and the rectangular dissection returned by the algorithm when using B and B^* .

From the algorithm we obtain:

Theorem 3.4. *For a 2-connected planar graph G with n vertices and n^* faces there is a visibility representation with horizontal segments whose endpoints are on integer points (x, y) with $0 \leq x \leq n^*$ and $0 \leq y \leq n - 1$. Such a representation can be computed in linear time.*

A sketch of the proof of correctness for the algorithm can be found in [DETT99]. A sweep like proof may, however, be simpler. This could be accomplished by an inductive proof of the statement that the set of segments with y -coordinate $\leq k$ is a visibility representation of the graph induced by vertices v with $y(v) \leq k$.

The bound on the size of the representation accounted for a visibility between s and t .

Quite some research has been put into more compact visibility representations. The basic idea is that a carefully chosen bipolar orientation may have a longest $s \rightarrow t$ path of length much less than n . Almost optimal bounds are known:

- Zhang and He [ZH05] show that there are planar graphs on n vertices requiring size at least $(\lfloor \frac{2n}{3} \rfloor) \times (\lfloor \frac{4n}{3} \rfloor - 3)$ for a visibility representation.
- He and Zhang [HZ08] show that every planar graph with n vertices has a visibility representation with height at most $\frac{2n}{3} + 2\lceil \sqrt{n/2} \rceil$. They use properties of Schnyder woods to construct an appropriate bipolar orientation.
- Fan et al. [FLLY07] show that every planar graph with n vertices has a visibility representation with width at most $\lfloor \frac{4n}{3} \rfloor - 2$.
- Sadasivam and Zhang [SZ10] show that it is NP-hard to find the bipolar orientation of G which minimizes the length of the longest $s \rightarrow t$ path.

3.3 Bipolar orientations and separating decompositions

Let again Q be a plane quadrangulation with color classes consisting of black and white vertices. The *black graph* G_b of Q is the graph on the set V_b of black vertices of G where $u, v \in V_b$ are connected by an edge for every face f of Q incident to u and v , i.e., there is a bijection between faces of Q and edges of G_b . The graph G_b inherits a plane embedding from the plane embedding of Q . Note that in general G_b may have multiple edges.

Let G be a plane graph; the *angle graph* of G is the graph Q with vertex set consisting of vertices and faces of G , and edges corresponding to incidences between a vertex and a face, Q inherits a plane embedding from G . If there are no multiple incidences, i.e., if G is 2-connected, then Q is a quadrangulation. The angle graph construction $G \rightarrow Q_G$ is the inverse of the black graph construction $Q \rightarrow G_b$. More precisely, $Q \leftrightarrow G_b$ is a bijection between plane quadrangulations with a black-white coloring and plane 2-connected multigraphs. An example is given in Figure 15.

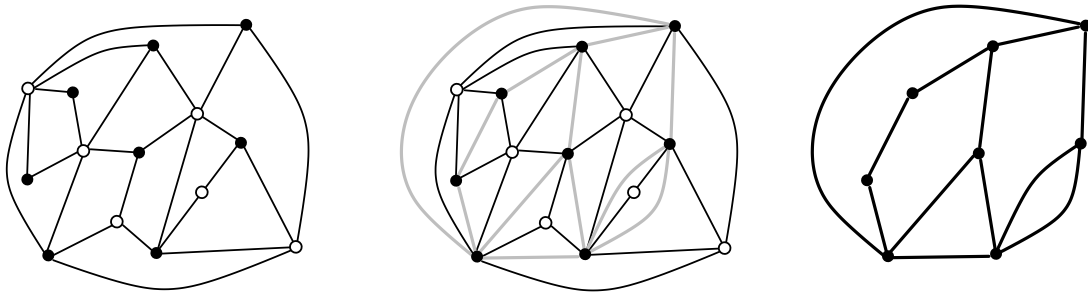


Figure 15: A quadrangulation Q and its black graph G_b .

Changing the role of the color classes and we can associate the *white graph* G_w with Q . By symmetry $Q \leftrightarrow G_w$ is again a bijection between plane quadrangulations Q with a black-white coloring and plane 2-connected multigraphs. The induced bijection $G_b \leftrightarrow G_w$ is nothing but the traditional plane duality.

The study of the oriented counterpart of these classical bijections goes at least back to [RT86] and was continued in [Ros89], [dFOdMR95], [dFOdM01], and [FFNO11]. Let B be a plane bipolar orientation on black vertices with the root edge (s, t) on the outer face. Let Q be the angle graph of the underlying undirected graph of B . Facts **V** and **F** of bipolar orientations yield two special angles for every $v \neq s, t$ of B and for every face of B . At vertices the special angles are the *left* and the *right* angle. At faces the special angles are the *source* and the *sink* angle. Using the correspondence between angles of B and edges of Q we define a separating decomposition on Q : The edge incident to v in Q that corresponds to the left special angle is outgoing blue, the edge that corresponds to the right special angle is outgoing red. The edge incident to f in Q that corresponds to the source is red outgoing and the edge corresponding to the sink is blue outgoing. The rules are illustrated in Figure 16. It is easily verified that they yield a separating decomposition of Q as defined in 3.1.1.

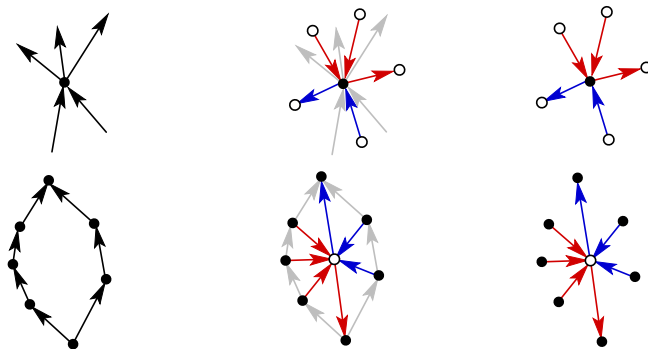


Figure 16: From a bipolar orientation to a separating decomposition and back.

Starting from a separating decomposition S on Q we obtain the unique bipolar orientation B on G_b inducing S by using the rules backwards: At a vertex v the two outgoing edges of S split the edges of G_b into two blocks: the block where Q may have blue edges is the block of incoming edges in the bipolar orientation, the edges of the other block are the outgoing edges in the bipolar orientation. The oriented bijection fits together well with oriented duality, there is only a slight asymmetry concerning the outer vertices and edges.

The correspondence between bipolar orientations and separating decomposition allows to use the construction from Subsection 3.1.1 for visibility representations and the algorithm from Subsection 3.2.2 for segment contact representations.

4 Square dissections

In this section we are mainly interested in *square dissections*, these are rectangular dissections where all the small rectangles are squares. Again we are going to look at different graphs associated with such a dissection.

4.1 Squarings and electricity

The theory of square layouts goes back to a seminal paper *The dissection of rectangles into squares* by Brooks, Smith, Stone and Tutte from 1940 [BSST40]. The story of the

collaboration of the four students has been told at several places. In [Tut87] Tutte tells his memories. The aim of the four students was to find squarings of a square such that the sizes of all small squares are different. Such a squaring is called *perfect*, see Figure 17.

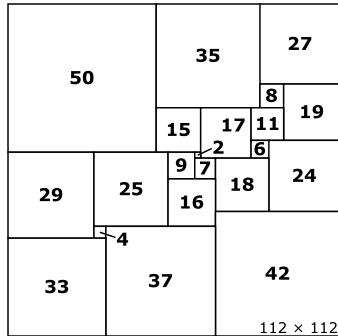


Figure 17: The essentially unique perfect squaring with the smallest number of squares. It was discovered in 1978 by A. J. W. Duijvestijn.

They based the search for perfect squarings on the following idea: start with a rectangular dissection and try to produce a combinatorially equivalent squaring.

The squaring process could be simplified with some observations: The initial rectangular dissection R can be described by the visibility graph G of the horizontal segments. The rectangles of the dissection are in bijection with the edges of G (this may require multiple edges in G). On G there is the natural up-wards pointing bipolar orientation B (the root edge (s, t) can be omitted).

With an arc a of B associate the width $w(a)$ of the rectangle corresponding to a in R . The function w on the arcs of B respects the flow conservation law in every vertex except s and t , it is an st -flow.

Symmetrically, from the dual graph G^* with the dual bipolar orientation B^* and the height $h(a^*)$ of the rectangle associated to a^* an s^*t^* -flow is obtained.

If the dissection R were a squaring, then $w(a) = h(a^*)$ for every dual pair $a \leftrightarrow a^*$ of arcs.

It can be verified that if we find a function f on G such that with $f(e^*) = f(e)$ we have a st -flow on G and a s^*t^* -flow on the dual G^* , then orienting the edges such that $f(a) \geq 0$ yields a bipolar orientation on G . Based on this bipolar orientation a squaring of G such that f describes the size of the squares can be constructed with a weighted variant of the algorithm from 3.2.2.

Physicists may recognize a flow with primal-dual flow conservation as an electrical flow. Primal and dual flow conservation correspond to the two Kirchhoff laws. To find an appropriate f we can thus build G as an electrical network such that each edge has unit resistance and then attach the poles of a battery to s and t and measure the electrical current in each edge. The solution can, of course, also be obtained analytically. In fact Kirchhoff's theorem [Kir47] relates the solution to the enumeration of certain spanning trees.

Theorem 4.1. *Let G be a plane graph with special vertices s and t on the outer face. For an edge $e = \{u, v\}$ of G let $w_e = |n_{uv} - n_{vu}|$, where n_{xy} is the number of spanning trees T of G such that T contains a path s, \dots, x, y, \dots, t . If $w_e > 0$ for all e , then there is a square layout R with G as visibility graph such that the square associated to e has side length w_e .*

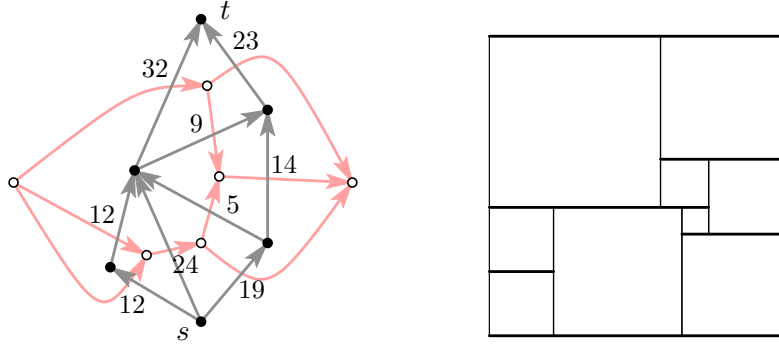


Figure 18: A flow with primal and dual flow conservation and the corresponding squaring.

Sketch of a proof. We use the duality $T \leftrightarrow T^*$ between spanning trees of G_b and its dual $G_b^* = G_w$. If a tree T contributes to n_{uv} , then T^* contains the edge $\{s^*, t^*\}$. Let f_l and f_r be the faces left and right of (u, v) in G_b and define $S = T - \{u, v\} + \{s, t\}$ and $S^* = T^* - \{s^*, t^*\} + \{f_l, f_r\}$. The pair S, S^* is again a dual pair of spanning trees. The mapping $(T, T^*) \leftrightarrow (S, S^*)$ yields a bijection between pairs contributing to n_{uv} and pairs contributing to the dual count $n_{f_l f_r}^*$. From the existence of this bijection it can be concluded that $w_{\{u, v\}} = w_{\{f_l, f_r\}}$. \square

Our proof is taking advantage of the planarity of G . However, for general graphs the same definition of w_e yields the electrical current in e (up to normalization), see e.g. [Bol02].

If $w_e = 0$ for some edge e we still get a square layout R . In R the horizontal segments corresponding to the two end vertices of e are merged into a single segment. If all edges with zero flow are isolated this may be tolerated as consistent with the notion of visibility representation, however, as shown in Figure 19 more complex parts of a graph can disappear because all their edges have zero flow. A sufficient condition that ensures that edges with zero flow are at least isolated is that the graph is 3-connected (see Proposition 4.5).

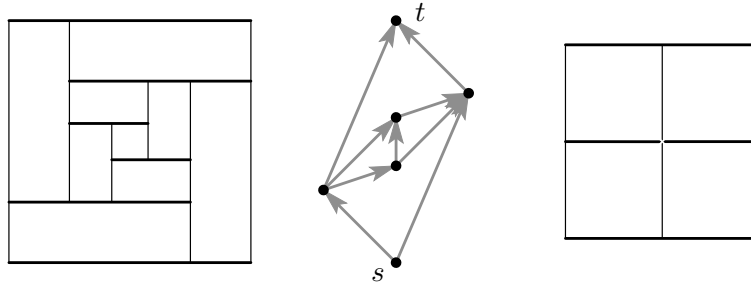


Figure 19: A rectangular dissection R , the bipolar orientation of the visibility graph G obtained from R and the squaring of G with 5 invisible zero squares in the center.

4.2 Squarings and separating decompositions

We now come to a different approach to square layouts. Associate a variable with every inner face of a quadrangulation Q . In these variables we set up a system of linear equations

such that a non-negative solution of the system yields a squaring. To define the system of equations enhance the quadrangulation Q with a separating decomposition S . We aim for a squaring that induces S with its segment contacts.

For a vertex v of Q let $\mathcal{F}_r(v)$ be the set of faces incident to v in the angle between the two outgoing edges of v where incoming edges are red and let $\mathcal{F}_b(v)$ be the other incident faces of v , i.e., the faces in the angle with blue incoming edges.

Suppose that such a square representation inducing S exists and let x_a be the sidelength of the square representing a face a of Q . Every inner vertex v implies an equation that has to be fulfilled by the sidelengths:

$$\sum_{a \in \mathcal{F}_r(v)} x_a = \sum_{a \in \mathcal{F}_b(v)} x_a. \quad (1)$$

A quadrangulation with n vertices has $n - 2$ faces, hence, we have a system of $n - 4$ linear equations (inner vertices) in $n - 3$ variables (inner faces). To forbid the trivial solution of the homogenous system we let $\mathcal{F}_r(t)$ be the set of bounded faces incident to t and add the equation $\sum_{a \in \mathcal{F}_r(t)} x_a = 1$. Rewriting the equations (1) as $\sum_{a \in \mathcal{F}_r(v)} x_a - \sum_{a \in \mathcal{F}_b(v)} x_a = 0$ and collecting all of them in a matrix A_S we find that the vector of sidelengths is a solution to the system

$$A_S \cdot x = \mathbf{e}_1. \quad (2)$$

Given any separating decomposition S of Q we can consider the corresponding system (2). In the following we show:

- The matrix A_S is non-degenerate, hence, there is a unique solution x to the system.
- If the solution vector x is non-negative, then there is a squaring with sidelengths given by the components of x . If the solution vector x is positive, then the separating decomposition induced by the segment contacts of the squaring is S .

Theorem 4.2. *The matrix A_S is non-degenerate, i.e., $\det(A_S) \neq 0$.*

Proof. The idea for the proof is to show that $|\det(A_S)|$ is the number of perfect matchings of an auxiliary graph H_S . Let \hat{A}_S be the matrix obtained from A_S by replacing each -1 by 1 and note that \hat{A}_S only depends on Q and not on S . It has a 1 for every incidence between an inner face (variable) and an inner vertex or t (equation). Regard $\hat{A} = \hat{A}_S$ as the adjacency matrix of a bipartite graph H . From what we said before the graph H is obtained from the angle graph $A(Q)$ of Q by removing the vertices s, s^*, t^* and the vertex corresponding to the outer face of Q together with the incident edges. This implies:

- H is planar.
- All inner faces of H are quadrangles.

Consider the Leibniz-expansions of $\det(A_S)$. The non-vanishing summands $\prod_i a_i \sigma(i)$ are in bijection with the perfect matchings M_σ of H . The contribution of a perfect matching M to $\det(A_S)$ is either $+1$ or -1 , it will be denoted $\text{sign}_S(M) = \text{sign}(\pi_M) \prod_{ij \in M} [A_S]_{ij}$.

The proof of the theorem relies on the following two claims:

Claim A. The graph H has a perfect matching.

Claim B. If M and M' are perfect matchings of H , then $\text{sign}_S(M) = \text{sign}_S(M')$.

We first prove Claim A by verifying the Hall condition for H . Let (X, R) be the vertex bipartition of H where X corresponds to the set of inner faces of Q and $R = V(Q) \setminus \{s, s^*, t^*\}$.

The Hall condition for the full angle graph $A(Q)$ is easily verified: Consider a set F of inner faces of Q and let Z be the set of connected components of $\mathbb{R}^2 \setminus \bigcup_{f \in F} \bar{f}$, where \bar{f} is the closure of face f . The set $F \cup Z$ is the set of faces of a planar bipartite graph whose vertices and edges are those incident to elements of F in Q . This graph has at least $\frac{1}{2}(4|F| + 4|Z|)$ edges and hence, by Euler's formula at least $|F| + |Z| + 2 \geq |F| + 3$ vertices. This implies the Hall condition for H , as H equals $A(Q)$ after the removal three vertices.

For the proof of Claim B we need a slight extension of Claim A, namely that every edge of H takes part in some perfect matching. This can be verified by showing that $V(Q) \setminus \{s, s^*, t^*\}$ is the unique non-empty subset of vertices of Q such that the Hall condition for the set is tight in H . Now we use some facts about α -orientations of planar graphs from [Fel04b]:

- Perfect matchings of H are in bijection with orientations of H with $\text{out-deg}(x) = 1$ for all $x \in X$ and $\text{out-deg}(r) = \text{deg}(r) - 1$ for all $r \in R$. We refer to these orientations as α_M -orientations.
- It is possible to move from any α_M -orientation to any other α_M -orientation by *flips* of the following type: select an inner face whose boundary is an oriented cycle and revert the orientation of all edges of this cycle. This is true (see [Fel04b]) because α_M -orientations have no rigid edges, this follows because every edge takes part in some perfect matching.

Consider two perfect matchings M and M' of H such that the corresponding α_M -orientation differ by a single flip. Since all the faces of H are quadrangles the permutations π_M and $\pi_{M'}$ differ by a single transposition, whence $\text{sign}(\pi_M) = -\text{sign}(\pi_{M'})$. To obtain $\text{sign}_S(M) = \text{sign}_S(M')$ we need $\prod_{ij \in M} [A_S]_{ij} = -\prod_{ij \in M'} [A_S]_{ij}$. Since all entries of A_S are $+1$ or -1 it is enough to show that if we multiply the four entries of A_S associated to the four edges of a face of H the result will always be -1 . A face of H is a cycle of the form (x_1, r_1, x_2, r_2) with $x_i \in X$ and $r_j \in R$. In Q we have the edge $r_1 r_2$ which is oriented in the separating decomposition S , say as $r_1 \rightarrow r_2$. From the definition of the equations based on S we find that $[A_S]_{r_1 x_1} = -[A_S]_{r_1 x_2}$ and $[A_S]_{r_2 x_1} = [A_S]_{r_2 x_2}$. From this we obtain $\prod_{ij=1,2} [A_S]_{r_i x_j} = -1$. Since we can move between any two matchings with flips and since flips leave the sign unaffected we have proved Claim B. \square

The theorem tells us that the linear system (2) has a unique solution x_S . The solution, however, need not be non-negative. What we do next is to show that based on a solution x_S containing negative entries we can modify S to obtain a new separating decomposition S' such that the solution $x_{S'}$ of the system corresponding to S' is non-negative.

Consider a rectangular dissection R representing S and color gray all rectangles whose value in the solution vector x_S is negative. Let Γ be the boundary of the gray area in R . Here is a simple but useful lemma:

Lemma 4.3. *The boundary Γ contains no complete segment.*

Proof. Suppose Γ contains the complete segment corresponding to a vertex v of Q . Then we have $x_a < 0$ for all $a \in \mathcal{F}_r(v)$ and $x_a \geq 0$ for all $a \in \mathcal{F}_b(v)$ or the other way round. In either case we get a contradiction because the entries of x_S satisfy equation (1). \square

Let s_0 be any segment which contributes to Γ . From the lemma we know that at some interior point of segment s_0 the boundary snaps off and continues on another segment s_1 . Again, the boundary has to leave s_1 at some interior point to continue on s_2 . Because this procedure always follows the boundary of the gray region it has to turn back to segment s_0 . Figure 20 shows an example.

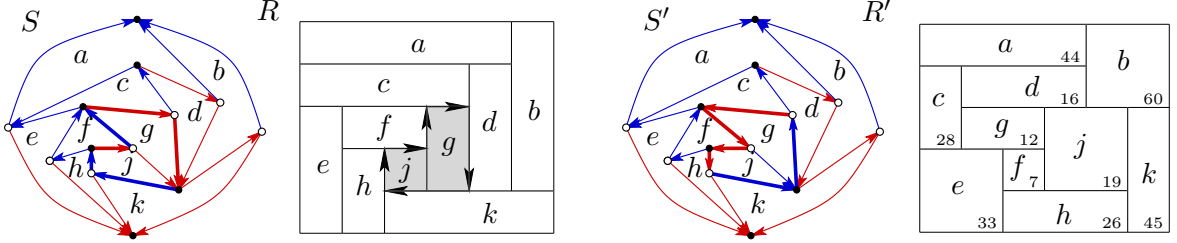


Figure 20: The separating decomposition S corresponds to the rectangular layout R . In the solution x_S of $A_S \cdot x = \mathbf{e}_1$ the gray rectangles are those with a negative value. The boundary Γ consists of a single cycle whose reversal yields S' . The dissection R' corresponding to S' can be squared, small numbers are multiples of the entries of the solution $x_{S'}$.

Recall that a 2-orientation of a quadrangulation with white and black vertices is an orientation of the edges such that the two black vertices s and t on the outer face have out-degree zero and all the other vertices have out-degree two. In [dFODM01] it was shown that separating decompositions and 2-orientations of Q are in bijection. Reverting a directed cycle in a 2-orientation yields another 2-orientation.

Now Γ corresponds to some directed cycles in S reverting the cycles yields another 2-orientation and via the bijection another separating decomposition S' .

Lemma 4.4. *Let S' be obtained from the separating decomposition S by reverting the cycles of the boundary Γ between faces with positive and negative x_S values. There are no negative entries in the solution vector $x_{S'}$ of the system $A_{S'} \cdot x = \mathbf{e}_1$.*

Proof. Let O be a 2-orientation of a quadrangulation Q and let C be a directed cycle in O . Let O' denote the 2-orientation obtained from O by reverting C . The relation between the separating decompositions S and S' corresponding to O and O' was investigated in [FHKO10]. There it is shown that all edges outside of the cycle keep their color while all edges in the interior of the cycle change their color.

Now consider the matrices A_S and $A_{S'}$. The rows correspond to the outer vertex t and the inner vertices of Q , the columns correspond to bounded faces of Q . An entry $a_{v,f}$ is non-zero, more precisely $a_{v,f} = \pm 1$, if v is a boundary vertex of f . Only the sign of an entry depends on the separating decomposition, it is positive if f belongs to $\mathcal{F}_r(v)$ (the set of faces incident to v in the angle between the two outgoing edges of v where incoming edges are red) and it is negative if f belongs to $\mathcal{F}_b(v)$.

From the above it follows that if f is inside of C and v is incident to f , then f belongs to $\mathcal{F}_b(v)$ with respect to S if and only if f belongs to $\mathcal{F}_r(v)$ with respect to S' . In other words

if $A_S = (a_{v,f})$ and $A_{S'} = (a'_{v,f})$ then

$$a'_{v,f} = \begin{cases} -a_{v,f} & f \text{ inside of } C \\ a_{v,f} & \text{otherwise.} \end{cases}$$

The solution $x_{S'}$ of $A_{S'} \cdot x = \mathbf{e}_1$ can, therefore, be obtained from the solution x_S of $A_S \cdot x = \mathbf{e}_1$ by changing the sign of all entries of x_S that correspond to faces f inside of C .

Since S' was obtained by reversing the boundary enclosing all negative faces of x_S it follows that $x_{S'}$ is non-negative. \square

Given a non-negative solution it remains to actually construct the squaring with the given sizes. We omit the details but note that the squaring can be constructed using a weighted variant of the algorithm in 3.2.2.

The electricity approach to squarings implies that to a rectangular dissection there corresponds a unique squaring. This must be the same as the squaring that we obtain by following the approach of this subsection.

There is again the issue of squares of size zero. If we color faces of S corresponding to zero squares gray and consider the boundary Γ of the gray region, then Lemma 4.3 holds again. Hence, again a cycle of Γ is a directed cycle in S . Looking at the square representation we see that a region of zero squares is incident to at most four non-zero squares. A cycle in Γ must, therefore be of length at most four. As a consequence we get:

Proposition 4.5. *If Q contains no separating 4-cycle, then the square dissection contains a segment contact representation of Q (we have to allow that two horizontal resp. vertical segments share an endpoint).*

4.3 Trapezoidal dissections and Markov chains

The connection between square dissections and electrical networks was used by Dehn [Deh03] in 1903 to show that if an $A \times B$ rectangle admits a squaring using finitely many squares then A/B is rational. Tutte used network methods in his investigations of dissections using equilateral triangles [Tut48]. In [SST00] dissections into triangles are constructed using a generalized ‘unsymmetrical’ electricity. Since there is a well-known connection between electrical networks and random walks (see e.g. [DS84]) it is consequent to base the construction of dissections on random walks. This approach has been taken by Kenyon [Ken98].

For the description of Kenyon’s ideas it is convenient to start with a tiling of a rectangle into trapezoids with horizontal upper and lower sides, this is a *trapezoidal dissection*. With a trapezoidal dissection \mathcal{T} associate the *t-visibility graph* $G_{\mathcal{T}}$: the vertices of $G_{\mathcal{T}}$ are the horizontal segments of the dissection, the edges of $G_{\mathcal{T}}$ correspond to the trapezoids of \mathcal{T} . The lower and upper segment of the enclosing rectangle are denoted s and t . An example is shown in Figure 21.

For a trapezoid T whose horizontal sides are on segments i and j we let $\text{height}(T)$ be the distance between segments i and j and $\text{width}_i(T)$ be the length of the side of T contained in segment i , note that $\text{width}_i(T) = 0$ is possible. Define an unsymmetrical weighting on edges: $m(i, j) = \frac{\text{width}_i(T)}{\text{height}(T)}$. These weights are used to define a random walk (Markov chain) on $G_{\mathcal{T}}$ by taking the probability $p(i, j)$ of a transition from i to j proportional to $m(i, j)$, i.e., $p(i, j) = \frac{1}{\sum_j m(i, j)} m(i, j)$.

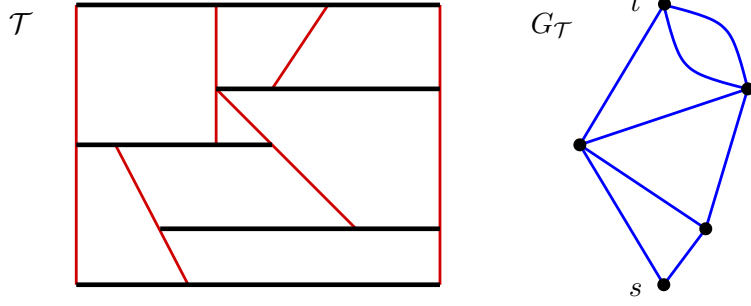


Figure 21: A trapezoidal dissection \mathcal{T} and its t-visibility graph $G_{\mathcal{T}}$.

Consider a stationary distribution π of p , that is, a distribution such that for all i :

$$\pi(i) = \sum_j \pi(j)p(j, i).$$

In most cases the Markov chain p induced by \mathcal{T} will be aperiodic, in that case p is ergodic and π is unique. For an edge (i, j) define $\pi(i, j) = \pi(i)p(i, j)$ and note that $\pi(i, j)$ is the probability of presence of the edge in the random walk. More formally, it is the stationary distribution for the induced random walk on the line graph of $G_{\mathcal{T}}$.

Define a function ω on the faces of $G_{\mathcal{T}}$ by:

(W₀) $\omega(f_{\infty}) = 0$, where f_{∞} is the outer face.

(W₁) $\omega(f) - \omega(f') = \pi(i, j) - \pi(j, i)$ when (i, j) is an edge with f on the left and f' on the right side.

Lemma 4.6. *The function w is well defined.*

Proof. First note that if edges (i, j) and (i', j') have f on the left and f' on the right, then the removal of the two edges cuts $G_{\mathcal{T}}$ into two components H and H' each containing at least one vertex. The random walk has to commute in both directions between these components equally often, i.e., $\pi(i, j) + \pi(j', i') = \pi(j, i) + \pi(i', j')$. Therefore $\omega(f) - \omega(f')$ is independent of the edge chosen for the definition.

The value of $\omega(f)$ can be determined by taking a dual path from f_{∞} to f . To show that the result is independent of the path it is enough to show that summing up the differences $\pi(i, j) - \pi(j, i)$ on a dual path around vertex i results in zero. This follows from

$$\sum_j \pi(i, j) = \sum_j \pi(j, i).$$

To prove this note that $\pi(i) = \pi(i) \sum_j p(i, j) = \sum_j \pi(i)p(i, j) = \sum_j \pi(i, j)$ and because π is the stationary distribution also $\pi(i) = \sum_j \pi(j)p(j, i) = \sum_j \pi(j, i)$. \square

The function $\omega(x)$ is the expected relative counterclockwise winding number of the random walk around face x . Clearly the winding number has to comply with the two properties (W₀) and (W₁), but as shown in the lemma, this determines the function.

For a face f of $G_{\mathcal{T}}$ let s_f be the corresponding line segment in \mathcal{T} and define $w(f) = 1/\text{slope}(s_f)$ and $w(f) = 0$ if s_f is vertical. Recall that $m(i, j) = \text{width}_i(T)/\text{height}(T)$.

Proposition 4.7. *Up to a scalar factor the functions $w(f)$ and $m(i, j)$ defined on \mathcal{T} equal the winding number $\omega(f)$ and the edge stationary distribution $\pi(i, j)$ of the Markov chain.*

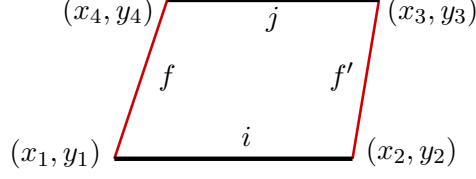


Figure 22: An example trapezoid.

Proof. Consider a trapezoid from \mathcal{T} as shown in Figure 22. Taking the coordinates from there we have

$$m(i, j) = \frac{x_2 - x_1}{y_4 - y_1}, \quad m(j, i) = \frac{x_3 - x_4}{y_4 - y_1}, \quad w(f) = \frac{x_4 - x_1}{y_4 - y_1}, \quad w(f') = \frac{x_3 - x_2}{y_3 - y_2} = \frac{x_3 - x_2}{y_4 - y_1}.$$

It follows that $w(f) - w(f') = m(i, j) - m(j, i)$. Summing up this equation along the dual cycle around a vertex we obtain $\sum_j m(i, j) = \sum_j m(j, i)$. This implies that $m(i, j)$ is a scalar multiple of an edge stationary distribution. In turn from (W_0) and (W_1) it follows that the values $w(f)$ are multiples of the winding numbers. \square

Proposition 4.8. *For a vertex i of $G_{\mathcal{T}}$ let $y(i)$ be the y -coordinate of the corresponding segment in \mathcal{T} and let $p(i, j)$ be as above. For all $i \notin \{s, t\}$ the function y is harmonic with respect to p , i.e.,*

$$y(i) = \sum_j y(j)p(i, j).$$

Proof. Let $I^+ = \{j : y(j) > y(i)\}$ and $I^- = \{j : y(j) < y(i)\}$ and note that if T_{ij} is a trapezoid connecting segments i and j then $\text{height}(T_{ij}) = y(i) - y(j)$ if $j \in I^-$ and $\text{height}(T_{ij}) = -(y(i) - y(j))$ if $j \in I^+$. We now have:

$$\begin{aligned} y(i) - \sum_j y(j)p(i, j) &= \sum_j (y(i) - y(j))p(i, j) \\ &= \sum_{j \in I^-} (y(i) - y(j))p(i, j) + \sum_{j \in I^+} (y(i) - y(j))p(i, j) \\ &= \frac{1}{\sum_j m(i, j)} \left(\sum_{j \in I^-} \text{width}_i(T_{ij}) - \sum_{j \in I^+} \text{width}_i(T_{ij}) \right) = 0. \end{aligned}$$

The last equation follows because the trapezoids with segment i on the low side and those with the segment on the high side can both be used to partition the segment. \square

From the tiling \mathcal{T} we have obtained a Markov chain p on $G_{\mathcal{T}}$ such that $m(i, j) \sim \pi(i, j)$ and $w(f) \sim \omega(f)$ and the y -coordinates of the vertex segments are a function which is p -harmonic for all vertices $i \notin \{s, t\}$ of $G_{\mathcal{T}}$.

To revert the process let $G = (V, E)$ be a planar graph with s and t on the outer face and let p be a Markov chain on G . From these data we obtain the edge stationary distribution $\pi : E \rightarrow \mathbb{R}$, the winding numbers $\omega(f)$ on the faces of G and the unique p -harmonic function $y : V \rightarrow \mathbb{R}$ with $y(s) = 0$ and $y(t) = 1$.

Note that the probability $h(i)$ that a random walk started at i reaches t before reaching s has the properties required for $y(i)$. Uniqueness, i.e., the fact that $h(i) = y(i)$, follows from the maximum principle for discrete harmonic functions, see e.g. [BL03].

From the data we build a trapezoidal dissection \mathcal{T} such that the trapezoid T_{ij} corresponding to an edge (i, j) of G with $y(i) < y(j)$ is a horizontal translate of the trapezoid with corners

$$\begin{aligned} (\omega(x)(y(j) - y(i)), y(j)), & \quad ((\pi(i, j) + \omega(x'))(y(j) - y(i)), y(j)), \\ (0, y(i)), & \quad (\pi(i, j)(y(j) - y(i)), y(i)). \end{aligned}$$

The proof that these trapezoids fit nicely together to a tiling of a rectangle is done inductively. For the organization of the inductive argument the following lemma is useful:

Lemma 4.9. *The orientation B of G with $(i, j) \in B$ iff $y(i) < y(j)$ is a bipolar orientation with source s and sink t .*

Proof. This follows from the maximum principle for discrete harmonic functions, i.e, from the fact that such a function assumes its maximum at boundary vertices. \square

As with squarings it may happen that trapezoids degenerate and have 0 area. We say that a Markov chain p on G is generic if this does not happen. In particular this implies that $p(i, j) + p(j, i) > 0$ for all edges (i, j) of G and $y(i) \neq y(j)$ for every pair i, j of adjacent vertices.

Theorem 4.10 (Kenyon '98). *Let G be planar with s and t on the outer face. If p is a generic Markov chain on G and \mathcal{T} is the trapezoidal dissection associated to (G, p) by the above construction, then $G = G_{\mathcal{T}}$ and p can be recovered from \mathcal{T} .*

Some special cases are particularly interesting:

- If p is reversible, i.e., if $\pi(i)p(i, j) = \pi(j)p(j, i)$ for all edges, then all trapezoids in the dissection are rectangles. (This is the case of planar electrical networks with edges of varying resistance).
- If $p(i, j) = \frac{1}{\deg(i)}$, then $\pi(i) = \frac{\deg(i)}{2m}$ and the aspect ratios $m(i, j) \sim \pi(i)p(i, j)$ of all rectangles are equal. Hence, after scaling the dissection we obtain a squaring.

Kenyon [Ken98] also considers dissections of more general trapezoidal shapes than rectangles. Markov chains that yield dissections into non-trapezoidal shapes are considered in [KS04].

5 Square duals

In this section we review a result of Schramm [Sch93]. He proves the existence of rectangular duals where all rectangles are squares for a large class of triangulations. The approach is based on *extremal length*. To begin with we need some definitions. Throughout $G = (V, E)$ is a prescribed graph.

- Any function $m : V \rightarrow \mathbb{R}^+$ is called a *discrete metric* on G .
- The length of a path γ in G is: $\ell_m(\gamma) = \sum_{x \in \gamma} m(x)$.

- For $A, B \subset V$ let $\Gamma(A, B)$ be the set of A, B paths.
The distance between A and B is defined as: $\ell_m(A, B) = \min_{\gamma \in \Gamma(A, B)} \ell_m(\gamma)$.
- The *extremal length* of A, B is $L(A, B) = \sup_m \frac{\ell_m(A, B)^2}{\|m\|^2} = \sup_m \frac{\ell_m(A, B)^2}{\sum_v m(v)^2}$.
- A metric realizing the extremal length is an *extremal metric*.

Proposition 5.1. *For G and $A, B \subset V$ there is an (up to scaling) unique extremal metric.*

Proof. If m is extremal, then so is every positive scalar multiple of m . Therefore, we only have to look at metrics m with $\ell_m(A, B) = 1$.

These metrics form a polyhedral set P described by the finitely many linear inequalities of the form $\sum_{x \in \gamma} m(x) \geq 1$ with $\gamma \in \Gamma(A, B)$.

The extremal metric is the unique $m \in P$ with minimal norm $\|m\| = \sqrt{\sum_v m(v)^2}$. \square

Proposition 5.2. *A squaring with A and B representing top and bottom of the dissection induces an extremal metric on the rectangular dual graph G of the squaring.*

Proof. Let $h = \text{height}(R)$ and $w = \text{width}(R)$ we may assume $h \cdot w = 1$. Let $s : V \rightarrow \mathbb{R}$ be the metric where $s(v)$ is the side length of the square of vertex v . Since $\|s\|^2 = \sum s(v)^2 = h \cdot w = 1$ we have $\|s\| = 1$.

Let m be any metric. For $t \in [0, w]$ the squaring induces a path γ_t consisting of the vertices v whose representing square is intersected by the vertical line $x = t$. By definition: $\ell_m(A, B) \leq \sum_{v \in \gamma_t} m(v)$.

$$\begin{aligned} w \cdot \ell_m(A, B) &\leq \int_0^w \sum_{v \in \gamma_t} m(v) dt = \int_0^w \sum_{v \in V} m(v) \delta_{[v \in \gamma_t]} dt = \sum_{v \in V} m(v) \int_0^w \delta_{[v \in \gamma_t]} dt \\ &= \sum_{v \in V} m(v) s(v) = \langle m, s \rangle \leq \|m\| \cdot \|s\| = \|m\| \end{aligned}$$

Hence:

$$\frac{\ell_m(A, B)^2}{\|m\|^2} \leq \frac{1}{w^2} = h^2 = \frac{h^2}{\|s\|^2}.$$

This shows that the metric s is extremal. \square

The proposition together with the uniqueness of extremal metrics implies that for a given G there can, up to scaling, only be a single square dual.

We next show the converse of the proposition: Extremal lengths can be used to get squarings. Let G be an inner triangulation of a 4-gon, i.e., a triangulation with one outer edge removed. Call the four vertices of the outer face s, a, t, b in counterclockwise order.

Now let m be an extremal metric with respect to a and b such that $\|m\| = 1$ and define $h = l_m(a, b)$. For an inner vertex v of G let $x(v)$ be the length of a shortest $s \rightarrow v$ path and $y(v)$ be the length of a shortest $a \rightarrow v$ path, the length of a path is of course taken with respect to m .

Proposition 5.3 (Schramm '93). *The squares $Z_v = [x(v) - m(v), x(v)] \times [y(v) - m(v), y(v)]$ for inner vertices of G together with four appropriate squares for the outer vertices yield a square contact representation of G in the rectangle $R = [0, h^{-1}] \times [0, h]$. Moreover, if G has no separating 3- and 4-cycles then there are no degenerate squares, i.e., $m(v) > 0$ for all v .*

Proof. The first claim is that all edges are represented as contacts or intersections. Let (u, v) be an interior edge of G and $u, v \neq t$. From $x(v) \leq x(u) + m(v)$ and $x(u) \leq x(v) + m(u)$ and the corresponding inequalities for $y(u)$ and $y(v)$ it follows that $Z_u \cap Z_v \neq \emptyset$. The same property for edges of the form (u, t) is non-trivial. Schramm [Sch93] has a direct argument while Lovász [Lov09] argues with blocking polyhedra and shows that m is an extremal metric with respect to s and t . We skip this part of the proof.

The next claim is that the squares cover R , i.e., $R \subseteq \bigcup_v Z_v$. Suppose that there is a point p in R that is not contained in a Z_v . For each v choose a representative point $q_v \in Z_v$ and draw the edges of G such that an edge (u, v) is represented by a curve connecting q_u and q_v that stays in $Z_u \cup Z_v$. The simplest choice for the edge may be to represent it as union of straight segments $[q_u, q_{uv}]$ and $[q_{uv}, q_v]$ for some $q_{uv} \in Z_u \cap Z_v$. With triangle (u, v, w) in G let $T_{(u,v,w)}$ be the topological triangle enclosed by the edges connecting q_u, q_v and q_w in R . By following a generic ray starting at p and considering situations where the ray crosses a curve representing an edge, it can be verified that p is contained in an odd number of the triangles $T_{(u,v,w)}$. Therefore, there is at least one triangle $T_{(u,v,w)}$ covering p . Consider $S = Z_u \cup Z_v \cup Z_w$. The boundary of $T_{(u,v,w)}$ is a closed curve γ contained in S such that $p \notin S$ is in the interior of γ . Since this is impossible for a union S of three squares we have a contradiction. This proves the claim.

Since $\text{area}(R) = 1 = \|m\|^2 = \sum_v m(v)^2 = \sum_v \text{area}(Z_v)$ it follows that intersections of the squares are confined to their boundaries and we have a tiling of R with squares. We also know that all edges of G are represented by contacts. With a counting argument it can be shown that these have to be all contacts between the squares except for:

- Additional corner to corner contacts at points where four squares meet.
- Contacts where at least one of the two participating squares is a degenerate square of size 0.

This shows that the extremal length yields a square contact representation of G .

Finally we show that in the absence of separating 3- and 4-cycles all squares have non-zero area. Let W be the set of vertices of a connected component of the subgraph of G induced by vertices with degenerate squares, i.e., with $m(v) = 0$. All the vertices of W are represented by the same point p in R . This point p must also be contained in all Z_v where $v \notin W$ is a neighbor of some $w \in W$. These at most four vertices v form a cycle separating W from the outer vertices. \square

5.1 Square duals and transversal structures

We now propose an alternative method to approach the problem of finding a square dual for an inner triangulation of a 4-gon. The approach is similar to what we have done in 4.2. First we encode a rectangular dual as a graph with additional structure, in this case a transversal structure. From the transversal structure we extract a system of linear equations such that a non-negative solution yields the metric information needed for the square dual. If the

solution has negative variables we do not know how to use the solution to get a square dual. In this respect the situation is more complicated than in 4.2. However, the signs in the solution provide a rule for changing the transversal structure. With the new transversal structure we can proceed as before. Unfortunately we can not yet prove that the procedure stops, i.e., that at some iteration the solution is non-negative and we get the square dual we are looking for.

Let G be an inner triangulation of a 4-gon with outer vertices s, a, t, b in counterclockwise order. A *transversal structure* for G is an orientation and 2-coloring of the inner edges of G such that

- (1) All edges incident to s, a, t and b are blue outgoing, red outgoing, blue ingoing, and red ingoing, respectively.
- (2) The edges incident to an inner vertex v come in clockwise order in four non-empty blocks consisting solely of red ingoing, blue ingoing, red outgoing, and blue outgoing edges, respectively.

Transversal structures have been studied in [KH97], [Fus07], and [Fus09]. The relevance of transversal structures in our context comes from the following simple proposition. See Figure 23.

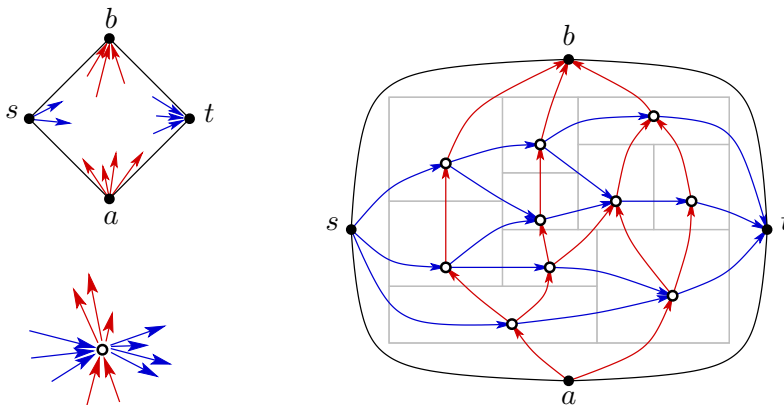


Figure 23: The two local conditions and an example of a transversal structure together with a corresponding dissection.

Proposition 5.4. *Transversal structures of an inner triangulation G of a 4-gon with outer vertices s, a, t, b are in bijection with combinatorially different rectangular dissections R with rectangular dual $G \setminus \{s, a, t, b\}$ where the rectangles of vertices adjacent to s, a, t and b touch the left, lower, right, and upper boundary of R , respectively.*

Based on a transversal structure T we want to write down a system of linear equations such that a non-negative solution of the system yields a square dissection with T as underlying transversal structure. For an inner vertex v let x_v be a variable intended to represent the size of the square representing v . For a directed edge (u, v) in T let x_{uv} be a variable representing the length of the contact between the rectangles representing u and v . The edges incident to a vertex v are partitioned into the four nonempty classes $\mathcal{R}^+(v)$, $\mathcal{B}^+(v)$,

$\mathcal{R}^-(v)$, and $\mathcal{B}^-(v)$, where the letter indicates the color of the edge and the sign whether the edge is outgoing or incoming. With vertex v we associate four equations:

$$\sum_{(u,v) \in \mathcal{R}^+(v)} x_{uv} = x_v, \quad \sum_{(u,v) \in \mathcal{B}^+(v)} x_{uv} = x_v, \quad \sum_{(v,u) \in \mathcal{R}^-(v)} x_{vu} = x_v, \quad \sum_{(v,u) \in \mathcal{B}^-(v)} x_{vu} = x_v.$$

To forbid the trivial solution we require that the width of the enclosing rectangle is 1. This is done by adding the equation $\sum_{(u,b) \in \mathcal{R}^-(b)} x_{ub} = 1$. Collecting the coefficients of the equations in a matrix A_T we find that the vector x of lengths of a square dissection is a solution to the system

$$A_T \cdot x = \mathbf{e}_1. \quad (3)$$

Fact 1. If the solution vector x is non-negative, then there is a square dissection with the lengths given by x . If the solution vector x is positive the transversal structure corresponding to the square dissection is T .

Fact 2. The matrix A_T is non-degenerate, hence, there is a unique solution x to the system.

A proof of Fact 2 can be given along the lines of the proof of Theorem 4.2. The idea is to interpret A_T as the adjacency matrix of a planar bipartite graph H . Terms in the Leibniz-expansions of $\det(A_T)$ correspond to perfect matchings in H . It can be shown that H admits a perfect matching, moreover, the sign of all perfect matchings is equal because all inner faces of H have length 10, i.e., have residue 2 modulo 4.

To deal with the case where the solution vector x has negative entries we need more insight into transversal structures. Recall from 2.3 the definition of the *trimmed angle graph* $\check{A}(G)$ associated with a plane graph G . The vertex set of this graph consists of the primal vertex set V together with the dual vertex set save the dual of the unbounded face, i.e., $V^* \setminus \{f_\infty\}$. Edges of $\check{A}(G)$ correspond to incidences between vertices and bounded faces of G or equivalently to the internal angles of G . Inner faces of $\check{A}(G)$ correspond to the inner edges of G .

Given a transversal structure T on G we define an orientation of $\check{A}(G)$ as follows: Orient the edge $\{v, f\}$ as $v \rightarrow f$ when the two edges incident to f and v have different colors, otherwise the edge is oriented $f \rightarrow v$. It can be verified that

- $\text{out-deg}(v) = 4$ for all inner vertices of G , $\text{out-deg}(v) = 0$ for the four vertices of the outer face, and $\text{out-deg}(f) = 1$.

An orientation of $\check{A}(G)$ obeying the above rules for the out-degrees is called an α_4 -orientation (note that up to changing the role of the color classes α_4 -orientations are identical to the α_{skel} orientations from 2.3).

Fact 3. Let G be an inner triangulation of a 4-gon with outer vertices s, a, t, b . Transversal structures of G are in bijection with α_4 -orientations of $\check{A}(G)$.

Fact 4. Let x be the solution to the system of equations corresponding to the transversal structure T and let $E^-(x)$ be the set of edges whose value in x is negative. The boundary $\partial^-(x)$ of the union of all faces of $\check{A}(G)$ corresponding to edges in $E^-(x)$ decomposes into directed cycles with respect to the α_4 -orientation corresponding to T .

Reverting a directed cycle in an α_4 -orientation yields another α_4 -orientation. Hence reverting all edges of $\partial^-(x)$ yields another α_4 -orientation which corresponds to a new transversal structure T' of G .

The approach for computing a square dual for G is this:

- Compute a transversal structure T of G and the matrix A_T .
- Compute a solution x_T of $A_T \cdot x = \mathbf{e}_1$.

If all entries of x_T are non-negative we are done; based on x_T we can build the square dissection for G . If there are negative entries in x_T we can use the α_4 -orientation to transform T into another transversal structure T' and iterate. We conjecture that independent of the choice of T the sequence $T \rightarrow T' \rightarrow T'' \rightarrow \dots$ has a finite length, i.e., there is a k such that the solution $x_{T^{(k)}}$ of the system corresponding to $T^{(k)}$ is non-negative.

There is strong experimental support for the truth of the conjecture.

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