Geodesic Embeddings and Planar Graphs

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Abstract. Schnyder labelings are known to have close links to order dimension and drawings of planar graphs. It was observed by Ezra Miller that geodesic embeddings of planar graphs are another class of combinatorial or geometric objects closely linked to Schnyder labelings. We aim to contribute to a better understanding of the connections between these objects. In this article we prove

- a characterization of 3-connected planar graphs as those graphs admitting rigid geodesic embeddings,
- a bijection between Schnyder labelings and rigid geodesic embeddings,
- a strong version of the Brightwell-Trotter theorem.

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1 The Players of the Game

1.1 Planar Maps

A *planar map* M is a simple planar graph G together with a fixed planar embedding of G in the plane. A *suspension* M^σ of M is obtained by selecting three different vertices a_1, a_2, a_3 in clockwise order from the outer face of M and adding a half-edge that reaches into the outer face to each of these special vertices. The *closure* M^σ_∞ of a suspension M^σ of M is obtained by adding a new vertex v_∞, this new vertex is used as second endpoint of each of the half-edges of M^σ.

1.2 Schnyder Labelings and Woods

We review definitions and some properties of Schnyder labelings and Schnyder woods for 3-connected planar maps. Originally, these objects have been introduced for planar triangulation [7] and [8]. A comprehensive treatment for the case of 3-connected planar maps can be found in [4]. This includes the proof for the existence of these objects and results related to drawings of planar graphs.

Let M^σ be the suspension of a 3-connected planar map. A *Schnyder labeling* with respect to a_1, a_2, a_3 is a labeling of the angles of M^σ with the labels $1, 2, 3$ (alternatively: red, green, blue) satisfying three rules. Throughout we assume a cyclic structure on the labels so that $i + 1$ and $i - 1$ is always defined.

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(A1) The two angles at the half-edge of the special vertex a_i have labels $i+1$ and $i-1$ in clockwise order.

(A2) *Rule of vertices:* The labels of the angles at each vertex form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's.

(A3) *Rule of faces:* The labels of the angles at each interior face form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's. At the outer face the same is true in counterclockwise order.

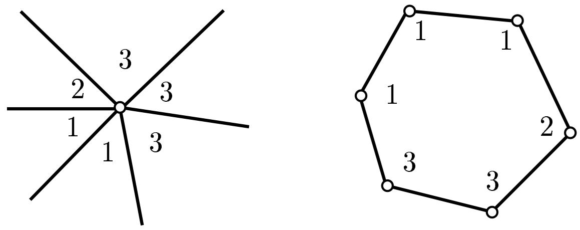

Figure 1: Rule of vertices and rule of faces

Lemma 1. *Let G be a plane graph with a Schnyder labeling, then the four angles of each edge contain all three labels 1,2,3. Thus every edge has one of the two types shown in Figure 2.*

Figure 2: The two types of labeling for an edge.

A proof of the lemma is given in the appendix. There it is also shown that in a Schnyder labeling all interior angles at the special vertex a_i are labeled i.

Let M^σ be the suspension of a 3-connected planar map. A *Schnyder wood** rooted at a_1, a_2, a_3 is an orientation and labeling of the edges of M^σ with the labels $1, 2, 3$ satisfying the following rules.

(W1) Every edge e is oriented by one or two opposite directions. The directions of edges are labeled such that if e is bioriented the two directions have distinct labels.

(W2) The half-edge at a_i is directed outwards and labeled i.

(W3) Every vertex v has outdegree one in each label. The edges e_1, e_2, e_3 leaving v in labels 1,2,3 occur in clockwise order. Each edge entering v in label i enters v in the clockwise sector from e_{i+1} to e_{i-1}. See Figure 3.

(W4) There is no interior face whose boundary is a directed cycle in one label.

*Schnyder wood is more suggestive than triorientation.

A Schnyder labeling of the angles of a plane graph induces a Schnyder wood. If an edge has different angular labels i and j at one of its ends, we direct the edge from this end towards the other and give this direction the third label k. Conversely, a Schnyder wood induces a Schnyder labeling. The proof of the following theorem is placed back into the appendix.

Theorem 1. *Let M^σ be the suspension of a 3-connected planar map. The above correspondence is a bijection between the Schnyder labelings (axioms A1,A2,A3) and Schnyder woods (axioms W1,..,W4) of M^σ.*

Henceforth, we bend terminology and reuse the term Schnyder labeling to denote a Schnyder angle labeling together with a compatible Schnyder wood.

Let G be a planar graph with a Schnyder labeling. With T_i we denote the digraph induced by the edges having a direction labeled i and oriented in this direction. Since every inner vertex has outdegree one in T_i every v is the starting vertex of a unique i-path $P_i(v)$ in T_i. In the remainder of this subsection we review results from [4]. The digraph T_i is acyclic and, even more, T_i is a tree with root a_i.

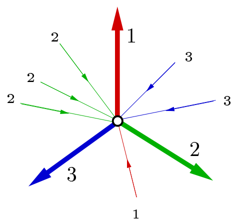

Figure 3: Edge orientations and labels at a vertex.

The orientation $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is also acyclic (disregarding bidirected edges in labels $i-1$ and $i+1$). It follows that for $i \neq j$ the paths $P_i(v)$ and $P_j(v)$ have v as the only common vertex. Therefore, $P_1(v), P_2(v), P_3(v)$ divide G into three regions $R_1(v)$, $R_2(v)$ and $R_3(v)$, where $R_i(v)$ denotes the region bounded by and including the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$, see Fig. 4.

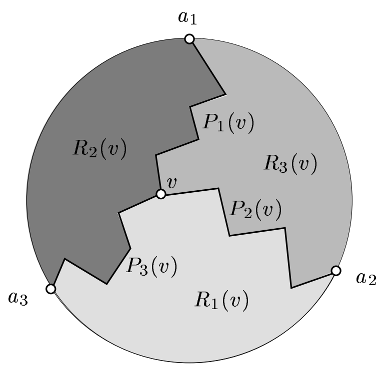

Figure 4: The three regions of a vertex

Lemma 2. *For any two vertices u and v of a Schnyder labeled graph*

- *there are i and j with $R_i(u) \subset R_i(v)$ and $R_j(v) \subset R_j(u)$,*
- *$R_i(u) \subseteq R_i(v)$ iff $u \in R_i(v)$.*

The *region vector* of a vertex v is the vector (v_1, v_2, v_3) defined by

$$v_i = \text{The number of faces of } M \text{ contained in region } R_i(v).$$

Given three non-collinear points α_1, α_2 and α_3 in the plane. These points and the region vectors can be used to define an embedding of M in the plane. A vertex v is mapped to the point

$$\mu : v \to v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3,$$

An edge $\{u, v\}$ is mapped by μ to the line segment connecting $\mu(u)$ and $\mu(u)$.

Theorem 2. *The drawing $\mu(M)$ of a 3-connected plane map is convex, i.e., the boundary of every face is a convex polygon.*

1.3 Geodesic Embeddings

With the notation in this part we widely follow Miller [6]. Consider $\mathbb{Z}^3$ (or $\mathbb{N}^3$) as subsets of $\mathbb{R}^3$ and the *dominance order* on these sets, i.e, with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ we have u dominates v, in symbols $u \geq v$ if $u_i \geq v_i$ for $i = 1, 2, 3$. Use $u \vee v$ and $u \wedge v$ to denote the *join* (component-wise maximum) and *meet* (component-wise minimum) of $u, v \in \mathbb{R}^3$.

Let $\mathcal{V} \subset \mathbb{N}^3 \subset \mathbb{R}^3$ be an antichain, i.e., a set of pairwise incomparable elements. The *filter* generated by $\mathcal{V}$ in $\mathbb{R}^3$ is the set

$$\langle \mathcal{V} \rangle = \{\alpha \in \mathbb{R}^3 \mid \alpha \geq v \text{ for some } v \in \mathcal{V}\}.$$

The boundary $\mathcal{S}_\mathcal{V}$ of $\langle \mathcal{V} \rangle$ is the *orthogonal surface* generated by $\mathcal{V}$. Orthogonal projection onto the plane $x + y + z = 0$ yields a picture of $\mathcal{S}_\mathcal{V}$ in the plane. This picture is a rhombic tiling of the plane, see Figure 5.

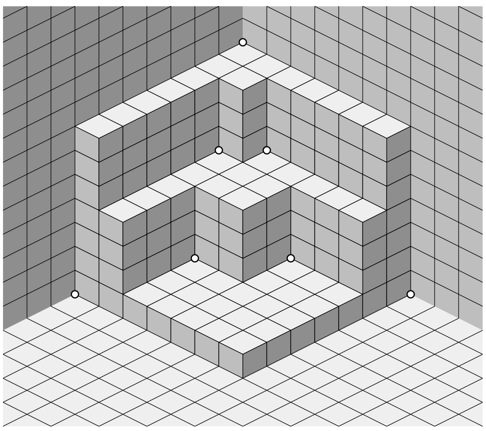

Figure 5: An orthogonal surface and its generator set $\mathcal{V}$

If $u, v \in \mathcal{V}$ and $u \vee v \in \mathcal{S}_\mathcal{V}$ then $\mathcal{S}_\mathcal{V}$ contains the union of the two line segments joining u and v to $u \vee v$; we refer to such arcs as *elbow geodesics* in $\mathcal{S}_\mathcal{V}$.

The *orthogonal arc* of $v \in \mathcal{V}$ in direction of the standard basis vector e_i is the intersection of the ray $v + \lambda e_i$, $\lambda \geq 0$, with $\mathcal{S}_\mathcal{V}$. Clearly every vector $v \in \mathcal{V}$ has exactly three orthogonal arcs, one parallel to each coordinate axis, although some orthogonal arcs may be unbounded while others are bounded. Observe that $u \vee v$ must share two coordinates with at least one (and perhaps both) of u and v, so every elbow geodesic contains at least one orthogonal arc.

Making compatible choices of elbow geodesics containing all orthogonal rays yields a planar map. A plane drawing $M \hookrightarrow \mathcal{S}_\mathcal{V}$ is a *geodesic embedding* in $\mathcal{S}_\mathcal{V}$, if the following two axioms are satisfied:

(Vertex axiom) There is a bijection between the vertices of M and $\mathcal{V}$.

(Elbow geodesic axiom) Every edge of M is an elbow geodesic in $\mathcal{S}_\mathcal{V}$, and every bounded orthogonal arc in $\mathcal{S}_\mathcal{V}$ is part of an edge of M.

An antichain $\mathcal{V}$ in $\mathbb{Z}^3$ is called *axial* if it contains exactly three unbounded orthogonal arcs. Clearly every finite antichain contains at least three unbounded orthogonal arcs, one in each direction. An *axial geodesic embedding* of a suspended map M^σ is a geodesic embedding $M \hookrightarrow \mathcal{S}_\mathcal{V}$ such that the three half edges of M^σ map onto the three unbounded orthogonal rays of $\mathcal{S}_\mathcal{V}$.

Theorem 3. *Let $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$ be an axial geodesic embedding, then the closure M^σ_∞ is 3-connected and the embedding induces a Schnyder labeling of M^σ. Conversely, every Schnyder labeling of a map M^σ yields an axial geodesic embedding of M^σ.*

Proof. Given the embedding $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$, the Schnyder labeling of the angles of M^σ is the labeling by the three different shades in the tiling figure. The vertex and face rules are easily verified. Alternatively, color the edges by the direction of the orthogonal arcs they contain, see Figure 6.

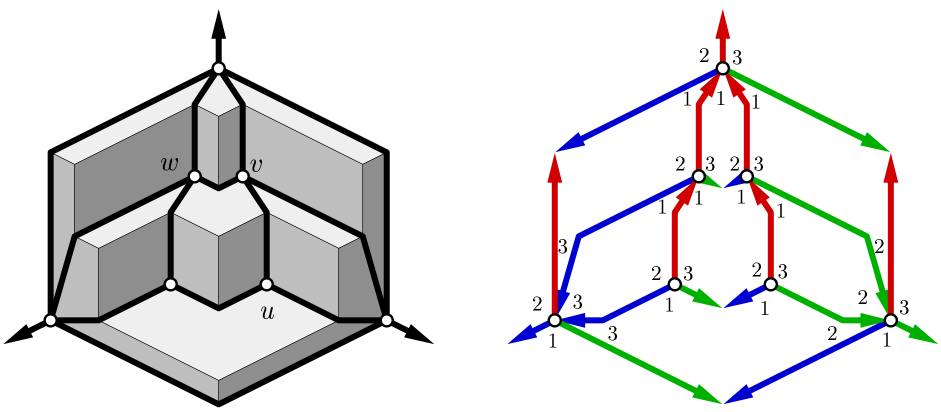

Figure 6: The Schnyder labeling induced by a geodesic embedding.

3-connectedness is obtained from the existence of three internally disjoint path between any two vertices u and v (Menger's theorem). Such path can be constructed by considering the path $P_i(w)$ for $w = u, v$ and $i = 1, 2, 3$ together with the three edges incident to v_∞.

Given a Schnyder labeling of M^σ embed every vertex v in $\mathbb{N}^3$ at its region vector (v_1, v_2, v_3). Since $v_1 + v_2 + v_3 = f - 1$ is independent of v (it is the number of bounded faces of M) this maps the vertex set of M to an antichain in $\mathbb{N}^3$. If $e = \{u, v\}$ is an edge of M and $x \notin e$ a vertex, then for some i edge e is contained in region $R_i(x)$. This implies $R_i(u) \subseteq R_i(x)$ and $R_i(v) \subseteq R_i(x)$ hence, $u_i \leq x_i$ and $v_i \leq x_i$. Together with $x_1 + x_2 + x_3 = f - 1$ this shows that with $e = \{u, v\}$ the join $u \vee v$ and hence the elbow geodesic $[u, v]$ is on the surface $\mathcal{S}_\mathcal{V}$. In this construction the orthogonal arc of v in direction e_i is used by the edge which leaves v in color i. □

1.4 Order Dimension

The *dimension* of an order P is the least k such that P admits an order preserving embedding in $\mathbb{N}^k$ equipped with the dominance order. For more on order dimension see [9], [10], [3] or [4]. With a graph $G = (V, E)$ associate its incidence order $P(G)$ on ground set $V \cup E$ and with relations $v < e$ iff vertex v is one of the endvertices of e. Schnyder's celebrated theorem [7]. characterizes planar graphs in terms of order dimension.

Theorem 4. *A Graph G is planar if and only if the dimension of the incidence order $P(G)$ is at most 3.*

Theorem 3 is not a direct consequence of Schnyder's Theorem. At first, in an embedding of $P(G)$ in $\mathbb{N}^3$ the edges need not fall onto the orthogonal surface $\mathcal{S}_V$ generated by the vertices. Even if we push the edges down to the join of their incident vertices to fulfill this condition, the property that the orthogonal arcs of the vertices are used by the connecting elbow geodesics can fail. An example can be found in Figure 6 of [6].

Conversely, it might seem that Theorem 3 yields a proof of Schnyder's theorem. At least we have shown that if we embed edge $e = \{u, v\}$ at $u \vee v$ then both vertices u and v are dominated by the edge. However, referring to Figure 6, we find that $u \vee v$ also happens to dominate w, therefore, we can only claim to have embedded some extension of $P(G)$ which is not even the incidence order of a graph.

To overcomes this difficulty we demand an additional property from elbow geodesics, this property called rigidity was proposed by Miller [6]. The main contribution of this paper is to show how Schnyder labelings can be used to obtain rigid geodesic embeddings of 3-connected planar maps. As a consequence this leads to a strong version of a theorem by Brightwell and Trotter. The Brightwell–Trotter Theorem (see [3] or [9]) is the following generalization of Schnyder's Theorem.

Theorem 5. *The incidence order of vertices, edges and bounded faces of a planar map has dimension at most 3.*

An elbow geodesic connecting vertices u and v is $\mathcal{S}_\mathcal{V}$ is *rigid* if u and v are the only vertices in $\mathcal{V}$ which are dominated by $u \vee v$. A geodesic embedding of a map M in a surface $\mathcal{S}_\mathcal{V}$ is called rigid if all the edges of M are mapped to rigid elbow geodesics.

Let $M \hookrightarrow \mathcal{S}_\mathcal{V}$ be a rigid geodesic embedding. By the above remarks this yields an embedding of the vertex-edge incidence order of M in $\mathbb{N}^3$.

The following result (Proposition 2.4 from Miller [6]) shows that in this case even the full incidence order of vertices, edges and bounded faces of M is embedded in $\mathbb{N}^3$.

Proposition 1. *Let $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$ be a rigid embedding, and F a bounded region of M. If α_F is the join of the vertices of F, then $w \in F \Leftrightarrow w \leq \alpha_F$.*

Proof. The embedding $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$ induces a Schnyder labeling of M^σ (Theorem 3). Associated to a vertex v there are three special paths $P_i(v)$. The vertices visited by path P_i are strictly increasing in coordinate i and non-increasing (weakly decreasing) in coordinates $i-1$ and $i+1$. Suppose face F is contained in region $R_i(v)$. For $w \in R_i(v)$ let w' be the last vertex of $P_i(w)$ in $R_i(v)$. From $w' \in P_{i-1}(v) \cup P_{i+1}(v)$ follows $v_i \geq w'_i \geq w_i$. Hence, for every v there is a coordinate i such that $v_i \geq [\alpha_F]_i$. This proves that $\alpha_F = \bigwedge_{w \in F} w$ is one of the maxima of the surface $\mathcal{S}_\mathcal{V}$.

Let $X_F = \{v \in V : v \leq \alpha_F\}$. Since $\alpha_F \in \mathcal{S}_\mathcal{V}$ every $w \in X_F$ is on one of the descents from α_F, i.e., there is an i with $w_i = [\alpha_F]_i$. Let $[X_F]_i = \{w \in X_F : w_i = [\alpha_F]_i\}$, each $[X_F]_i$ induces a path in M. The three path are joined together by three additional edges to form a simple cycle, the boundary of face F. Therefore, X_F is the set of vertices of F. □

With this proposition we immediately obtain a proof of the Brightwell Trotter theorem. Actually, the result implied by Proposition 1 is stronger than the original. In our embedding all vertices, edges and faces sit in a single surface. An alternative way of stating this is: For every incident pair $v \in F$ there is a coordinate i such that $v_i = (\alpha_F)_i$.

1.5 Rigid Geodesic Embeddings

A nice property of rigid embeddings is that they uniquely determine a planar map. A fixed planar map, however, can be determined by different rigid embeddings. Figure 7 shows an example.

Consider two rigid geodesic embeddings to be *combinatorially equivalent* if they induce the same Schnyder labeled planar graph. In geometric terms this equivalence can be stated as follows: Two rigid surfaces $\mathcal{S}_\mathcal{V}$ and $\mathcal{S}_\mathcal{W}$ are equivalent if there is a bijection $\pi : \mathcal{V} \to \mathcal{W}$ such that an elbow geodesic $[v, w]$ in $\mathcal{S}_\mathcal{V}$ contains a piece of the orthogonal ray $v + \lambda e_i$ iff $[\pi(v), \pi(w)]$ is an elbow geodesic in $\mathcal{S}_\mathcal{W}$ containing a piece of the orthogonal ray $\pi(v) + \lambda e_i$.

The following theorem is a special case of our main Theorem 9. The restriction to planar triangulations allows a short proof.

Theorem 6. *Let M^σ be a suspended planar triangulation. There is a bijection between equivalence classes of rigid geodesic embeddings and Schnyder labelings of M^σ.*

Proof. By our definition of equivalence for rigid surfaces the map from rigid geodesic embeddings to Schnyder labelings is injective. It remains to show that

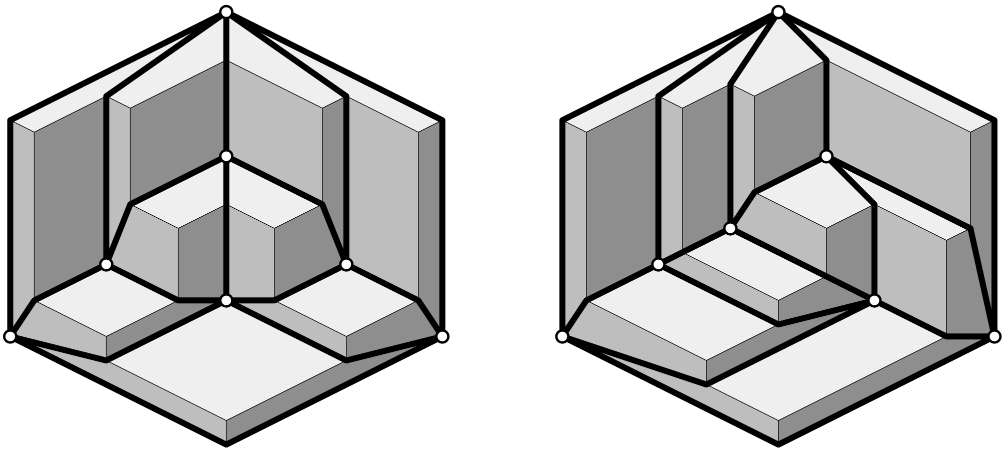

Figure 7: A planar map with two combinatorially different rigid embeddings

every Schnyder labeling of M^σ is induced by some rigid geodesic embedding. From the proof of Theorem 3 it follows that every Schnyder labeling is induced by some geodesic embedding. We thus only have to take care of rigidity.

Claim 1. No elbow geodesic representing a interior edge contains two orthogonal arcs.

This follows from counting. We need three trees covering all edges and each of the three outer edges is contained in two of the trees.

Claim 2. Every elbow geodesic is rigid.

Let $[u, v]$ be an elbow geodesic using an orthogonal ray leaving u. The following argument is illustrated in Figure 8. If $[u, v]$ is non-rigid, then there is some w

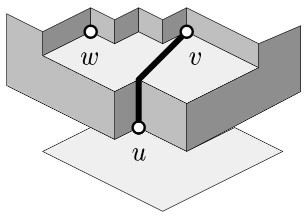

Figure 8: A non-rigid edge

below $u \vee v$. On the surface the only direction where it goes down from $u \vee v$ is on the plane patch descending to v. Therefore, v and w both sit on this same plane patch and are connected by a zig-zag path consisting only of orthogonal arcs. By the elbow geodesic axiom these arcs are contained in edges, in contradiction to Claim 1.

□

Brehm [2] has studied the structure of all Schnyder labelings of a suspended triangulated map. Define $A \prec B$ for Schnyder labelings A and B if the two labelings differ only locally around a triangle as shown in Figure 9.

Let $\mathbf{P}(M^\sigma)$ be the transitive closure of the $\prec$ relation on the set of all Schnyder labelings of a suspended planar triangulation M^σ. The main result

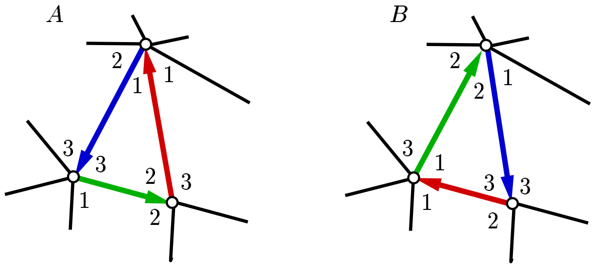

Figure 9: A flip from labeling A to labeling B.

of Brehm, which carries over to equivalence classes of rigid embeddings, is the following.

Theorem 7. *If M^σ is a 4-connected suspended planar triangulation then $\mathbf{P}(M^\sigma)$ is a distributive lattice.*

2 A Construction for Rigid Embeddings

For planar triangulations the existence of rigid embeddings immediately follows from Theorem 6 together with the existence of Schnyder labelings. In this section we show the existence of a rigid embeddings for every 3-connected planar map.

The idea is to use Schnyder labelings and modify the definition for the regions of a vertex. To begin with we define the *lifted i-path* $\widetilde{P}_i(v)$ of vertex v as a sequence of edges.

- The first edge of $\widetilde{P}_i(v)$ is the edge leaving v in color i.

Let (x, y) be an i-colored edge of $\widetilde{P}_i(v)$.

- If $y = a_i$ then the path ends.
- If $\{x, y\}$ is bidirected, such that (y, x) is colored $i-1$ and there is an unidirected edge entering y in color $i+1$, let (z, y) be the first such edge in clockwise direction from (x, y) at y. Then $\widetilde{P}_i(v)$ contains (y, z) and the edge leaving z in color i.
- Else, $\widetilde{P}_i(v)$ contains the edge leaving y in color i.

The definition of $\widetilde{P}_i(v)$ is illustrated in Figure 10. In the next lemma we collect some important properties of lifted path.

Lemma 3. *The following statements are valid for all $u, v \in V$.*

(1) *The lifted path $\widetilde{P}_i(v)$ is simple, it starts in v and ends in a_i.*

(2) *The lifted path $\widetilde{P}_i(v)$ and the unlifted path $P_{i+1}(v)$ are internally disjoint.*

(3) *Two path $\widetilde{P}_i(u)$ and $\widetilde{P}_i(v)$ never cross.*

Proof. Property (1) readily follows from the fact that $T_i \cup T_{i+1}^{-1}$ is acyclic.

Assume that vertex x is common to $\widetilde{P}_i(v)$ and $P_{i+1}(v)$. Going from v to x along $\widetilde{P}_i(v)$ and back on the reverse of $P_{i+1}(v)$ gives a cycle in $T_i \cup T_{i+1}^{-1}$. The contradiction proves (2).

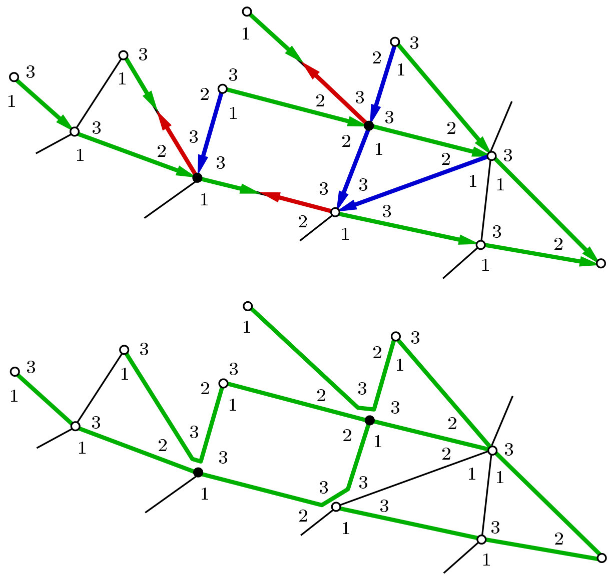

Figure 10: A partial Schnyder labeling and the induced lifted 2-paths.

For property (3) consider two paths $\widetilde{P}_i(u)$ and $\widetilde{P}_i(v)$ meeting in vertex x. If they both continue with the edge leaving x in color i, then they both contain $\widetilde{P}_i(x)$ and there is no crossing at or after x. Otherwise, one of the paths, say $\widetilde{P}_i(u)$ contains an $i+1$ colored edge e entering x. By definition $\widetilde{P}_i(u)$ enters x along the edge which follows e in counterclockwise direction at x. Hence, the path $\widetilde{P}_i(u)$ stays on the same side of $\widetilde{P}_i(v)$ and there is no crossing at x. $\triangle$

Let $\widetilde{R_i}(v)$ be the region bounded by the unlifted path $P_{i-1}(v)$ and the lifted path $\widetilde{P_{i+1}}(v)$ including the two paths, we call $\widetilde{R_i}(v)$ the *lifted i-region* of v.

Lemma 4. *For any vertices u and v of a Schnyder labeled graph and every i*

(1) $R_i(v) \subseteq \widetilde{R_i}(v)$.

(2) $R_i(u) = R_i(v)$ *iff* $\widetilde{R_i}(u) = \widetilde{R_i}(v)$.

(3) $\widetilde{R_i}(u) \subseteq \widetilde{R_i}(v)$ *iff* $u \in \widetilde{R_i}(v)$.

(4) *If* $R_i(u) \subset R_i(v)$ *then* $\widetilde{R_i}(u) \subset \widetilde{R_i}(v)$.

The lemma shows that the inclusion order with respect to the lifted i-regions $\widetilde{R_i}$ is an extension of the inclusion order of the i-regions R_i. A comparison of the lifted and unlifted 1-regions of the two black vertices in Figure 10 shows that it is a proper extension.
We now define three orders on the vertices of a planar Schnyder labeled graph. The relation S_i is defined such that $u \to v$ in S_i iff either

(a) $\widetilde{R_i}(u) \subset \widetilde{R_i}(v)$, or

(b) $\widetilde{R_i}(u) \not\subseteq \widetilde{R_i}(v)$, $\widetilde{R_i}(v) \not\subseteq \widetilde{R_i}(u)$ and $R_{i+1}(u) \subset R_{i+1}(v)$.

The open interior $R_i^o(v)$ of region $R_i(v)$, is $R_i(v) \setminus (P_{i-1}(v) \cup P_{i+1}(v))$. Note that with this notation the condition for $u \xrightarrow{b} v$ in S_i is equivalent to the statement that $u \in R_{i+1}^o(v)$ and the two paths $\widetilde{P_{i+1}}(u)$ and $P_{i-1}(v)$ intersect.

Lemma 5. *The relation S_i is acyclic, hence, an order relation.*

Proof. Call $u \to v$ an a-pair if in S_i by part (a) of the definition otherwise $u \to v$ is a b-pair. The a-relation and the b-relation are containment relations, hence acyclic. The a-relation is clearly transitive, we claim that the b-relation is also transitive. Let $u \xrightarrow{b} v \xrightarrow{b} w$ and let P be the path that joins a_{i-1} to v along the reverted $P_{i-1}(v)$ and continues along $\widetilde{P_{i+1}}(v)$ to a_{i+1}. This path P is crossed in the first part by $\widetilde{P_{i+1}}(u)$ and in the second part by $P_{i-1}(w)$. Therefore, the two paths $\widetilde{P_{i+1}}(u)$ and $P_{i-1}(w)$ cross in the interior of region $R_i(v)$. This shows that there is no inclusion between the lifted i-regions of u and w and furthermore $u \in R^o_{i+1}(w)$ this shows $u \xrightarrow{b} w$.

Transitivity and irreflexivity of $\xrightarrow{a}$ and $\xrightarrow{b}$ implies that a shortest cycle in S_i contains an a-pair followed by a b-pair. We claim that with $u \xrightarrow{a} v \xrightarrow{b} w$ we also have $u \to w$, in contradiction to the choice of the cycle.

From $u \xrightarrow{a} v \xrightarrow{b} w$ conclude $\widetilde{R_i}(u) \subset \widetilde{R_i}(v) \subset \widetilde{R_i}(w) \cup R_{i+1}(w)$. If $u \in \widetilde{R_i}(w)$ then $\widetilde{R_i}(u) \subseteq \widetilde{R_i}(v) \cap \widetilde{R_i}(w) \subset \widetilde{R_i}(w)$, hence, $u \xrightarrow{a} w$. Otherwise there is no inclusion between $\widetilde{R_i}(u)$ and $\widetilde{R_i}(w)$ and $u \in R^o_{i+1}(w)$ which implies $u \xrightarrow{b} w$. △

Note that the incomparability relation in S_i is an equivalence relation: u and v are incomparable in S_i iff $\widetilde{R_i}(u) = \widetilde{R_i}(v)$. Therefore, S_i is a weak-order, i.e., a series composition of antichains. In particular S_i is ranked. For a vertex v of a Schnyder labeled graph and $i = 1, 2, 3$, let $v_i = \mathsf{rank}(v, S_i)$.

Theorem 8. *Let M^σ be a suspended planar map with a Schnyder labeling. With $v_i = \mathsf{rank}(v, S_i)$ and $v \to (v_1, v_2, v_3)$ define a map from V to a subset $\mathcal{V}$ of $\mathbb{N}^3$. The surface $\mathcal{S}_\mathcal{V}$ generated by $\mathcal{V}$ supports a rigid geodesic embedding of M^σ.*

Recall the four properties that have to be verified for the embedding $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$.

(1) There is a bijection between the vertices of M and $\mathcal{V}$.

(2) Every edge is an elbow geodesic in $\mathcal{S}_\mathcal{V}$.

(3) Every edge is a rigid geodesic in $\mathcal{S}_\mathcal{V}$.

(4) Every orthogonal arc is part of an edge of M^σ.

Lemma 6. *Let $e = (u, v)$ be an edge and x be a vertex not on e, then there is an i, such that $u < x$ and $v < x$ in S_i.*

Proof. By symmetry we may assume that $e \in \widetilde{R_1}(x)$. From Lemma 4.(3) we know $\widetilde{R_1}(u) \subseteq \widetilde{R_1}(x)$ and $\widetilde{R_1}(v) \subseteq \widetilde{R_1}(x)$. If $\widetilde{R_1}(u) \neq \widetilde{R_1}(x)$ and $\widetilde{R_1}(v) \neq \widetilde{R_1}(x)$ this implies $u \xrightarrow{a} x$ and $v \xrightarrow{a} x$, i.e., $u < x$ and $v < x$ in S_1.

Now suppose $\widetilde{R_1}(v) = \widetilde{R_1}(x)$, it follows that there is a bidirected path in colors 2 and 3 connecting v and x. First assume that v is to the left of x. The situation is shown in the left part of Figure 11. Clearly v is on $P_3(x)$ as well as on $\widetilde{P_3}(x)$. From the relative positions of the outgoing edges of v and x in color 1 it follows that $\widetilde{R_2}(v) \subset \widetilde{R_2}(x)$. From $e \in \widetilde{R_1}(x)$ it follows that $u \in P_3(x)$ or (u, v) is unidirected in color 1. Therefore, either $\widetilde{P_3}(x)$ goes below u or $\widetilde{P_3}(x)$ contains u, more formally, either $u \in P_3(x)$ or u is contained in the

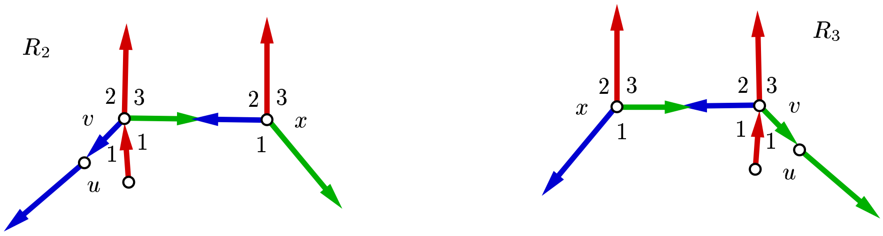

Figure 11: The two cases for $\widetilde{R_1}(v) = \widetilde{R_1}(x)$.

closed region enclosed by $P_3(x)$ and $\widetilde{P_3}(x)$. In conclusion $\widetilde{R_2}(u) \subset \widetilde{R_2}(x)$. From $\widetilde{R_2}(v) \subset \widetilde{R_2}(x)$ and $\widetilde{R_2}(u) \subset \widetilde{R_2}(x)$ it follows that $u < x$ and $v < x$ in S_2.

The last case is that $\widetilde{R_1}(v) = \widetilde{R_1}(x)$ and v is to the right of x. This situation is shown in the right part of Figure 11. The inclusion $\widetilde{R_3}(v) \subset \widetilde{R_3}(x)$ is obvious and shows $v < x$ in S_3. If $u \in P_2(x)$ then $\widetilde{R_3}(u) \subset \widetilde{R_3}(x)$ whence $u < x$ in S_3. If $u \notin P_2(x)$ then (u, v) is unidirected in color 1 and there is no inclusion between $\widetilde{R_3}(u)$ and $\widetilde{R_3}(x)$. However, we find u in the open interior of $R_1(x)$ and conclude that $R_1(u) \subset R_1(x)$. Therefore $u < x$ in S_3. △

If v and x are two vertices then there is some $u \neq x$ such that $\{u, v\}$ is an edge. This observation and the lemma allows the conclusion that the map $V \to \mathcal{V}$ is injective. Together with a symmetric argument using a neighbor y of x this shows that for vertices $v \neq x$ there are i and j such that $v < x$ in S_i and $x < v$ in S_j, i.e., $\mathcal{V}$ is an antichain in $\mathbb{N}^3$.

Let $\{u, v\}$ be an edge of M, the lemma implies that u and v are the only elements of $\mathcal{V}$ which are dominated by $u \vee v$. Therefore, $u \vee v$ and the geodesic $[u, v]$ are contained in $\mathcal{S}_{\mathcal{V}}$ and $[u, v]$ is rigid.

Lemma 7. *Every bounded orthogonal arc of $\mathcal{S}_{\mathcal{V}}$ is part of an edge of M^σ.*

Proof. The claim is that the orthogonal arc contained in $v + \lambda e_1$ is contained in $[v, w]$, where (v, w) is the outgoing edge in color 1 at v. Observe that with $w \in \widetilde{R_2}(v)$ and $w \in \widetilde{R_3}(v)$ we have $w_2 \leq v_2$ and $w_3 \leq v_3$. This already implies $v \vee w = (w_1, v_2, v_3)$. △

The lemma verified the fourth and last property of rigid embeddings. This concludes the proof of Theorem 8. □

The natural mapping from equivalence classes of rigid geodesic embeddings of a map M^σ to Schnyder labelings of M^σ is injective by definition of the equivalence classes. The proof of Theorem 8, in particular Lemma 7 shows that the mapping is surjective. This proves the generalization of Theorem 6 to general 3-connected planar maps:

Theorem 9. *Let M^σ be a suspended planar map. There is a bijection between equivalence classes of rigid geodesic embeddings and Schnyder labelings of M^σ.*

3 Concluding Remarks

From the connection with algebra described by Miller [6] there arises interest in a classification of grid surfaces. The bijections given in Theorem 6 and Theorem 9 translate parts of this problem into the realm of planar maps and Schnyder labelings. For the case of a planar triangulation M Theorem 7 provides good insight into the structure of all Schnyder labelings of M. A similar result holds true in the more general context of 3-connected planar maps. In [5] the notion of 'flips' between Schnyder labelings is generalized. Let $\mathbf{P}(M^\sigma)$ be the transitive closure of directed flips on the set of all Schnyder labelings of a suspended planar map M^σ. In [5] we prove the following structure theorem:

Theorem 10. *If M^σ is a suspended planar map then $\mathbf{P}(M^\sigma)$ is a distributive lattice.*

The proof makes use of a beautiful duality for geodesic embeddings. This duality is obtained by reverting the order on $\mathbb{N}^3$, i.e., exchanging the notions of above and below and the roles of minima and maxima on a surface $\mathcal{S}$. This duality gives a nice bijection between the Schnyder labelings of a suspended planar map and its suspension dual, i.e., the dual obtained by placing a dual vertex in each of the three parts of the unbounded face and connecting them pairwise by an edge. Figure 12 shows an example where the primal graph is a triangulation. Remarkably, the duality even preserves rigidity. This is because non-rigid edges come in primal-dual pairs.

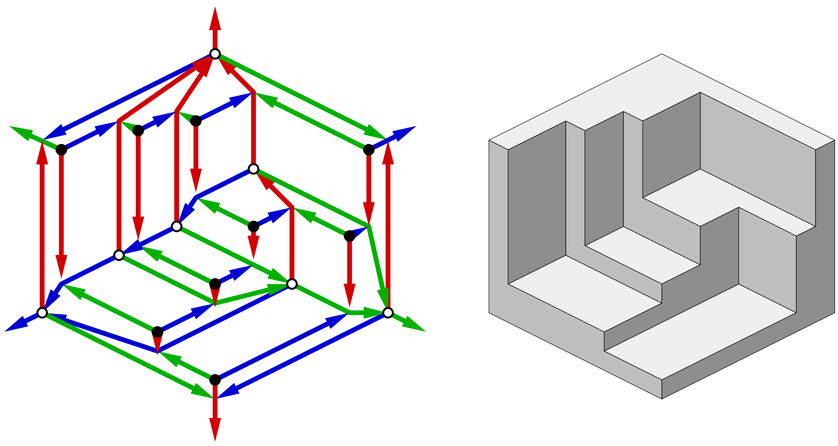

Figure 12: A dual pair of Schnyder labelings for a planar graph and its dual, on the right side the inducing surface.

Bonichon et. al. [1] investigate the structure of all planar triangulations with their Schnyder labelings on a given number of vertices. They propose 'colored flips' and show that these flips make the set connected.

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Appendix: Schnyder Labelings and Woods

Proof of Lemma 1

The proof is based on double counting and Euler's formula. Define the degree $d(v)$ of a vertex v as the number of edges incident with v whose angles at v have distinct labels. By the rule of vertices $d(v) = 3$ for every vertex v. Similarly, the degree $d(F)$ of a face F is the number of boundary edges of F whose angles in F have distinct labels. By the rules of faces $d(F) = 3$ for every face, in the case of the outer face the half-edges are counted. Therefore,

$$S = \sum_v d(v) + \sum_F d(F) = 3n + 3f = 3|E| + 6.$$

The same number S can be obtained by counting the changes of label around the edges. Each of the half-edges contributes two. Hence, the average contribution of full-edges is 3. Consider the four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of an edge in counterclockwise order. Define $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ so that $\alpha_2 = \alpha_1 + \epsilon_1$, $\alpha_3 = \alpha_2 + \epsilon_2$, $\alpha_4 = \alpha_3 + \epsilon_3$ and $\alpha_1 = \alpha_4 + \epsilon_4$. Form the rules of vertices and faces $\epsilon_j \in \{0, 1\}$, for all j. The cyclic nature of the linear system implies that $\sum_{j=1}^{4} \epsilon_j = 0 \mod 3$, hence, either $\sum_j \epsilon_j = 0$ or $\sum_j \epsilon_j = 3$. Since the contribution of an edge e to the degree sum S is $\sum_j \epsilon_j(e)$ we must have $\sum_j \epsilon_j(e) = 3$ for every full-edge e. Up to rotational symmetry this only leaves the two cases shown in Figure 2. □

Note that from the exterior labels at special vertices (A1) and the rule for the outer face we know all the edge labels at outer angles. Every outer angle on the clockwise outer path from a_i to a_{i+1} has label $i - 1$. The previous lemma together with rule (A2) applied to vertex a_i implies:

Corollary. *In a Schnyder labeling all interior angles at the special vertex a_i are labeled i.*

The equivalence of Schnyder angle labelings and Schnyder woods

Let a Schnyder angle labeling be given. Recall how this induces orientations and labeling of edges: If an edge has different angular labels i and j at one of its ends, we direct the edge from this end towards the other and give this direction the third label k.

Lemma 1 shows that this orientation and labeling obeys (W1). There is an immediate correspondence of (A1) and (W2). Also the two rules of vertices (A2) and (W3) correspond to each other. If there is an interior face whose boundary is directed in one label, then we infer that all interior angles of this face have the same label. Hence, the rule of faces (A3) forces (W4).

For the converse suppose a Schnyder wood is given. If the direction (u, v) is colored i then we color the angle at u to the left of edge uv with $i + 1$ and the angle to the right of uv with $i - 1$. If uv is unidirectional we also color the two adjacent angles at v with color i. From (W3) it follows that this coloring is well-defined, i.e., the colors assigned to an angle from its two adjacent edges coincide.

Again, the correspondence of (A1) and (W2) and of the two rules of vertices (A2) and (W3) is trivial. To show that the rule of faces (A3) is valid is more subtle.

We count color changes at vertices, faces and edges. The four angles of a full-edge are colored as in Figure 2 (by (W1)), therefore, $d(e) = 3$ for every full-edge and $d(e) = 2$ for each of the three half-edges. The contribution of vertices is $\sum_v d(v) = 3n$. The degree $d(F)$ of a bounded face F is the number of color changes at the angles when we cycle around F. If we cycle clockwise then an angle colored i is always followed by an angle colored i or $i + 1$. Consequently, $d(F)$ must be a multiple of 3. Rule (W4) enforces $d(F) \neq 0$. For the unbounded face we have $d(F) = 3$ anyway.

$$\sum_v d(v) + \sum_F d(F) = \sum_e d(e) \qquad \Longrightarrow \qquad 3n + \sum_F d(F) = 3|E| + 6$$

With Euler's Formula $\sum_F d(F) = 3f$ which is only possible if $d(F) = 3$ for every face F.

We have already seen that the colors go clockwise at bounded faces and counterclockwise at the outer face. This proves the rule of faces (A3). □

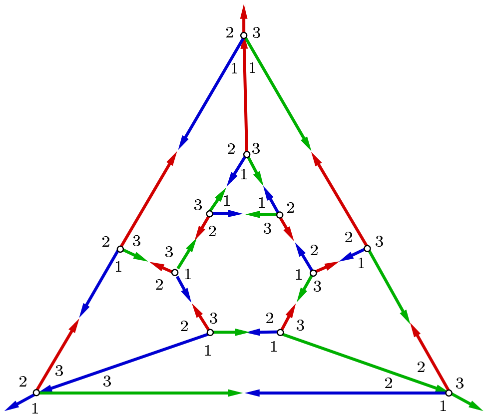

Figure 13: The example orientation and coloring of edges obeys (W1), (W2) and (W3) but not (W4). The induced angle labeling is not a Schnyder labeling.