Exploiting Air-Pressure to Map Floorplans on Point Sets

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Abstract. We prove a conjecture of Ackerman, Barequet and Pinter. Every floorplan with n segments can be embedded on every set of n points in generic position. The construction makes use of area universal floorplans also known as area universal rectangular layouts.

The notion of area used in our context depends on a non-uniform density function. We, therefore, have to generalize the theory of area universal floorplans to this situation. For the proof we use the air-pressure approach of Izumi, Takahashi and Kajitani. The method is then used to prove a result about accommodating points in floorplans that is slightly more general than the original conjecture. We close with some remarks on the counting problem that motivated the conjecture of Ackerman et al.

Mathematics Subject Classifications (2010) 05B45, 05C10, 52C15.

1 Introduction

In our context a *floorplan* is a dissection of a rectangle R into a finite set of interior-disjoint rectangles. A floorplan is *generic* if it has no cross, i.e., no point where four rectangles of the partition meet. A *segment* of a floorplan is a maximal nondegenerate interval that belongs to the union of the boundaries of the rectangles. In general we disregard the four segments from the boundary of R, i.e., we only consider inner segments. Segments are either horizontal or vertical. The segments of a generic floorplan are internally disjoint. Two floorplans F and F' are *weakly equivalent* if there exist bijections $\phi : S_H(F) \to S_H(F')$ and $\phi : S_V(F) \to S_V(F')$ between their horizontal and vertical segments such that segment s has an endpoint on segment t in F iff $\phi(s)$ has an endpoint on $\phi(t)$. A set P of points in $\mathbb{R}^2$ is *generic* if no two points from P have the same x or y coordinate. Section 2 provides a more comprehensive overview of definitions and notions related to floorplans.

Let P be a set of n points in a rectangle R and let F be a generic floorplan with n segments. A *cover map* from F to P is a floorplan F' that is weakly equivalent to F and has outer rectangle R such that every segment of F' contains exactly one point from P. Figure 2 shows an example.

In this paper we answer a question of Ackerman et al. [1] by proving Theorem 1. The proof of this theorem, as well as some variants and generalizations of it, is the subject of Section 4.
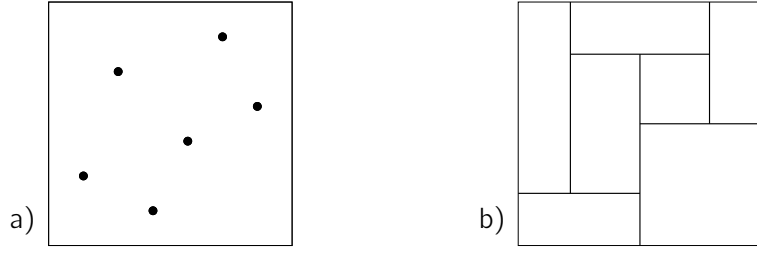

Figure 1: A generic set of six points and a generic floorplan with six segments.
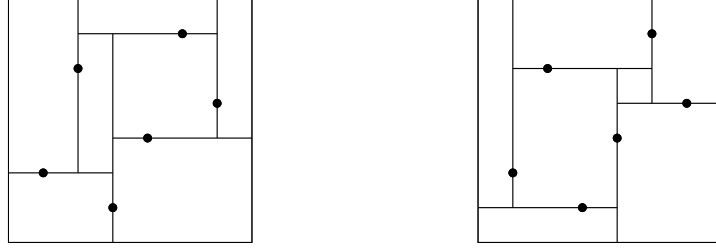

Figure 2: Two cover maps from the floorplan of Fig. 1.b to the point set of Fig. 1.a.

Theorem 1 *If P is a generic set of n points and F is a generic floorplan with n segments, then there is a cover map from F to P.*

The proof is based on results about area representations of floorplans. The following theorem is known, it has been proven with three quite different methods, see [15, 12, 6].

Theorem 2 *Let F be a floorplan with rectangles $R_1, \ldots, R_{n+1}$, let A be a rectangle and let $w : \{1, \ldots, n+1\} \to \mathbb{R}_+$ be a weight function with $\sum_i w(i) = \mathsf{area}(A)$. There exists a unique floorplan F' contained in A that is weakly equivalent to F such that for all i the area of the rectangle $R'_i = \phi(R_i)$ is exactly $w(i)$.*

In Section 3 we prove the generalization of Theorem 2 that will be needed for the proof of Theorem 1. In the generalized theorem (Theorem 3) the weight of a rectangle is measured as an integral over some density function.

2 Floorplans and Graphs

A *floorplan* is a partition of a rectangle into a finite set of interiorly disjoint rectangles. From a given floorplan F we can obtain several graphs and additional structure. We introduce some of these and close the section by introducing two types of equivalence for floorplans.

The skeleton graph. The *skeleton* $\mathsf{skel}(F)$ of a floorplan F is the union of the boundaries of the rectangles in the partition. Let $C(F)$ be the set of all corners of rectangles of F. The *skeleton graph* $G_{\mathsf{skel}}(F)$ of F has the corners of rectangles of F as vertices, i.e., $V(G_{\mathsf{skel}}(F)) = C(F)$. The edges of $G_{\mathsf{skel}}(F)$ are the connecting line segments. More formally, the edges correspond to the connected components of $\mathsf{skel}(F) \setminus C(F)$. The skeleton graph $G_{\mathsf{skel}}(F)$ has four vertices of degree two incident to the outer face. All the other vertices are of degree 3 or 4. If F is generic, then there are no vertices of degree 4.

The rectangular dual. Let $R(F)$ be the set of rectangles of a floorplan F. It is convenient to include the enclosing rectangle in the set $R(F)$. The *rectangular dual* of F is the graph $G^*(F)$ with vertex set $R(F)$ and edges joining pairs of rectangles that share a boundary segment. Usually the

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notion of a rectangular dual is used in the other direction, i.e., it is assumed that a planar graph G is given and the quest is for a floorplan F such that $G = G^*(F)$. It is tempting to think that the graph $G^*(F)$ is the dual graph of $G_{\text{skel}}(F)$. However, due to some issues about the multiplicity of edges incident to the outer face of $G_{\text{skel}}(F)$, i.e., to the enclosing rectangle, this is not true in general. For convenience a floorplan F is often extended with four rectangles that frame F as shown in Figure 3. In the dual of an extended floorplan F_+ omit the vertex that corresponds to the enclosing rectangle. With this twist in the definition of the dual we have: The dual $G_+^*(F)$ of the extended floorplan of a generic F is a 4-connected inner triangulation of a 4-gon. Indeed this is the characterization of the duals of extended generic floorplans. Buchsbaum et al. [3] and Felsner [7] provide many pointers to the literature related to floorplans and rectangular duals.
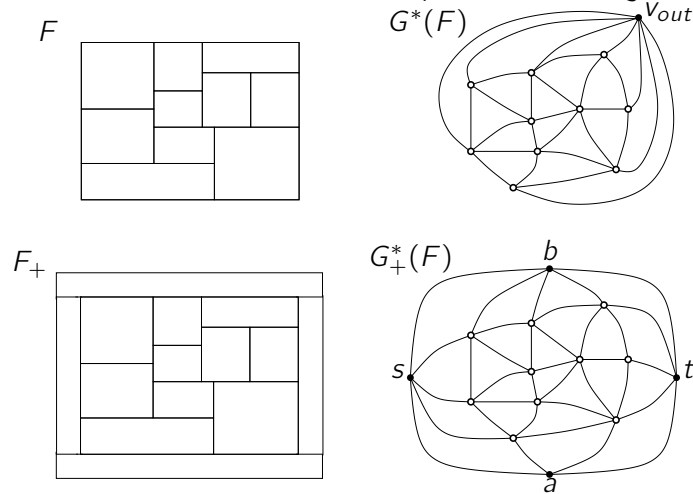

Figure 3: A floorplan F, the extended floorplan F_+, and their duals.

The transversal structure. The transversal structure (also known as regular edge labeling) associated with a floorplan F is an orientation and coloring of the edges of the extended dual $G_+^*(F)$.

Let G be an inner triangulation of a 4-gon with outer vertices s, a, t, b in counterclockwise order. A *transversal structure* for G is an orientation and 2-coloring of the inner edges of G such that

(1) All edges incident to s, a, t and b are blue outgoing, red outgoing, blue ingoing, and red ingoing, respectively.

(2) The edges incident to an inner vertex v come in clockwise order in four nonempty blocks consisting solely of red ingoing, blue ingoing, red outgoing, and blue outgoing edges, respectively.

Lemma 1 *A floorplan F induces a transversal structure on $G_+^*(F)$.*

The idea is to color the duals of horizontal edges of $G_{\text{skel}}(F)$ red and orient them upward and to color the duals of vertical edges of $G_{\text{skel}}(F)$ blue and orient them from left to right. See Figure 4. This coloring and orientation of edges is a transversal structure of $G_+^*(F)$

Lemma 2 *Every transversal structure of an inner triangulation G of a 4-gon with outer vertices s, a, t, b is induced by a floorplan F with $G = G_+^*(F)$*

Transversal structures have been studied in [10], [11], and in [13]. A proof of Lemma 2 can e.g. be found in [7].

The segment contact graph. Recall that we call a floorplan *generic* if it has no cross, i.e., no point where four rectangles meet. A *segment* of a floorplan is a maximal nondegenerate interval that
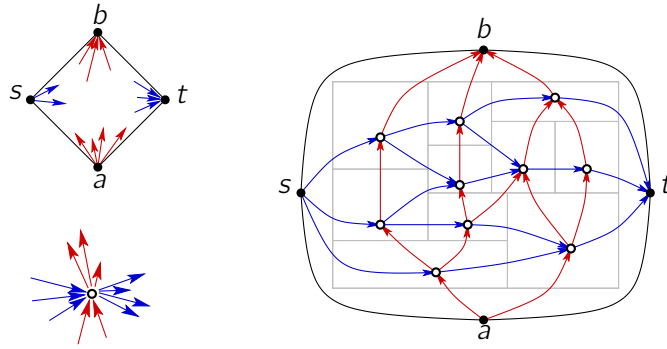
Figure 4: The two local conditions and an example of a transversal structure together with a corresponding floorplan.

belongs to the union of the boundaries of the rectangles. In the generic case intersections between segments only occur between horizontal and vertical segments and they involve an endpoint of one of the segments, i.e., they are contacts. If a floorplan has a cross at point p we can break one of the two segments that contain p into two to get a system of interiorly disjoint segments.

The *segment contact graph* $G_{\text{seg}}(F)$ of a floorplan F is the bipartite planar graph whose vertices are the segments of F and edges correspond to contacts between segments. From Figure 5 we see that $G_{\text{seg}}(F)$ is indeed planar and that the faces of $G_{\text{seg}}(F)$ are in bijection with the rectangles of F and are uniformly of degree 4. Therefore $G_{\text{seg}}(F)$ is a maximal bipartite planar graph, i.e., a quadrangulation.
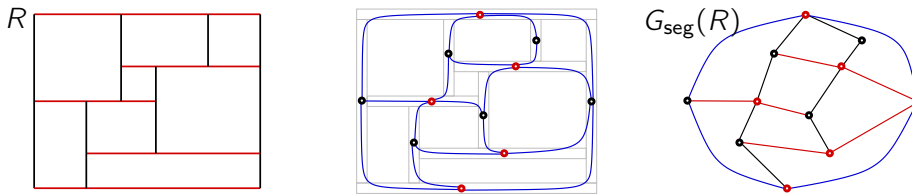

Figure 5: A floorplan F and two drawings of its segment contact graph $G_{\text{seg}}(F)$.

The separating decomposition. The separating decomposition associated to a floorplan is an orientation and coloring of the edges of the segment contact graph.

Let Q be a quadrangulation, we call the color classes of the bipartition white and black and name the two black vertices on the outer face s and t. A *separating decomposition* of Q is an orientation and coloring of the edges of Q with colors red and blue such that two conditions hold:

(1)   All edges incident to s are ingoing red and all edges incident to t are ingoing blue.

(2)   Every vertex $v \neq s, t$ is incident to a nonempty interval of red edges and a nonempty interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval of a color is outgoing and all the other edges of the interval are incoming. If v is black, the outgoing edge is the last one of its color in clockwise order (see Figure 6).
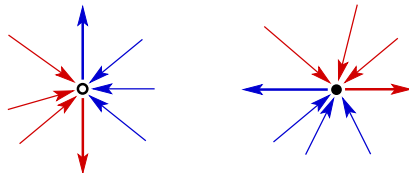

Figure 6: Edge orientations and colors at white and black vertices.

Separating decompositions have been studied in [5], [9], and [8]. To us they are of interest because of the following two lemmas.

Lemma 3 *A floorplan induces a separating decomposition on its segment contact graph* $G_{seg}(F)$.

Proof. Let $Q = G_{seg}(F)$ and let s be the horizontal bottom segment and t be the horizontal top segment of F. Consider them to be sligthly extended, so that their endpoints have no contact with another segment. Since s is horizontal all neighbors of s have to be vertical. From the choice of s and t as horizontal it follows that we want to think of the vertices of Q that correspond to horizontal segments as black vertices and vertices corresponding to vertical segments as white vertices.

An edge of Q corresponds to a contact where an endpoint of one segment is touching the interior of another segment. Orient each edge such that the vertex contributing the endpoint is its tail. This yields a 2-orientation of Q, i.e., an orientation where every vertex except for s and t has out-degree two. Color the edge corresponding to the left contact of a horizontal segment blue and the edge of the right contact red. Similarly, the edge induced by the top contact of a vertical segment is colored blue and the edge of the bottom contact is colored red. This construction yields a separating decomposition of Q. For an example see Figure 7. □
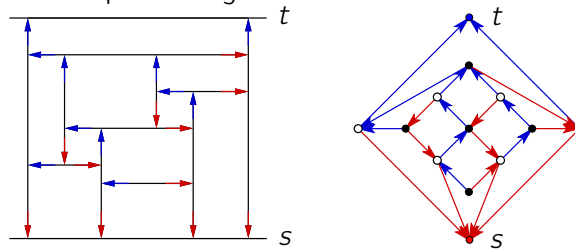

Figure 7: A floorplan F and the separating decomposition induced by F on its segment contact graph Q.

Proofs of the following lemma can be found in [7].

Lemma 4 *Every separating decomposition of a planar quadrangulation Q is induced by a floorplan F with $Q = G_{seg}(F)$.*

2.1 Notions of equivalence for floorplans

Definition 1 *Two floorplans are* weakly equivalent *if they induce the same separating decomposition.*

Definition 2 *Two floorplans are* strongly equivalent *if they induce the same transversal structure.*

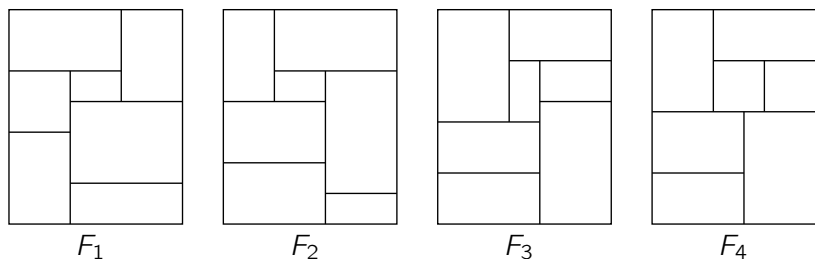**Definition 3** *Two floorplans are* dual equivalent *if they have the same rectangular dual.*

Figure 8: Four floorplans such that F_1, F_2 are dual equivalent, F_2, F_3 are weakly equivalent and F_3, F_4 are dual equivalent. There are no further equivalences.

Strong equivalence clearly implies dual equivalence. The following proposition is less trivial.

Proposition 1 *Strong equivalence implies weak equivalence.*

Proof (sketch). Define a relation on the edges of a transversal structure: $(u, v) \sim (x, y)$ iff the color of the two oriented edges is the same and $u = x$ or $v = y$. The classes of the transitive closure of this relation correspond to the segments of the floorplan. Moreover, if we look at the triangles incident to the edges of a generic red class (not in correspondence to a boundary segment) we see an arbitrary number of triangles with two red and one blue edge but exactly two triangles with two blue and one red edge. The blue edges of these two triangles belong to the classes of the out edges of a 2-orientation of G_{seg}. Since a separating decomposition is uniquely determined by the underlying 2-orientation all floorplans of a strong equivalence class belong to the same weak equivalence class. □

In the introduction we said that two floorplans F and F' are weakly equivalent if there exist bijections $\phi : S_H(F) \to S_H(F')$ and $\phi : S_V(F) \to S_V(F')$ between their horizontal and vertical segments such that segment s has an endpoint on segment t in F iff $\phi(s)$ has an endpoint on $\phi(t)$. We want to show that this yields the same equivalence classes as Definition 1. Clearly, if F and F' induce the same separating decomposition then they are weakly equivalent in the above sense. For the converse we start with two observations. First, observe that the segment contact graphs of F and F' are isomorphic, i.e., $G_{\text{seg}}(F) = G_{\text{seg}}(F')$.

Now define an orientation Q_F on $Q = G_{\text{seg}}(F)$ by orienting s to t iff segment s has an endpoint on segment t. Let $Q_{F'}$ be the orientation defined on Q using the segments of F'. Now observe that $Q_F = Q_{F'}$.

Since a separating decomposition is uniquely determined by the underlying 2-orientation (see [5] or [8]) we conclude that F and F' induce the same separating decomposition, i.e., $SD_F = SD_{F'}$. This implies that F and F' are weakly equivalent in the sense of Definition 1.

Eppstein et al. [6] use the term *layout* instead of floorplan. Their equivalent layouts correspond to strongly equivalent floorplans and order-equivalent layouts to weakly equivalent floorplans. Asinowski et al. [2] study independent notions of R-equivalence and S-equivalence for floorplans.

3 Realizing Weighted Floorplans via Air-Pressure

In this section we prove a generalization of Theorem 2 to situations where the "area" of a rectangle is replaced by the mass defined through a density distribution.

Let $\mu : [0, 1]^2 \to \mathbb{R}_+$ be a density function on the unit square whose total mass is 1, i.e., $\int_0^1 \int_0^1 \mu(x, y) dx dy = 1$. We assume that μ can be integrated over axis aligned rectangles and all fibers μ_x and μ_y can be integrated over intervals. Moreover, for all $0 \le a < b \le 1$ the function $g_{a,b}(x) = \int_a^b \mu_x(y) dy$ is continuous in x and $g_{a,b}(y) = \int_a^b \mu_y(x) dx$ is continuous in y. We also require that integrating μ over a nondegenerate rectangles and fibers over nondegenerate intervals always yield positive values. The *mass* of an axis aligned rectangle $R \subseteq [0, 1]^2$ is defined as $m(R) = \iint_R \mu(x, y) dx dy$.

Theorem 3 *Let* $\mu : [0, 1]^2 \to \mathbb{R}_+$ *be a density function of total mass 1. If F is a floorplan with rectangles* $R_1, \ldots, R_{n+1}$ *and* $w : \{1, \ldots, n+1\} \to \mathbb{R}_+$ *a positive weight function with* $\sum_1^{n+1} w(i) = 1$ *then there exists a unique floorplan F' in the unit square that is weakly equivalent to F such that* $m(R_i) = w(i)$ *for each rectangle* R_i.

Our proof follows the air-pressure paradigm as proposed by Izumi, Takahashi and Kajitani [12]. We first describe the idea. Consider a realization of F in the unit square and compare the mass $m(R_i)$ to the intended mass $w(i)$. The quotient of these two values can be interpreted as the pressure inside the rectangle. Integrating this pressure along a side of the rectangle yields the force by which R_i is pushing against the segment that contains the side. The difference of pushing forces

from both sides of a segment yields the effective force acting on the segment. The intuition is that shifting a segment in direction of the effective force yields a better balance of pressure in the rectangles. We will show that iterating such improvement steps drives the realization of F towards a situation with $m(R_i) = w(i)$ for all i, i.e., the procedure converges towards the floorplan F' whose existence we want to show.

In [12] the air-pressure paradigm was used for situations where the mass of a rectangle is its area. The authors observed fast convergence experimentally but they had no proof of convergence. Here we provide such a proof for the more general case of weights given by integrals over a density function.

A proof of Theorem 3 can also be given along the lines of the proof of Theorem 2 by Eppstein et al. in [6]. We digress to sketch the argument:

We restrict the considerations to floorplans in the unit square. A floorplan F with n segments can be encoded by a vector $z_F \in \mathbb{R}^n$. The vectors of the weak equivalence class of F are the points of an open polytope P_F°. Points on the boundary ∂P_F° of P_F° are degenerate floorplans, i.e., floorplans that have rectangles of width or height 0. Define a map $W : P_F^\circ \to \mathbb{R}^n$ such that $W(z)_i = m(R_i(F_z))$, i.e., the i-th coordinate of $W(z)$ is given by the mass of the i-th rectangle[1] in the floorplan F_z corresponding to z. The map W is contineous and injective. *Invariance of domain* implies that W is a homeomorphism. Consider the segment σ with endpoints $W(z_F)$ and the vector w of intended masses. The pullback $W^{-1}(\sigma)$ avoids ∂P_F°. Hence, there exists some $z^* \in P_F^\circ$ with $W(z^*) = w$ and $F^* = F_{z^*}$ is the floorplan we look for.

This approach has been detailed by Schrezenmaier [14]. The resulting proof is quite compact, however, it has the disadvantage of being purely existential. Schrezenmaier also has a java implementation of the air-pressure approach that solves instances of moderate size quickly.

Let $R_i = [x_l, x_r] \times [y_b, y_t]$ be a rectangle of F. Recall that the mass of R_i is $m(R_i) = \int_{x_l}^{x_r} \int_{y_b}^{y_t} \mu(x, y) dy dx$. The pressure $p(i)$ in R_i is the fraction of the intended mass $w(i)$ and the actual mass $m(R_i)$, i.e., $p(i) = \frac{w(i)}{m(R_i)}$. Let s be a segment of F and let R_i be one of the rectangles with a side in s. Let s be vertical with x-coordinate x_s and let $s \cap R_i$ span the interval $[y_b(i), y_t(i)]$. The (undirected) *force imposed on s by R_i* is the pressure $p(i)$ of R_i times the density dependent length of the intersection.

$$f(s, i) = \frac{w(i)}{m(R_i)} \int_{y_b(i)}^{y_t(i)} \mu(x_s, y) dy = p(i) \int_{y_b(i)}^{y_t(i)} \mu_{x_s}(y) dy.$$

The *force acting on s* is obtained as a sum of the directed forces imposed on s by incident rectangles.

$$f(s) = \sum_{R_i \text{ left of } s} f(s, i) - \sum_{R_i \text{ right of } s} f(s, i).$$

Symmetric definitions apply to horizontal segments.

Balance for rectangles and segments

Definition 4 *A segment s is in* balance *if $f(s) = 0$.*
A rectangle R_i is in balance *if $p(i) = 1$, i.e., if $m(R_i) = w(i)$.*

Lemma 5 *If all rectangles R_i of F are in balance, then all segments are in balance.*

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[1] The $n + 1$-st rectangle is ignored because its mass is determined by the others.

Proof. Since all rectangles are in balance we can eliminate the pressures from the definition of the $f(s, i)$. With this simplification we get for a vertical segment s

$$f(s) = \sum_{R_i \text{ left of } s} \int_{y_b(i)}^{y_t(i)} \mu_{x_s}(y)dy - \sum_{R_j \text{ right of } s} \int_{y_b(j)}^{y_t(j)} \mu_{x_s}(y)dy.$$

Hence $f(s) = M_s - M_s = 0$, where M_s is the integral of the fiber density μ_{x_s} along s. The symmetric argument applies to horizontal segments. $\qquad\square$

Interestingly, the converse of the lemma also holds.

Proposition 2 *If all segments of F are in balance, then all rectangles are in balance.*

Proof. Suppose that F balances all segments but not all rectangles. Choose some τ with $\min_i p(i) < \tau \leq \max_i p(i)$. Let T_τ be the union of all rectangles R_i whose pressure exceeds τ and let Γ_τ be the boundary of T_τ.

Claim. The boundary Γ_τ of T_τ contains no segment, i.e., $s \setminus \Gamma_\tau \neq \emptyset$ for all s.

Suppose Γ_τ contains the vertical segment s such that T_τ is left of s. Let I be a nontrivial interval on s that is defined as the intersection of a rectangle R_i that has its right edge on s and a rectangle R_j that has its left edge on s. The force acting on s along I is $p(i) \int_I \mu_{x_s}(y)dy - p(j) \int_I \mu_{x_s}(y)dy$. Since $\int_I \mu_{x_s}(y)dy > 0$ by our assumption on μ and $p(i) > p(j)$ by definition of T_τ, the force is positive. This holds for every interval I on s. Thus, the overall force $f(s)$ acting on s is also positive. This contradicts the assumption that s is in balance and completes the proof of the claim. $\qquad\triangle$

Let s_0 be any segment which contributes to Γ_τ. From the lemma we know that at some interior point of segment s_0 the boundary leaves s_0 and continues along another segment s_1. Again, the boundary has to leave s_1 at some interior point to continue on s_2. Because this procedure always follows the boundary of T_τ which is a region defined by a union of rectangles in F the sequence of segments has to get back to segment s_0, i.e., there is an index k such that $s_k = s_0$.

From the definition of the separating decomposition SD_F corresponding to F we find that $s_0 \leftarrow s_1 \leftarrow s_2 \leftarrow \ldots \leftarrow s_{k-1} \leftarrow s_0$ is a directed cycle in SD_F. The four segments of the enclosing square of F do not contribute to the boundary of T_τ simply because they cannot belong to a directed cycle of SD_F.

Recall the assumption that F balances all segments but not all rectangles. Let s be the vertical segment with maximal x-coordinate among all vertical segments that contribute to a boundary Γ_τ for some τ. From the choice of s it is clear that T_τ is to the left of s. Let $s = s_0$ and consider the cycle $s_0 \leftarrow \ldots \leftarrow s_{k-1} \leftarrow s_0$ corresponding to Γ_τ in SD_F and define s' as $s' = s_{k-1}$. The subsegment of s' to the left of the contact point p of s is part of the boundary Γ_τ. From the choice of τ and s it follows that to the right of p the rectangles on both sides of s' have the same pressure $p(i)$. Otherwise the right part of s' would belong to some boundary $\Gamma_{\tau'}$ and the vertical segment following s' on $\Gamma_{\tau'}$ is in contradiction the choice of s.

Now consider $f(s')$ and split the contributions to this force at p. On the left of p the pressure on the side of T_τ exceeds the pressure from the other side. On the right of p the rectangles on both sides of p have the same pressure. This contradicts our assumptions since now we have $f(s') \neq 0$. This completes the proof of the proposition. $\qquad\square$

Balancing segments and optimizing the entropy

Proposition 3 *If a segment s of F is unbalanced, then we can keep all the other segments at their position and shift s parallel to a position where it is in balance. The resulting floorplan F' is weakly equivalent to F.*

Proof. We consider the case of a vertical segment, the horizontal case is symmetric. Let x_s be the x-coordinate of s. With S_- and S_+ we denote the sets of rectangles in F that touch s from the left and right respectively. Let R_l be the rectangle with a left boundary of maximal x-coordinate x_l in S_- and let R_r be the rectangle with a right boundary of minimal x-coordinate x_r in S_+. Note that if t satisfies $x_l < t < x_r$ then segment s can be shifted parallel to the position $x_s = t$ and the resulting floorplan is weakly equivalent to F.

For $t \in (x_l, x_r)$ we define $h(t)$ as the force acting on s when the segment it is shifted to $x_s = t$. We observe:

- The pressure $p(i)$ depends continuously on t for all rectangles $R_i \in S_- \cup S_+$.

- The value of $\int_I \mu_t(y)dy$ is a continuous function of t for all nondegenerate intervals I.

Hence, $h(t)$ is a continuous function. With t approaching x_l from the right the area of R_l tends to zero. Hence, the mass $m(R_l)$ also tends to zero and the pressure $p(l)$ tends to infinity. Since $\int_{y_b(l)}^{y_t(l)} \mu_t(y)dy > 0$ we conclude that $h(t) \to +\infty$ with $t \to x_l$. Similar reasoning involving R_r shows that $h(t) \to -\infty$ with $t \to x_r$. It follows that there is some $t_0 \in (x_l, x_r)$ with $h(t_0) = 0$. This shows that upon shifting s to the position $x_s = t_0$ the force acting on s vanishes and s is in balance. $\square$

Definition 5 *The* entropy *of a rectangle R_i of F is defined as $-w(i)\log p(i)$. The* entropy *of the floorplan F is*

$$E = \sum_i -w(i)\log p(i)$$

The proof of Theorem 3 will be completed after showing the following

(1) The entropy E is always nonpositive.

(2) $E = 0$ if and only if all rectangles R_i of F are in balance.

(3) Shifting an unbalanced segment s into its balance position increases the entropy.

(4) The process of repeatedly shifting unbalanced segments into their balance position makes F converge to a floorplan F' such that the entropy of F' is zero.

(5) The solution is unique.

The first two of these statements are shown in the next lemma.

Lemma 6 *The entropy E is always nonpositive and $E = 0$ if and only if all rectangles R_i of F are in balance.*

Proof. We use that $p(i) > 0$ and hence $\log p(i) \geq (1 - \frac{1}{p(i)}) = (1 - \frac{m(R_i)}{w(i)})$. For the entropy of R_i we get $-w(i)\log p(i) \leq -w(i)(1 - \frac{m(R_i)}{w(i)}) = m(R_i) - w(i)$. This yields

$$E = \sum_i -w(i)\log p(i) \leq \sum_i m(R_i) - \sum_i w(i) = 1 - 1 = 0$$

The equality $E = 0$ is equivalent to equality for each summand. Hence $0 = \log p(i) = (1 - \frac{m(R_i)}{w(i)})$ and $m(R_i) = w(i)$ for all i. $\square$

Lemma 7 *Shifting an unbalanced segment s into its balance position increases the entropy.*

Proof. We consider a vertical segment s as in the proof of Proposition 3 and assume $f(s) > 0$. Let t_0 be the first zero of $h(t)$ right of x_s. For all $t \in [x_s, t_0)$ the force $h(t)$ acting on s is positive, i.e., pushing s to the right.

Let $E(t)$ be the entropy of the floorplan when s is shifted to $x_s = t$. We consider $E(t)$ as a function of t.

Claim. $\dfrac{d}{dt}E(t) = h(t)$.

Only rectangles touching s change their contribution to $E(t)$. Let $R_i = [x_l, t] \times [y_1, y_2]$ be a rectangle on the left of s, i.e., $R_i \in S_-$, and t is the x-coordinate of the right side of R_i. Hence

$$\frac{d}{dt}(-w(i)\log p(i)) \;=\; -w(i)\frac{1}{p(i)}p'(i) \;=\; -w(i)\frac{m(R_i)}{w(i)}\frac{d}{dt}\frac{w(i)}{m(R_i)} \;=\;$$

$$-w(i)\frac{m(R_i)}{w(i)}w(i)\frac{-m'(R_i)}{m^2(R_i)} = \frac{w(i)}{m(R_i)}m'(R_i) = \frac{w(i)}{m(R_i)}\frac{d}{dt}\int_{x_l}^{t}\int_{y_1}^{y_2}\mu(x,y)\,dy\,dx =$$

$$\frac{w(i)}{m(R_i)}\int_{y_1}^{y_2}\mu_t(y)\,dy \;=\; p(i)\int_{y_1}^{y_2}\mu_t(y)\,dy.$$

When $R_i \in S_+$ the mass $m(R_i)$ is decreasing with t so that $m'(R_i)$ is negative and $\frac{d}{dt}(-w(i)\log p(i)) = -p(i)\int_{y_1}^{y_2}\mu_t(y)\,dy$. Summing this over all rectangles incident to s we obtain that $\frac{d}{dt}E(t) = h(t)$. This is the claim. $\triangle$

While shifting s from the initial position x_s to t_0 we have $h(t) > 0$. The claim implies that the derivative of the entropy is positive and, hence, the entropy is increasing. $\square$

We continue with item (4) from our program. To this aim, however, we have to add a condition to the process of balancing segments. The iteration has to be performed such that no unbalanced segment can be ignored. A rule is called[2] *nonignoring* if it complies with this condition. Here are two examples of nonignoring selection rules:

- Choose the segment for balancing uniformly at random from the set of unbalanced segments.

- Always choose the segment so that the increase of the entropy is as large as possible.

Proposition 4 *Let $F_0, F_1, F_2, \ldots$ be a sequence of floorplans where F_{i+1} is obtained from F_i by balancing an unbalanced segment from F_i. If the selection of segments is nonignoring, then there is a subsequence $G_0, G_1, \ldots$ of floorplans that has a limit $G = \lim G_i$ and the entropy of the floorplan G is zero.*

Proof. Enumerate the inner segments of F as $s_1, s_2, \ldots, s_n$. A floorplan that is weakly equivalent to F can be encoded by the coordinate vector of the segments. This vector z in $\mathbb{R}^n$ has the value $z(i) = x_s$ if s_i is a vertical segment and $z(i) = y_s$ if s_i is horizontal. A sequence of floorplans is converging if the corresponding coordinate vectors converge in $\mathbb{R}^n$.

Consider the sequence of coordinate vectors $z_0, z_1, \ldots$ of the given sequence of floorplans. Since each of the coordinates of these vectors is from the interval $(0, 1)$, there is a convergent subsequence. Let $G_0, G_1, \ldots$ be the corresponding convergent sequence of floorplans and let e_i be the entropy of G_i. From Lemma 7 we know that the e_i form an increasing sequence of negative numbers. Hence, they converge to some value $-a$. The task is to show that $a = 0$.

Assume that the sequence $(e_i)_i$ converges to $-a \neq 0$. Consider the limit $G = \lim G_i$. Since the entropy of G is $-a < 0$ there is an unbalanced rectangle R_j in G (Lemma 6) and, hence, there is an unbalanced segment s in G (Proposition 2). Let Δ be the increase of the entropy that comes from balancing s in G. Now, for all i greater than a sufficiently large N the floorplan G_i is so close to G that balancing s in G_i implies an increase of entropy of at least $\Delta/2$. For all i greater than a

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[2]A formal definition can be extracted form the use of the notion in the proof.

sufficiently large M we also have $e_i > -a - \Delta/2$. It follows that the unbalanced segment s was not used for balancing in any G_i with $i \geq \max(M, N)$. This is in contradiction to the assumption that the process is nonignoring. $\qquad\square$

Actually, a stronger statement is true. The full sequence $F_0, F_1, F_2, \ldots$ is also converging. To prove this we need the uniqueness shown in Proposition 5 below. In fact if G is the unique floorplan that is weakly equivalent to F and has $m(R_i) = w(i)$ for all i, then it follows from the continuity of the entropy that there is an $\varepsilon > 0$ such that all floorplans whose entropy is larger than $-\varepsilon$ have a coordinate vector that is δ_ε close to the coordinate vector of G. This implies that $\lim F_i = G$.

Proposition 5 *For every floorplan F with $n+1$ rectangles and every positive weight function w : $\{1, \ldots, n+1\} \to \mathbb{R}_+$ with $\sum_i w(i) = 1$ there is a unique floorplan F' in the unit square that is weakly equivalent to F and has $m(R_i) = w(i)$ for all i.*

The proof of the proposition given here is a variant of the uniqueness proof given by Eppstein et al. [6].

Proof. Suppose that F' and G' are different floorplans that are weakly equivalent to F and both realize the weight function w. The argument will be based on an auxiliary graph A. The vertices of A are all rectangles of F and the segments that have different entries in the coordinate vectors $z_{F'}$ and $z_{G'}$ of F' and G'. The edges are defined as follows: If s and R are incident and shifting s from its position in F' to its position in G' moves s into R, then $s \to R$ in A. If the shift is moving s away from R, then $R \to s$ in A. The graph A can be viewed as a subgraph of the angle graph of $G_{\mathrm{seg}}(F)$, see Figure 9. In particular, A is bipartite and planar. We fix a plane embedding of $G_{\mathrm{seg}}(F)$ and consider A with the inherited drawing.
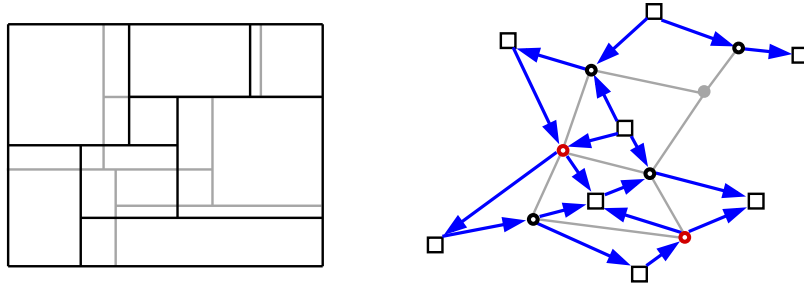

Figure 9: Floorplans F' in black and G' in gray together with the induced graph A and some edges of $G_{\mathrm{seg}}(F)$ in gray.

If one of the rectangle vertices of A is a source or a sink, then the representations of this rectangle in F' and G' are contained in each other. Hence, the mass of the rectangle in F' and G' can not be the same and the two floorplans do not realize the same weight function.

To prove the uniqueness it thus suffices to show that if A has edges, then it contains a source or a sink. Assume not, then A contains a cycle. Let C be a directed cycle with a minimal enclosed region in the drawing of A.

Claim. There is no edge e in the interior of C that belongs to A.

We show that such an edge e would imply the existence of a cycle enclosing a smaller region: This is obvious if head and tail of e belong to C, i.e., e is an inner chord for C. If the head of e is not on C we grow a forward path (A has no sink) until we reach C or close a cycle. If the tail of e is not on C we can grow a backward path (A has no source) until we reach C or close a cycle. If forward or backward path close a cycle this cycle encloses a smaller area, contradiction. We also find a smaller cycle if the two paths intersect. In the remaining case the two paths together form a chordal paths for C and again there is a smaller cycle. This proves the claim.

Assuming that C is a clockwise directed cycle we find a subpath $R' \to s \to R \to s'$ of C such that s is vertical, s' is horizontal, the left boundary of R is on s, and R's bottom boundary is on s'. All rectangles whose right boundary is on s have an edge pointing towards s and s has an edge pointing towards all rectangles with left boundary on s. Now R' and R are the lowest rectangles incident to s on their respective sides, otherwise there would be an edge e in the interior of C, a contradiction.

However, if R' and R are the lowest rectangles incident to s on their respective sides, then they share the segment defining their lower sides. Hence $R' \to s'$ is an edge in the interior of the cycle. Again a contradiction. $\qquad\square$

4 Accommodating Floorplans on Point Sets

Let P be a generic set of n points in a rectangle $\mathcal{R}$. Let F be a generic floorplan and S be a subset of the segments of F of size n. A *cover map* from (F, S) to P is a floorplan F' with outer rectangle $\mathcal{R}$ that is weakly equivalent to F such that every segment from $S' = \phi(S)$ contains exactly one point from P. The main result in this paper is the following generalization of Theorem 1.

Theorem 4 *If P is a generic set of k points in a rectangle $\mathcal{R}$ and F is a generic floorplan with $n > k$ segments and a prescribed subset S of the segments of size k, then there is a cover map F' from (F, S) to P.*

Proof. The idea is to use Theorem 3 as a tool for the proof. To this end we first transform the point set P into a suitable density distribution $\mu = \mu_P$ inside $\mathcal{R}$. This density is defined as the sum of a uniform distribution μ_1 with $\mu_1(q) = 1/\mathbf{area}(\mathcal{R})$ for all $q \in \mathcal{R}$ and a distribution μ_2 that represents the points of P. Choose some $\Delta > 0$ such that $\|p - p'\| > 3\Delta$ for all $p, p' \in P$, this is possible because P is generic. Define $\mu_2 = \sum_{p \in P} \mu_p$ where $\mu_p(q)$ takes the value $(\Delta^2 \pi)^{-1}$ on the disk $D_\Delta(p)$ of radius Δ around p and the value 0 for q outside of this disk.

In this section we use the following notation for densities. For a density ν over $\mathcal{R}$ and a rectangle $R \subseteq \mathcal{R}$ we let $\nu(R)$ be the integral of the density ν over R. Using this notation we can write $\mu_1(\mathcal{R}) = 1$ and $\mu_p(\mathcal{R}) = 1$ for all $p \in P$, hence the total mass of $\mathcal{R}$ is $\mu(\mathcal{R}) = 1 + k$.

Next we transform the floorplan F into a floorplan F_S depending on the set S of segments that has to cover the points of P. To this end we replace every segment in S by a thin rectangle. Equivalently, we double each vertical segment in S into a left and a right copy and each horizontal segment in S into a top and a bottom copy. Figure 10 shows an example of a floorplan F_S produced from F and a set S of its segments. Let $\mathcal{S}$ be the set of new rectangles obtained by inflating[3] segments from S.
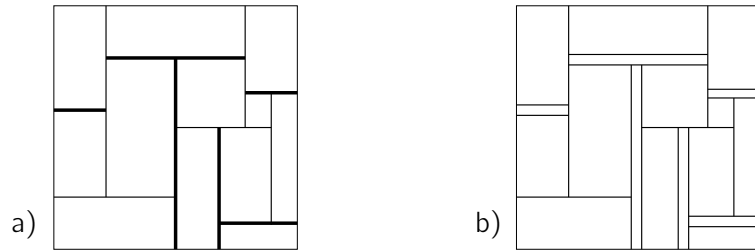

Figure 10: A floorplan F with a prescribed subset S of segments (bold and gray) and the floorplan F_S obtained by doubling the segments of S.

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[3]This is the term used in [2] for the doubling of a segment.

We define weights for the rectangles of F_S as follows. If F_S has r rectangles we define $w(R) = 1 + 1/r$ if $R \in S$ and $w(R) = 1/r$ for all the rectangles of F_S that came from rectangles of F. The total weight $\sum_R w(R) = 1 + k$ equals the total mass $\mu(\mathcal{R})$.

The data $\mathcal{R}$ with μ and F_S with w constitute, up to scaling of $\mathcal{R}$ and w, a set of inputs for Theorem 3. From the conclusion of the theorem we obtain a floorplan F'_S weakly equivalent to F_S such that $m(R) = \int\int_R \mu(x,y)dxdy = w(R)$ for all rectangles R of F'_S.

The definition of the weight function w and the density μ is so that F'_S should be close to a cover map from (F, S) to P. In fact only the rectangles $R \in S$ that have been constructed by inflating segments may contain a disk $D_\Delta(p)$ and each of these rectangles includes at most one of the disks. This suggests a correspondence $S \leftrightarrow P$. However, a rectangle $R \in S$ may use parts of several disks to accumulate mass. To find a one-to-one correspondence between S and P we define a bipartite graph G whose vertices are the points in P and the rectangles in S:

- A pair (p, R) is an edge of G iff $R \cap D_\Delta(p) \neq \emptyset$ in F'_S.

The proof of the theorem will be completed by proving two claims:

- G admits a perfect matching.

- From F'_S and a perfect matching M in G we can produce a floorplan F' that realizes the cover map from (F, S) to P.

For the first claim we check Hall's matching condition. Consider a subset A of S. Since F_S realizes the prescribed weights we have $m(A) = \mu(A) = \sum_{R \in A} \mu(R) = \sum_{R \in A} w(R) = |A|(1 + 1/r)$. Since $\mu_1(A) < 1$ and $\mu_p(A) \leq 1$ for all $p \in P$ there must be at least $|A|$ points $p \in P$ with $\mu_p(A) > 0$. These are the points that have an edge to a rectangle from A in G. We have thus shown that every set A of inflated segments is incident to at least $|A|$ points in G. Hence, there is an injective mapping $\alpha : S \to P$ such that $R \cap D_\Delta(\alpha(R)) \neq \emptyset$ in F'_S for all $R \in S$.

Finally, we construct the floorplan F' that realizes the cover map from (F, S) to P. Let s be a segment in S and let R_s be the rectangle in F'_S that corresponds to s. If s is horizontal we define s' to be the unique maximal horizontal segment in R_s whose y-coordinate is as close to the y-coordinate of the point $\alpha(R_s)$ as possible. Symmetrically, if s is vertical s' is the vertical segment spanning R_s whose x-coordinate is as close to $\alpha(R_s)_x$ as possible. For segments s of F that do not belong to S set $s' = s$. The collection $\{s' : s \text{ segment in } F\}$ of segments may fail to be a floorplan, see e.g. Figure 11.b. However, if s_1 and s_2 are segments of F such that s_1 has one of its endpoits on s_2 and $s_2 \in S$ then we can extend s'_1 into R_{s_2} to recover the contact with s'_2. Having done this for all qualifying pairs s'_1, s'_2 we have recovered the property that the segments form a floorplan, see Figure 11.c. This florplan is weakly equivalent to F but there may still be segments of S that do not cover the assigned point. But by construction the distance from a segment to its assigned point is at most Δ. Now it becomes important that Δ is small compared to the distances of points in P. Shift all segments orthogonally so that they cover their assigned points. Again this may spoil the floorplan property, see e.g. Figure 11.d. However, enlarging or shortening segments by an amount of at most Δ at the ends, so that they properly touch each other, finally generates the floorplan F' that realizes the from (F, S) to P. $\qquad \square$

The topic of [1] was the study of the number $Z(P)$ of rectangulations of a generic point set P. In our terminology this is the total number of cover maps from floorplans with n segments to a generic point set P with n points. Theorem 4 implies that this number is at least as large as the number of weak equivalence classes of floorplans. This is the Baxter number B_{n+1} which is known to be of order $\Theta(8^{n+1}/(n+1)^4)$. In [1] an upper bound for $Z(P)$ of order $O(20^n/n^4)$ is shown.

For a floorplan F with n segments and a point set P with n points let $m(F, P)$ be the number of different cover maps from F to P. Note that $Z(P) = \sum_F m(F, P)$, where the sum is over weak
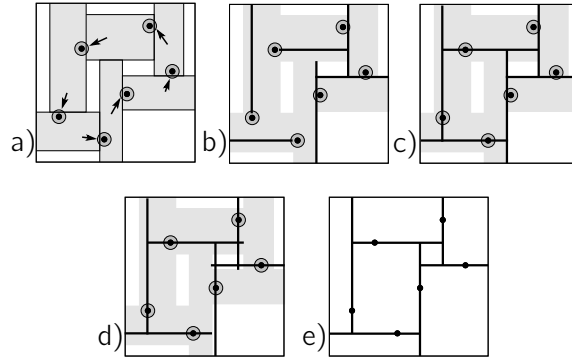
Figure 11: a) A solution F'_S for the instance from Fig. 1. The arrows indicate a matching α. b) Segments $s \in S$ shifted to their optimal position in R_s. c) Enlarged segments recover the contacts. d) Some segments s are moved outside R_s to cover the corresponding points $\alpha(R_s)$. Small final adjustments (clipping and enlarging) yields the F' shown in e).

equivalence classes of floorplans. Let $M(P) = \max_F m(F, P)$, since $Z(P) \leq M(P)B_{n+1}$ bounds on $M(P)$ are of interest. There is a set P_4 of four points such that the windmill floorplan W admits two cover maps on P_4. Nesting this example yields $M(P) \geq 2^{n/4}$. However, $M(P)$ can be even larger. This will be shown with the example sketched in Figure 12.

The left part of the figure shows a floorplan F mapped on a point set P. The floorplan has two classes of rectangles (actually squares) defined by translational symmetry. One class is the green class (horizontal waves) and the other class is the white class. Consider the graph W with the white class as vertices where two vertices are adjacencent iff there is a segment incident to both. Note that W is isomorphic to a grid. The pink rectangles (vertical waves) are an independent set in W. As shown in the right part of the figure there is a cover map $F \to P$ where the pink rectangles (vertical waves) are blown up. This can be done with every independent set of W.

If P has n points then there are $n/2$ rectangles in the white class. We are then interested in the number of independent sets in an $\sqrt{n/2} \times \sqrt{n/2}$ grid. It is known [4] that the $a \times b$ grid has at least 1.503^{ab} independent sets. In our case this yields $1.503^{n/2}$ which is clearly more than $2^{n/4}$. Actually, the bound can be improved by selecting squares in two phases. First, an independent set from the white class is choosen for blow up and then from the green squares (horizontal waves) a 'compatible' independent set is choosen for shrinking. We have no quantification of the improvement.
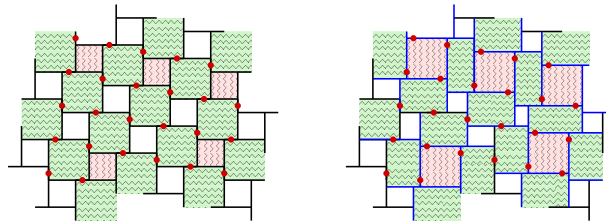

Figure 12: A floorplan F and a point set P such that $m(F, P)$ is large.

A related question that was asked by Eyal Ackerman is this: If for every point of P we prescribe whether its covering segment has to be horizontal or vertical, then for a fixed F there is at most one compatible cover map $F \to P$? A counterexample to this question was independently found by Andrei Asinowski, Maarten Loeffler, and Hendrik Schrezenmaier, see e.g [14].

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