Intersection Graphs of L-Shapes and Segments in the Plane*

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Abstract. An L-shape is the union of a horizontal and a vertical segment with a common endpoint. These come in four rotations: L, Γ, ⅃ and ⅂. A k-bend path is a simple path in the plane, whose direction changes k times from horizontal to vertical. If a graph admits an intersection representation in which every vertex is represented by an L, an L or Γ, a k-bend path, or a segment, then this graph is called an {L}-graph, {L, Γ}-graph, B_k-VPG-graph or SEG-graph, respectively. Motivated by a theorem of Middendorf and Pfeiffer [Discrete Mathematics, 108(1):365–372, 1992], stating that every {L, Γ}-graph is a SEG-graph, we investigate several known subclasses of SEG-graphs and show that they are {L}-graphs, or B_k-VPG-graphs for some small constant k. We show that all planar 3-trees, all line graphs of planar graphs, and all full subdivisions of planar graphs are {L}-graphs. Furthermore we show that all complements of planar graphs are B_{19}-VPG-graphs and all complements of full subdivisions are B_2-VPG-graphs. Here a full subdivision is a graph in which each edge is subdivided at least once.

Keywords: intersection graphs, segment graphs, co-planar graphs, k-bend VPG-graphs, planar 3-trees.

1 Introduction and Motivation

A **segment intersection graph**, SEG-graph for short, is a graph that can be represented as follows. Vertices correspond to straight-line segments in the plane and two vertices are adjacent if and only if the corresponding segments intersect. Such representations are called SEG-*representations* and, for convenience, the class of all SEG-graphs is denoted by SEG. SEG-graphs are an important subject of study strongly motivated from an algorithmic point of view. Indeed, having an intersection representation of a graph (in applications graphs often come

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along with such a given representation) may allow for designing better or faster algorithms for optimization problems that are hard for general graphs.

More than 20 years ago, Middendorf and Pfeiffer [23], considered intersection graphs of **axis-aligned L-shapes** in the plane, where an axis-aligned L-shape is the union of a horizontal and a vertical segment whose intersection is an endpoint of both. In particular, L-shapes come in four possible rotations: L, Γ, $\lrcorner$, and $\urcorner$. For a subset X of these four rotations, e.g., $X = \{\mathsf{L}\}$ or $X = \{\mathsf{L}, \Gamma\}$, we call a graph an *$X$-graph* if it admits an *X-representation*, i.e., vertices can be represented by L-shapes from X in the plane, each with a rotation from X, such that two vertices are adjacent if and only if the corresponding L-shapes intersect. Similarly to SEG, we denote the class of all X-graphs by X. The question if an intersection representation with polygonal paths or pseudo-segments can be *stretched* into a SEG-representation is a classical topic in combinatorial geometry and Oriented Matroid Theory. Middendorf and Pfeiffer prove the following interesting relation between intersection graphs of segments and L-shapes.

Theorem 1 (Middendorf and Pfeiffer [23]). *Every* $\{\mathsf{L}, \Gamma\}$*-representation has a combinatorially equivalent* SEG*-representation.*

This theorem is best-possible in the sense that there are examples of $\{\mathsf{L}, \urcorner\}$-graphs which are no SEG-graphs [6, 23], i.e., such $\{\mathsf{L}, \urcorner\}$-representations cannot be stretched. We feel that Theorem 1, which of course implies that $\{\mathsf{L}, \Gamma\} \subseteq$ SEG, did not receive a lot of attention in the active field of SEG-graphs. In particular, one could use Theorem 1 to prove that a certain graph class $\mathcal{G}$ is contained in SEG by showing that $\mathcal{G}$ is contained in $\{\mathsf{L}, \Gamma\}$. For example, very recently Pawlik *et al.* [24] discovered a class of triangle-free SEG-graphs with arbitrarily high chromatic number, disproving a famous conjecture of Erdős [17], and it is in fact easier to see that these graphs are $\{\mathsf{L}\}$-graphs than to see that they are SEG-graphs. To the best of our knowledge, the stronger result $\mathcal{G} \subseteq \{\mathsf{L}, \Gamma\}$ has never been shown for any non-trivial graph class $\mathcal{G}$. In this paper we initiate this research direction. We consider several graph classes which are known to be contained in SEG and show that they are actually contained in $\{\mathsf{L}\}$, which is a proper subclass of $\{\mathsf{L}, \Gamma\}$.

Whenever a graph is not known (or known not) to be an intersection graph of segments or axis-aligned L-shapes, one often considers natural generalizations of these intersection representations. Asinowski *et al.* [3] introduced **intersection graphs of axis-aligned k-bend paths** in the plane, called B_k-VPG-graphs. An (axis-aligned) k-bend path is a simple path in the plane, whose direction changes k times from horizontal to vertical. Clearly, B_1-VPG-graphs are precisely intersection graphs of all four L-shapes; the union of B_k-VPG-graphs for all $k \geq 0$ is exactly the class STRING of intersection graphs of simple curves in the plane. Now if a graph $G \notin$ SEG is a B_k-VPG-graph for some small k, then one might say that G is "not far from being a SEG-graph".

1.1 Our Results and Related Work

Let us denote the class of all planar graphs by PLANAR. A recent celebrated result of Chalopin and Gonçalves [5] states that PLANAR $\subset$ SEG, which was conjectured by Scheinerman [25] in 1984. However, their proof is rather involved and there is not much control over the kind of SEG-representations. Here we give an easy proof for a non-trivial subclass of planar graphs, namely *planar 3-trees*. A *3-tree* is an edge-maximal graph of treewidth 3. Every 3-tree can be built up starting from the clique K_4 and adding new vertices, one at a time, whose neighborhood is the so-far constructed graph is a triangle.

Theorem 2. *Every planar* 3*-tree is an* $\{\mathsf{L}\}$*-graph.*

It remains open to generalize Theorem 2 to planar graphs of treewidth 3 (i.e., subgraphs of planar 3-trees). On the other hand it is easy to see that graphs of treewidth at most 2 are $\{\mathsf{L}\}$-graphs. Chaplick and the last author show in [8] that planar graphs are B_2-VPG-graphs, improving on an earlier result of Asinowski *et al.* [3]. In [8] it is also conjectured that PLANAR $\subset \{\mathsf{L}\}$, which with Theorem 1 would imply the main result of [5], i.e., PLANAR $\subset$ SEG.

Considering line graphs of planar graphs, one easily sees that these graphs are SEG-graphs. Indeed, a straight-line drawing of a planar graph G can be interpreted as a SEG-representation of the line graph $L(G)$ of G. We prove the following strengthening result.

Theorem 3. *The line graph of every planar graph is an* $\{\mathsf{L}\}$*-graph.*

Kratochvíl and Kuběna [20] consider the class of all complements of planar (co-planar) graphs, CO-PLANAR for short. They show that CO-PLANAR are intersection graphs of convex sets in the plane, and ask whether CO-PLANAR $\subset$ SEG. As the Independent Set Problem in planar graphs is known to be NP-complete [14], Max Clique is NP-complete for any graph class $\mathcal{G} \supseteq$ CO-PLANAR , e.g., intersection graphs of convex sets. Indeed, the longstanding open question whether Max Clique is NP-complete for SEG [21] has recently been answered affirmatively by Cabello, Cardinal and Langerman [4] by showing that every planar graph has an even subdivision whose complement is a SEG-graph. The subdivision is essential in the proof of [4], as it still remains an open problem whether CO-PLANAR $\subset$ SEG [20]. The largest subclass of CO-PLANAR known to be in SEG is the class of complements of partial 2-trees [13]. Here we show that all co-planar graphs are “not far from being SEG-graphs”.

Theorem 4. *Every co-planar graph is a* B_{19}*-VPG graph.*

Theorem 4 implies that Max Clique is NP-complete for B_k-VPG-graphs with $k \geq 19$. On the other hand, the Max Clique problem for B_0-VPG-graphs can be solved in polynomial time, while Vertex Colorability remains NP-complete but allows for a 2-approximation [3]. Middendorf and Pfeiffer [23] show that the complement of any *even subdivision* of any graph, i.e., every edge is subdivided with a non-zero even number of vertices, is an $\{\mathsf{L}, \urcorner\}$-graph. This implies that MAX CLIQUE is NP-complete even for $\{\mathsf{L}, \urcorner\}$-graphs.

We consider *full subdivisions* of graphs, that is, a subdivision H of a graph G where each edge of G is subdivided at least once. It is not hard to see that a full subdivision H of G is in STRING if and only if G is planar, and that if G is planar, then H is actually a SEG-graph. Here we show that this can be further strengthened, namely that H is in an $\{\mathsf{L}\}$-graph. Moreover, we consider the complement of a full subdivision H of an arbitrary graph G, which is in STRING but not necessarily in SEG. Here, similar to the result of Middendorf and Pfeiffer [23] on even subdivisions we show that such a graph H is "not far from being SEG-graph".

Theorem 5. *Let H be a full subdivision of a graph G.*

(i) If G is planar, then H is an $\{\mathsf{L}\}$-graph.
(ii) If G is any graph, then the complement of H is a B_2-VPG-graph.

The graph classes considered in this paper are illustrated in Figure 1. We shall prove Theorems 2, 3, 4 and 5 in Sections 2, 3, 4 and 5, respectively, and conclude with some open questions in Section 6.

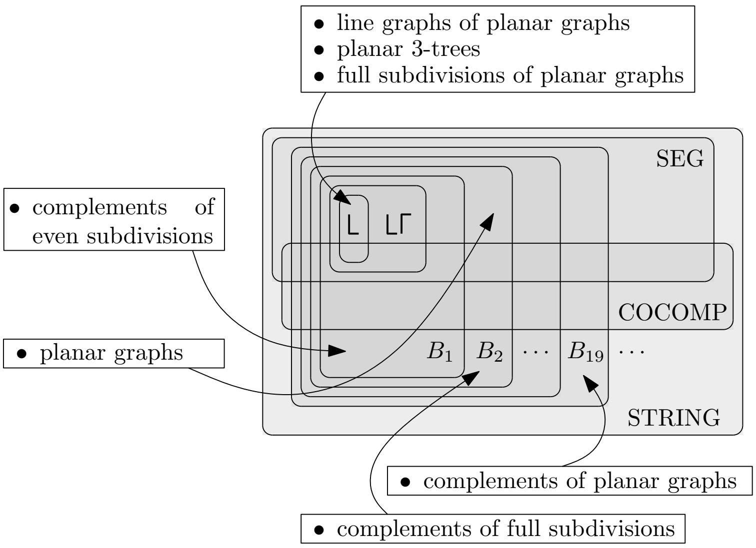

Fig. 1. Graph classes considered in this paper.

1.2 Related Representations

In the context of *contact representations*, where distinct segments or k-bend paths may not share interior points, it is known that every contact SEG-representation has a combinatorially equivalent contact B_1-VPG-representation, but not vice versa [19]. Contact SEG-graphs are exactly planar Laman graphs and their subgraphs [10], which includes for example all triangle-free planar graphs. Very recently, contact $\{\mathsf{L}\}$-graphs have been characterized [7]. Necessary and sufficient conditions for stretchability of a contact system of pseudo-segments are known [1, 11].

Let us also mention the closely related concept of *edge*-intersection graphs of paths in a grid (EPG-graphs) introduced by Golumbic *et al.* [15]. There are some notable differences, starting from the fact that *every* graph is an EPG-graph [15]. Nevertheless, analogous questions to the ones posed about VPG-representations of STRING-graphs are posed about EPG-representations of general graphs. In particular, there is a strong interest in finding representations using paths with few bends, see [18] for a recent account.

2 Proof of Theorem 2

Proof. Let G be a plane 3-tree with a fixed plane embedding. We construct an $\{\mathsf{L}\}$-representation of G satisfying the additional property that for every inner triangular face $\{a, b, c\}$ of G there exists a subset of the plane, called the *private region of the face*, that intersects only the L-paths for a, b and c, and no other L-path. More precisely, a private region of $\{a, b, c\}$ is an axis-aligned rectilinear polygon having one of the shapes depicted in Figure 2(a), such that the L-paths for a, b and c intersect the polygon as shown in figure.

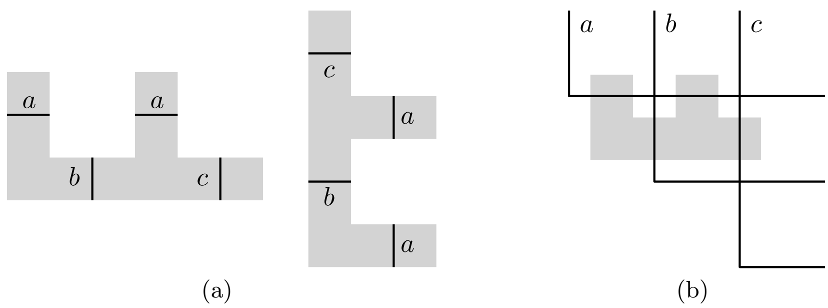

Fig. 2. (a) The two possible shapes of a private region for inner facial triangle $\{a, b, c\}$. (b) An $\{\mathsf{L}\}$-representation of the plane 3-tree on three vertices together with a private region for the only inner face.

Indeed, we prove the following stronger statement by induction on the number of vertices in G.

Claim. Every plane 3-tree admits an $\{\mathsf{L}\}$-representation together with a private region for every inner face, such that the private regions for distinct faces are disjoint.

As induction base ($|V(G)| = 3$) consider the graph G consisting only of triangle a, b, c. Then there is an essentially unique $\{\mathsf{L}\}$-representation of G and it is not difficult to find a private region for the unique inner face of G. We refer to Figure 2(b) for an illustration.

Now let us assume that $|V(G)| \geq 4$. Because G is a 3-tree there exists an inner vertex v of degree exactly three. In particular, the three neighbors a, b, c of v form an inner facial triangle in the plane 3-tree $G' = G \setminus v$. By induction G' admits an $\{\mathsf{L}\}$-representation with a private region for each inner face so that distinct private regions are disjoint.

Consider the private region R for $\{a, b, c\}$. By flipping the plane along the main diagonal if necessary, we can assume without loss of generality that R has the shape shown in the left of Figure 2(a). (Note that such a flip does not change the type of the L-paths.) Now we introduce an L-path for vertex v completely inside R as depicted in Figure 3(a). Since R does not intersect any other L-path this is an $\{\mathsf{L}\}$-representation of G.

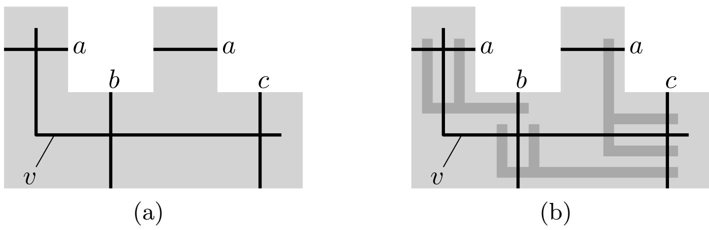

Fig. 3. (a) Introducing an L-shape for vertex v into the private region for the triangle $\{a, b, c\}$. (b) Identifying a pairwise disjoint private regions for the facial triangles $\{a, b, v\}$, $\{a, c, v\}$ and $\{b, c, v\}$.

Finally we identify three private regions for the three newly created inner faces $\{a, b, v\}$, $\{a, c, v\}$ and $\{b, c, v\}$. This is shown in Figure 3(b). Since these regions are pairwise disjoint and completely contained in the private region for $\{a, b, c\}$ we have identified a private region for every inner face so that distinct regions are disjoint. (Note that $\{a, b, c\}$ is not a facial triangle in G and hence does not need a private region.) This proves the claim and thus conclude the proof of the theorem.

3 Proof of Theorem 3

Proof. Without loss of generality let G be a maximally planar graph with a fixed plane embedding. (Line graphs of subgraphs of G are induced subgraphs of $L(G)$.) Then G admits a so-called *canonical ordering*, namely an ordering $v_1, \ldots, v_n$ of the vertices of G such that

– Vertices v_1, v_2, v_n form the outer triangle of G in clockwise order. (We draw G such that v_1, v_2 are the highest vertices.)

– For $i = 3, \ldots, n$ vertex v_i lies in the outer face of the induced embedded subgraph $G_{i-1} = G[v_1, \ldots, v_{i-1}]$. Moreover, the neighbors of v_i in G_{i-1} form a path on the outer face of G_{i-1} with at least two vertices.

We shall construct an $\{\mathsf{L}\}$-representation of $L(G)$ along a fixed canonical ordering $v_1, \ldots, v_n$ of G. For every $i = 2, \ldots, n$ we shall construct an $\{\mathsf{L}\}$-representation of $L(G_i)$ with the following additional properties.

For every outer vertex v of G_i there is a bottomless rectangle $R(v)$, i.e., an axis-aligned rectangle with bottom-edge at $-\infty$, such that:

– $R(v)$ intersects the horizontal segments of precisely those paths for edges in G_i incident to v.
– $R(v)$ does not contain any bends or endpoints of any path for an edge in G_i and does not intersect any $R(w)$ for $w \neq v$.
– the left-to-right order of the bottomless rectangles matches the order of vertices on the counterclockwise outer v_1, v_2-path of G_i.

For $i = 2$, the graph G_i consist only of the edge $v_1 v_2$. Hence an $\{\mathsf{L}\}$-representation of the one-vertex graph $L(G_2)$ consists of only one L-shape and two disjoint bottomless rectangles $R(v_1)$, $R(v_2)$ intersecting its horizontal segment.

For $i \geq 3$, let $(w_1, \ldots, w_k)$ be the counterclockwise outer path of G_{i-1} that corresponds to the neighbors of v_i in G_{i-1}. The corresponding bottomless rectangles $R(w_1), \ldots, R(w_k)$ appear in this left-to-right order. See Figure 4 for an illustration. For every edge $v_i w_j$, $j = 1, \ldots, k$ we define an L-shape $P(v_i w_j)$ whose vertical segment is contained in the interior of $R(w_j)$ and whose horizontal segment ends in the interior of $R(w_k)$. Moreover, the upper end and lower end of the vertical segment of $P(v_i w_j)$ lies on the top side of $R(w_j)$ and below all L-shapes for edges in G_{i-1}, respectively. Finally, the bend and right end of $P(v_i w_j)$ is placed above the bend of $P(v_i w_{j+1})$ and to the right of the right end of $P(v_i w_{j+1})$ for $j = 1, \ldots, k-1$, see Figure 4.

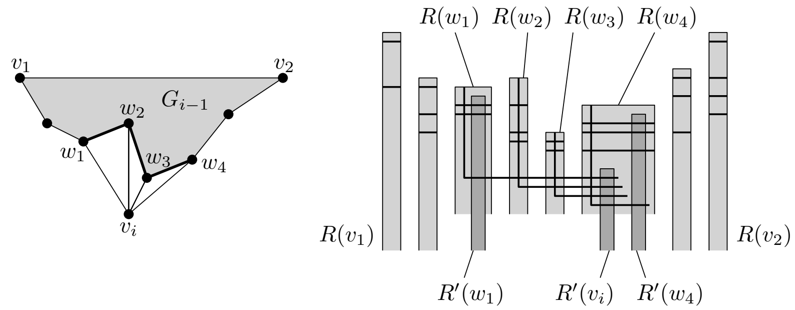

Fig. 4. Along a canonical ordering a vertex v_i is added to G_{i-1}. For each edge between v_i and a vertex in G_{i-1} an L-shape is introduced with its vertical segment in the corresponding bottomless rectangle. The three new bottomless rectangles $R'(w_1), R'(v_i), R'(w_k)$ are highlighted.

It is straightforward to check that this way we obtain an $\{\mathsf{L}\}$-representation of $L(G_i)$. So it remains to find a set of bottomless rectangles, one for each outer vertex of G_i, satisfying our additional property. We set $R'(v) = R(v)$ for every $v \in V(G_i) \setminus \{v_i, w_1, \ldots, w_k\}$. We define a bottomless rectangle $R'(w_1) \subset R(w_1)$ such that $R'(w_1)$ is crossed by all horizontal segments that cross $R(w_1)$ and additionally the horizontal segment of $P(v_i w_1)$. Similarly, we define $R'(w_k) \subset R(w_k)$. And finally, we define $R'(V_i) \subset R(w_k)$ in such a way that it is crossed by the horizontal segments of exactly $P(v_i w_1), \ldots, P(v_i w_k)$. Note that for $1 < j < k$ the outer vertex w_j of G_{i-1} is not an outer vertex of G_i. Then $\{R'(v) \mid v \in v(G_i)\}$ has the desired property. See again Figure 4. □

4 Proof of Theorem 4

Proof. Let $G = (V, E)$ be any planar graph. We shall construct a B_k-VPG representation of the complement $\bar{G}$ of G for some constant k that is independent of G. Indeed, $k = 19$ is enough. To find the VPG representation we make use of two crucial properties of G: A) G is 4-colorable and B) G is 5-degenerate. Indeed, our construction gives a B_{2d+9}-VPG representation for the complement of any 4-colorable d-degenerate graph.

Consider any 4-coloring of G with color classes V_1, V_2, V_3, V_4. Further let $\sigma = (v_1, \ldots, v_n)$ be an order of the vertices of V witnessing the degeneracy of G, i.e., for each v_i there at most 5 neighbors v_j of v_i with and $j < i$. We call these neighbors the *back neighbors of* v_i.

Consider any ordered pair of color classes, say (V_1, V_2), and denote $W = V_1 \cup V_2$, together with the vertex orders inherited from the order of vertices in V, i.e., $\sigma|_{V_1} = \sigma_1 = (v_1, \ldots, v_{|V_1|})$ and $\sigma|_{V_2} = \sigma_2 = (w_1, \ldots, w_{|V_2|})$. For $v \in V_1$ we denote the number of back neighbors of v in V_2 by $\deg_2^*(v)$. Further consider the axis-aligned rectangle $R = [0, A] \times [0, A]$, where $A = 2(|W| + 2)$. For illustration we divide R into four quarters $[0, A/2] \times [0, A/2]$, $[0, A/2] \times [A/2, A]$, $[A/2, A] \times [0, A/2]$ and $[A/2, A] \times [A/2, A]$.

We define a monotone increasing path $Q(v)$ for each $v \in W$ as follows. See Figure 5 for an illustration.

- For $v \in V_1$ let $i_1 < \cdots < i_k$ be the indices of back neighbors of v in V_2 and $i^* = \max\{j \in \{0\} \cup [\sigma^{-1}(v)] \mid v_j \in V_2\}$, that is, i^* is the largest index of a vertex in V_2 that comes before v in σ or $i^* = 0$ if there is no such vertex.
  Then we define the path $Q(v)$ so that it starts at $(1, 0)$, uses the horizontal lines at $y = 2i_j - 1$ for $j = 1, \ldots, k$, $y = 2i^* + 1$ and $y = A - 2\sigma_1(v)$ in that order, uses the vertical lines at $x = 1$, $x = 2i_j + 1$ for $j = 1, \ldots, k$ and $x = A - 2\sigma_1(v)$ in that order, and finally ends at $(A, A - 2\sigma_1(v))$.
  Note that $Q(v)$ avoids the top-left quarter of R, has exactly one bend at $(A - 2\sigma_1(v), A - 2\sigma_1(v))$ in the top-right quarter, goes above the point $(2i, 2i)$ in the bottom-left quarter if and only if $i \neq i_1, \ldots, i_k$ and $i \leq i^*$.
- For $w_i \in V_2$ the path $P(w_i)$ is defined analogous after rotating the rectangle R by 180 degrees and swapping the roles of V_1 and V_2.

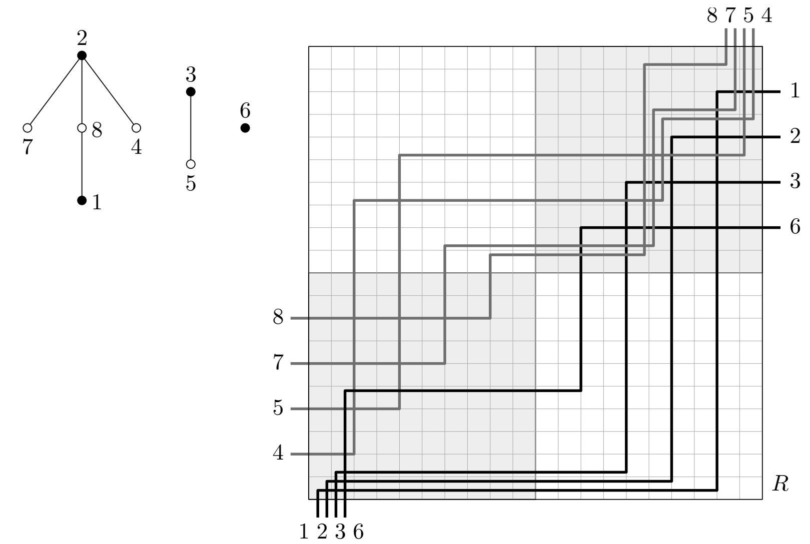

Fig. 5. The induced subgraph $G[W]$ for two color classes $W = V_1 \cup V_2$ of a planar graph G and a VPG representation of its complement $\bar{G}[W]$ in the rectangle $[0, 2(|W|+2)] \times [0, 2(|W|+2)]$.

It is straightforward to check that $\{Q(v) \mid v \in W\}$ is a VPG representation of $\bar{G}[W]$ completely contained in R, where each $Q(v)$ starts and ends at the boundary of R and has at most $3 + 2k$ bends, where k is the number of back neighbors of v in W.

Now we have defined for each pair of color classes $V_i \cup V_j$ a VPG-representation of $\bar{G}[V_i \cup V_j]$. For every vertex $v \in V$ we have defined three Q-paths, one for each colors class that v is not in. In total the three Q-paths for the same vertex v have at most $9 + 2k \leq 19$ bends, where $k \leq 5$ is the back degree of v. It remains to place the six representations of $\bar{G}[V_i \cup V_j]$ non-overlapping and to "connect" the three Q-paths for each vertex in such a way that connections for vertices of different color do not intersect. This can easily be done with two extra bends per paths, basically because K_4 is planar (we refer to Figure 6 for one way to do this). Finally, note that the first and last segment of every path in the representation can be omitted, yielding the claimed bound. □

5 Proof of Theorem 5

Proof. Let G be any graph and H be a subdivision of G in which each edge is subdivided at least once. Without loss of generality we may assume that every edge of G is subdivided exactly once or twice.

(i) Assuming that G is planar, we shall find an $\{\mathsf{L}\}$-representation of H as follows. Assume without loss of generality that G is maximally planar. We

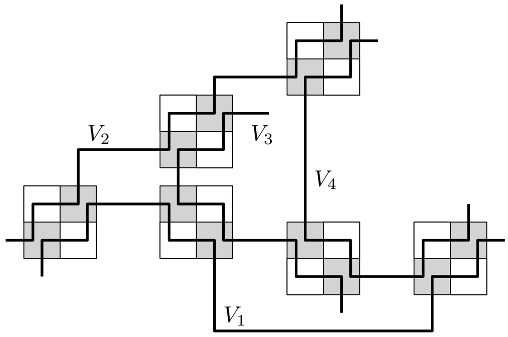

Fig. 6. Interconnecting the VPG representations of $\bar{G}[V_i \cup V_j]$ by adding at most two bends for each vertex. The set of paths corresponding to color class V_i is indicated by a single path labeled V_i, $i = 1, 2, 3, 4$.

consider a bar visibility representation of G, i.e., vertices of G are disjoint horizontal segments in the plane and edges are disjoint vertical segments in the plane whose endpoints are contained in the two corresponding vertex segments and which are disjoint from all other vertex segments. Such a representation for a planar triangulation exists e.g. by [22]. See Figure 7 for an illustration.

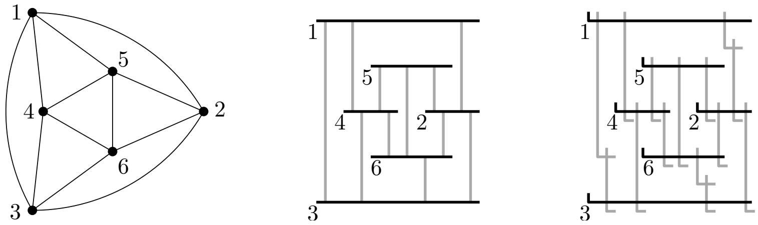

Fig. 7. A planar graph G on the left, a bar visibility representation of G in the center, and an $\{\mathsf{L}\}$-representation of a full division of G on the right. Here, the edges $\{1, 2\}$, $\{1, 3\}$ and $\{3, 6\}$ are subdivided twice.

It is now easy to interpret every segment as an L, and replace an segment corresponding to edge that is subdivided twice by two L-shapes. Let us simply refer to Figure 7 again.

(ii) Now assume that $G = (V, E)$ is any graph. We shall construct a B_2-VPG representation of the complement $\bar{H}$ of $H = (V \cup W, E')$ with monotone increasing paths only. First, we represent the clique $\bar{H}[V]$. Let $V = \{v_1, \ldots, v_n\}$ and define for $i = 1, \ldots, n$ the 2-bend path $P(v_i)$ for vertex v_i to start at $(i, 0)$, have bends at (i, i) and $(i + n, i)$, and end at $(i + n, n + 1)$. See Figure 8 for an illustration. For convenience, let us call these paths *v-paths*.

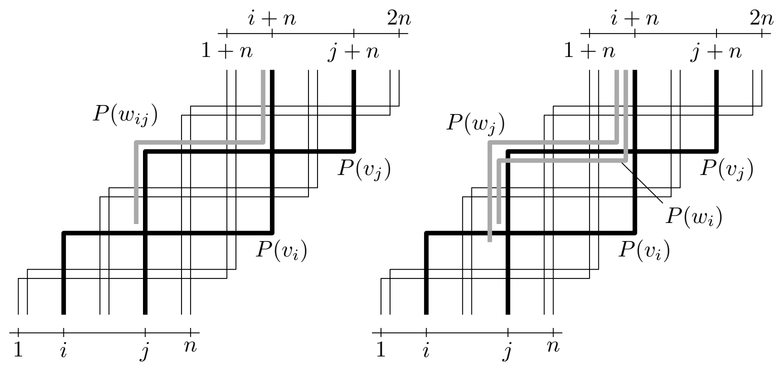

Fig. 8. Left: Inserting the path $P(w_{ij})$ for a single vertex w_{ij} subdividing the edge v_iv_j in G. Right: Inserting the paths $P(w_i)$ and $P(w_j)$ for two vertices w_i, w_j subdividing the edge v_iv_j in G.

Next, we define for every edge of G the 2-bend paths for the one or two corresponding subdivision vertices in $\bar{H}$. We call these paths *w-paths*. So let v_iv_j be any edge of G with $i < j$. We distinguish two cases.

Case 1. The edge v_iv_j is subdivided by only one vertex w_{ij} in H. We define the w-path $P(w_{ij})$ to start at $(j - \frac{1}{4}, i + \frac{1}{4})$, have bends at $(j - \frac{1}{4}, j + \frac{1}{4})$ and $(i + n - \frac{1}{4}, j + \frac{1}{4})$, and end at $(i + n - \frac{1}{4}, n + 1)$, see the left of Figure 8.

Case 2. The edge v_iv_j is subdivided by two vertices w_i, w_j with $v_iw_i, v_jw_j \in E(H)$. We define the start, bends and end of the w-path $P(w_i)$ to be $(j - \frac{1}{4}, i + \frac{1}{4})$, $(j - \frac{1}{4}, j - \frac{1}{4})$, $(i + n - \frac{1}{4}, j - \frac{1}{4})$ and $(i + n - \frac{1}{4}, n + 1)$, respectively. The start, bends and end of the w-path $P(w_j)$ are $(j - \frac{1}{2}, i - \frac{1}{4})$, $(j - \frac{1}{2}, j + \frac{1}{4})$, $(i + n - \frac{1}{2}, j + \frac{1}{4})$ and $(i + n - \frac{1}{2}, n + 1)$, respectively. See the right of Figure 8.

It is easy to see that every w-path $P(w)$ intersects every v-path, except for the one or two v-paths corresponding to the neighbors of w in H. Moreover, the two w-paths in Case 2 are disjoint. It remains to check that the w-paths for distinct edges of G mutually intersect. To this end, note that every w-path for edge v_iv_j starts near (j, i), bends near (j, j) and $(i + n, j)$ and ends near $(i+n, n)$. Consider two w-paths P and P' that start at (j, i) and (j', i'), respectively, and bend near (j, j) and (j', j'), respectively. If $j = j'$ then it is easy to check that P and P' intersect near (j, j). Otherwise, let $j' > j$. Now if $j > i'$, then P and P' intersect near (j', i), and if $j \leq i'$, then P and P' intersect near $(i + n, j')$.
Hence we have found a B_2-VPG-representation of $\bar{H}$, as desired. Let us remark, that in this representation some w-paths intersect non-trivially along some horizontal or vertical lines. However, this can be omitted by a slight and appropriate perturbation of endpoints and bends of w-paths. □

6 Conclusions and Open Problems

Motivated by Middendorf and Pfeiffer's theorem (Theorem 1 in [23]) that every $\{\mathsf{L},\mathsf{\Gamma}\}$-representation can be stretched into a SEG-representation, we considered the question which subclasses of SEG-graphs are actually $\{\mathsf{L},\mathsf{\Gamma}\}$-graphs, or even $\{\mathsf{L}\}$-graphs. We proved that this is indeed the case for several graph classes related to planar graphs. We feel that the question whether PLANAR $\subset \{\mathsf{L},\mathsf{\Gamma}\}$, as already conjectured [8], is of particular importance. After all, this, together with Theorem 1, would give a new proof for the fact that PLANAR $\subset$ SEG.

Open Problem 1 *Each of the following is open.*

(i) *When can a* B_1-VPG-*representation be stretched into a combinatorially equivalent* SEG-*representation?*
(ii) *Is* $\{\mathsf{L},\mathsf{\Gamma}\} = \text{SEG} \cap B_1\text{-VPG}$*?*
(iii) *Is every planar graph an* $\{\mathsf{L}\}$-*graph, or* B_1-VPG-*graph?*
(iv) *Does every planar graph admit an even subdivision whose complement is an* $\{\mathsf{L}\}$-*graph, or* B_1-VPG-*graph?*
(v) *Recognizing* B_k-*VPG graphs is known to be NP-complete for each* $k \geq 0$ *[6]. What is the complexity of recognizing* $\{\mathsf{L}\}$-*graphs, or* $\{\mathsf{L},\mathsf{\Gamma}\}$-*graphs?*

Sometimes it is of particular interest to find SEG-representations using only few different slopes for the segments. While bipartite and triangle-free planar graphs are (contact) SEG-graphs using only two slopes [12] and three slopes [9], respectively, open conjectures of Scheinerman [25] and West [27] state, that 3-colorable and general planar graphs have SEG-representations with only 3 slopes and 4 slopes, respectively. Can Middendorf and Pfeiffer's theorem be used to obtain SEG-representations with few slopes?

Recall that $\cup_{k \geq 0} B_k$-VPG = STRING [3]. Chaplick *et al.* [6] prove that B_k-VPG $\subsetneq B_{k+1}$-VPG for all $k \geq 0$ and also that SEG $\not\subseteq B_k$-VPG for each $k \geq 0$, even if SEG is restricted to three slopes only. Another natural subclass of STRING, which is in no inclusion-relation with SEG, is the class COCOMP of co-comparability graphs [16]. However, one can prove a result similar to the previous one concerning B_k-VPG-graphs and STRING-graphs:

There is no $k \in \mathbb{N}$ such that B_k-VPG $\supset$ COCOMP. A proof can be given along the "degrees of freedoms" approach of Alon and Scheinerman [2], i.e., by counting the graphs in the respective sets.

First, Alon and Scheinerman consider the number $P(n,t)$ of t-dimensional posets on n elements. They show that for fixed t the growth of the logarithm of this number behaves like $nt \log n$. Let $CC(n,t)$ be the number of cocomparability graphs of t-dimensional posets on n elements. With easy adaptations of the Alon and Scheinerman proof we obtain $\log CC(n,t) \geq n(t-1-o(1)) \log n$.

Every path of a B_k-VPG-representation can be encoded by $k+3$ numbers. The question whether two paths intersect can be answered by looking at the signs of few low degree polynomials in $2k+6$ variables evaluated at the encodings of the two paths. This means that the class B_k-VPGhas $k+3$ degrees of freedom.

Alon and Scheinerman show how to use Warren's Theorem [26] to get an upper bound on the size of such a class. The logarithm of the number of B_k-VPGgraphs on n vertices is $\leq O(1)nk \log n$.

Comparing the numbers we find that there are cocomparability graphs of $(k+2)$-dimensional posets that have no B_k-VPG-representation. On the other hand it is easy to see, that the co-comparability graph of a d-dimensional poset is a B_{d-1}-VPG-graph. Is there a similar parameter ensuring few bends in representations of SEG-graphs? We know that the number of slopes in the SEG-representation is not the right answer here.

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