Pentagon contact representations*

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Abstract. Representations of planar triangulations as contact graphs of a set of internally disjoint homothetic triangles or respectively of a set of internally disjoint homothetic squares have received quite some attention in recent years. In this paper we investigate representations of planar triangulations as contact graphs of a set of internally disjoint homothetic pentagons. Surprisingly such a representation exists for every triangulation whose outer face is a 5-gon. We relate these representations to *five color forests*. These combinatorial structures resemble Schnyder woods respectively transversal structures. In particular there is a bijection to certain α-orientations and consequently a lattice structure on the set of five color forests of a given graph. This lattice structure plays a role in an algorithm that is supposed to compute a contact representation with pentagons for a given graph. Again similar algorithms have been proposed for contact representations with homothetic triangles and with squares.

1 Introduction

A *pentagon contact system* $\mathcal{S}$ is a finite system of convex pentagons in the plane such that any two pentagons intersect in at most one point. If all pentagons of $\mathcal{S}$ are regular pentagons with a horizontal segment at the bottom, we call $\mathcal{S}$ a *regular pentagon contact representation*. Note that in this case any two pentagons of $\mathcal{S}$ are homothetic. The contact system is *non-degenerate* if every contact involves exactly one corner of a pentagon. The *contact graph* $G^*(\mathcal{S})$ of $\mathcal{S}$ is the graph that has a vertex for every pentagon and an edge for every contact of two pentagons in $\mathcal{S}$. Note that $G^*(\mathcal{S})$ inherits a crossing-free embedding into the plane from $\mathcal{S}$. For a given plane graph G and a pentagon contact system $\mathcal{S}$ with $G^*(\mathcal{S}) = G$ we say that $\mathcal{S}$ is a *pentagon contact representation* of G.

We will only consider the case that G is an *inner triangulation of a 5-gon*, i.e., the outer face of G is a 5-gon with vertices $a_1, \dots, a_5$ in clockwise order and all inner faces are triangles. Our interest lies in regular pentagon contact representations of G with the additional property that $a_1, \dots, a_5$ are represented by line segments $s_1, \dots, s_5$ which together form a pentagon with all internal angles equal to $(3/5)\pi$. The line segment s_1 is always horizontal and $s_1, \dots, s_5$ is the clockwise order of the segments of the pentagon. Figure 1 shows an example.

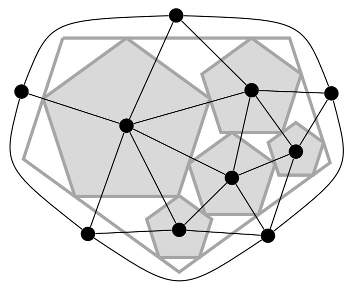

Figure 1: A regular pentagon contact representation of the black graph

Triangle contact representations have been introduced by De Fraysseix et al. [3]. They observed that Schnyder woods can be considered as combinatorial encodings of triangle contact representations of triangulations and essentially showed that any Schnyder wood can be used to construct a corresponding triangle contact system. They also showed that the triangles can be requested to be isosceles with a hor-

*Due to lack of space, proofs are only sketched or completely missing. A full version –in preparation– is available at `http://page.math.tu-berlin.de/~felsner/Paper/fcf-long.pdf`.

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izontal basis. Representations with homothetic triangles can degenerate in the presence of separating triangles. As an application of the *Monster Packing Theorem*, a strong result of Schramm, it was shown by Gonçalves et al. [7] that 4-connected triangulations admit contact representations with homothetic triangles. A more combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations which are based on a Schnyder woods was described by Felsner [5]. On the basis of this approach Schrezenmaier reproved the existence of homothetic triangle representations in his Masters thesis [11].

Representations of graphs using squares or more precisely graphs as a tool to model packings of squares already appear in classical work of Brooks et al. [1] from 1940. Schramm [10] proved that every 5-connected inner triangulation of a 4-gon admits a square contact representation. Again there is a combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations, see Felsner [6]. In this instance the role of Schnyder woods is taken by *transversal structures*. As in the case of homothetic triangles this approach comes with an algorithm which works well in practice, however, the proof that the algorithm terminates with a solution is still missing. On the basis of the non-algorithmic aspects of the approach Schrezenmaier [11] reproved Schramm's Squaring Theorem.

In this paper we investigate representations of planar triangulations as contact graphs of a set of internally disjoint homothetic pentagons. From Schramm's *Monster Packing Theorem* it easily follows that such a representation exists for every triangulation whose outer face is a 5-gon. We relate such representations to *five color forests*. The main part of the paper is reserved to show in what extent these combinatorial structures resemble Schnyder woods respectively transversal structures. At the end of the paper we propose an algorithm for computing homothetic pentagon representations on the basis of systems of equations and local changes in the corresponding five color forests. The idea of looking for pentagon contact representations and a substantial part of the work originate in the Bachelors Thesis of Steiner [12].

2 Five Color Forests and α-orientations

In this section G will always be a triangulation with outer face $a_1, \dots, a_5$ in clockwise order.

In the following definition colors $1, \dots, 5$ play a role. We will always consider these colors modulo 5. This means that, for example, -1 and 4 denote the same color.

Definition 1. A *five color forest* of G is an orientation and coloring of the inner edges of G in the colors $1, \dots, 5$ with the following properties (see Fig. 2 for an illustration):

(F1) All edges incident to a_i are oriented towards a_i and colored in the color i.

(F2) For every inner vertex v the incoming edges build five (possibly empty) blocks B_i, $i = 1, \dots, 5$, of edges of color i and the clockwise order of these blocks is $B_1, \dots, B_5$. Moreover v has at most one outgoing edge of color i and such an edge has to be located between the blocks B_{i+2} and B_{i-2}.

(F3) For every inner vertex and for $i = 1, \dots, 5$ the block B_i is nonempty or one of the outgoing edges of colors $i-2$ and $i+2$ exists.

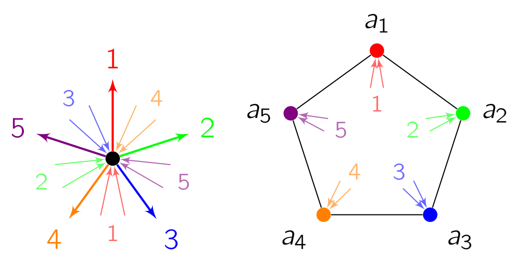

Figure 2: The local conditions of a five color forest

The following theorem shows the key correspondence between five color forests and pentagon contact representations.

Theorem 1 *Every regular pentagon contact representation induces a five color forest on its contact graph.*

Proof. Let $\mathcal{S}$ be a non-degenerate regular pentagon contact representation of $G = G^*(\mathcal{S})$. We color the corners of all pentagons of $\mathcal{S}$ with the colors $1, \dots, 5$ in clockwise order, starting with color 1 at the corner opposite to the horizontal segment. Let e be an inner edge of G. Then e corresponds to the contact of a corner of a pentagon A and an edge of a pentagon B in $\mathcal{S}$. We orient the edge e from the vertex corresponding to A to the vertex corresponding to B and color it in the color of the corner of A involved in the contact. This yields a five color forest. Figure 3 shows an example. □

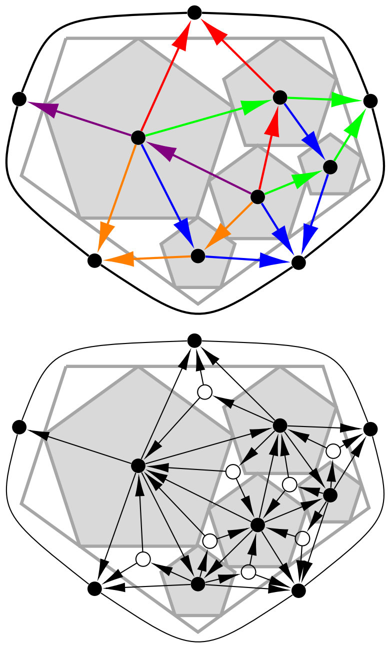

Figure 3: The induced five color forest and α-orientation of a pentagon contact representation

Our goal is to connect the setting of five color forests with the well studied setting of orientations of a given undirected planar graph with prescribed outdegrees.

Definition 2. Let H be an undirected graph and $\alpha : V(H) \to \mathbb{N}$. Then an orientation H' of H is called an α-*orientation* if $\text{outdeg}(v) = \alpha(v)$ for all vertices $v \in V(H')$.

In a five color forest, every inner vertex has outdegree at most 5, but not exactly 5 since the definition allows missing outgoing edges. The following lemma allows us to handle this issue.

Lemma 1 *Let G be endowed with a five color forest and let f be a face of G that does not contain an outer edge. Then in exactly one of the three interior angles of f an outgoing edge is missing in the cyclic order of the respective vertex.*

Now we are able to define an extension of G and a function α such that every five color forest of G can be extended to an α-orientation of this extension.

Definition 3. The *stack extension* $G^\star$ of G is the extension of G that contains an extra vertex in every inner face that is incident to at most one of the outer vertices. These new vertices are connected to all three vertices of the respective face. We call the new vertices *stack vertices* and the vertices of G *normal vertices*.

From now on, if we talk about α-orientations, we always mean α-orientations of $G^\star$ with

$$\alpha(v) = \begin{cases} 2 & \text{if } v \text{ is a stack vertex,} \\ 5 & \text{if } v \text{ is an inner normal vertex,} \\ 0 & \text{if } v \text{ is an outer normal vertex,} \end{cases}$$

where we keep the five outer edges undirected. A five color forest on G induces an α-orientation of $G^\star$ in a canonical way by keeping the orientation of the edges of G and defining the missing edge of Lemma 1 as the unique incoming edge of every stack vertex.

Observation 1 *The coloring of the inner edges of a five color forest can be extended to a coloring of the inner edges of the induced α-orientation that fulfills the properties of a five color forest at all normal vertices.*

We want to prove that the canonical mapping from five color forests to α-orientations is a bijection. For this purpose we need to reconstruct the colors of the inner edges of G if we are given an α-orientation. The idea of this construction is to start with an inner edge e of G and follow a properly defined path that at some point reaches one of the five outer vertices. Then the color of this outer vertex will be the color of e. This approach is similar to the proof of the bijection of Schnyder Woods and 3-orientations in [2].

Now we will define these paths. Actually, we always want to go on with the opposite outgoing edge, but if we run into a stack vertex, we need to be careful. The paths we will define are not unique, but we will see that all paths starting with the same edge e end at the same outer vertex.

Definition 4. Let $e = uv$ be an inner edge such that u is a normal vertex. We will recursively define a set $\mathcal{P}(e)$ of paths starting with e by distinguishing several cases concerning v.

- If v is an outer vertex, i.e. $v = a_i$ for some i, the set $\mathcal{P}(e)$ contains only one path, the path only consisting of the edge e.

- If v is an inner normal vertex, let e' be the opposite outgoing edge of e at v, i.e. the third outgoing edge in clockwise or counterclockwise direction, and we define $\mathcal{P}(e) := \{e + P : P \in \mathcal{P}(e')\}$.

- If v is a stack vertex, let $e'_1 = vv'_1$ and $e'_2 = vv'_2$ be the left and right outgoing edge of v. Further let e''_1 be the second outgoing edge of v'_1 after e'_1 in counterclockwise direction and e''_2 the second outgoing edge of v'_2 after e'_2 in clockwise direction. Note that e''_i is well defined if v'_i is not an outer vertex, and that not both of v'_1 and v'_2 can be outer vertices. If both of e''_1 and e''_2 are well defined, we define $\mathcal{P}(e) := \{e+e'_1+P : P \in \mathcal{P}(e''_1)\} \cup \{e+e'_2+P : P \in \mathcal{P}(e''_2)\}$. If only e''_i is well defined, we define $\mathcal{P}(e) := \{e + e'_i + P : P \in \mathcal{P}(e''_i)\}$.

The following lemma shows that these paths are suitable to define the the color of of an edge e.

Lemma 2 *(i) The paths $P \in \mathcal{P}(e)$ do not cycle.*

(ii) Let $P_1, P_2 \in \mathcal{P}(e)$ be two paths starting with the same edge e. Then P_1 and P_2 end in the same outer vertex.

(iii) Let v be a normal vertex and $e_1 = vv_1, e_2 = vv_2$ be two different outgoing edges. Further let $P_1 \in \mathcal{P}(e_1)$ and $P_2 \in \mathcal{P}(e_2)$ be two paths. Then v is the only common vertex of P_1 and P_2.

Now we are able to prove the the main result of this subsection.

Theorem 2 *The canonical mapping from five color forests to α-orientations is a bijection.*

Proof. Let $\mathcal{F}$ be the set of five color forests of G and $\mathcal{A}$ the set of α-orientations of $G^\star$. Further let $\phi : \mathcal{F} \to \mathcal{A}$ be the canonical map and $\psi : \mathcal{A} \to \mathcal{F}$ the map that keeps the orientation of the edges as in the α-orientation and colors every edge e in the color of the end vertex of the paths in $\mathcal{P}(e)$. Then ψ is well-defined and the inverse function of ϕ. □

It has been shown in [4] that the set of all α-orientations of a planar graph carries the structure of a distributive lattice. By applying this theory, we can derive the following result about five color forests.

Theorem 3 *The set of all five color forests of G carries the structure of a distributive lattice. In this lattice a five color forest F_1 covers a five color forest F_2 if and only if the α-orientation corresponding to F_1 can be obtained from the α-orientation corresponding to F_2 by the reorientation of a counterclockwise oriented facial cycle.*

3 An Algorithm

In this section we will propose an algorithm to compute a regular pentagon contact representation of a given graph G.

If we know a five color forest of G that is induced by a regular pentagon representation, the following system of linear equations allows us to compute this representation: Every inner vertex v of G gets a variable x_v representing the edge length of the corresponding pentagon and every inner face f gets four variables $x_f^{(1)}, \dots, x_f^{(4)}$ standing for the four edge lengths of the corresponding quadrilateral in clockwise order where the concave corner is located between the edges corresponding to $x_f^{(1)}$ and $x_f^{(2)}$ (see Fig. 4). For the five inner faces which are incident to two outer vertices of G we simply set $x_f^{(1)} = 0$ since these faces are represented by triangles, not by quadrilaterals. Now every inner vertex v naturally induces five equations, namely that x_v is equal to the sum of the lengths of the face edges building the five edges of the pentagon corresponding to v. For geometric reasons the three convex corners of the quadrilaterals corresponding to the faces of G are always exactly $(1/5)\pi$. This implies for every inner face f the two equations

$$x_f^{(3)} = x_f^{(1)} + \phi x_f^{(2)} \quad , \quad x_f^{(4)} = \phi x_f^{(1)} + x_f^{(2)}$$

where ϕ denotes the golden ratio. Finally, we add one more equation to our system stating that the sum of the lengths of the face edges building the line segment corresponding to the outer vertex a_1 of G is exactly 1.

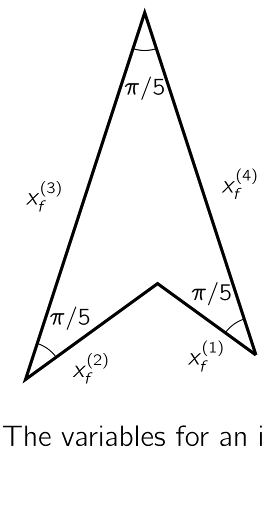

Figure 4: The variables for an inner face f

Theorem 4 *The system of linear equations is uniquely solvable. The solution is nonnegative if and only if the five color forest the system is based on is induced by a regular pentagon contact representation of G.*

The basic idea of our algorithm is to start with an arbitrary five color forest of G and to solve the system of linear equations. If the solution is nonnegative, we can construct the regular pentagon contact representation from the edge lengths given by the solution and are done. If the solution has negative entries, we would like to change the five color forest locally and proceed with the new one. The following theorem shows that such local changes locally change the sign of the solution.

Theorem 5 *Let F be a five color forest, g an oriented facial cycle in the corresponding α-orientation, f the face of G contained in g and $x_f^{(i)}$ with $i \in \{1, 2\}$ the variable corresponding to the edge surrounded by g. If we flip g in the α-orientation, i.e. change its orientation, and solve the system of linear equations for the five color forest corresponding to the new α-orientation, the sign of the variable $x_f^{(i)}$ changes.*

Finally, the following theorem shows that it is always possible to flip a facial cycle in the α-orientation such that locally a sign switches from negative to positive.

Theorem 6 *If the solution of the system of linear equations has negative entries, there is an oriented facial cycle in the α-orientation, that surrounds an edge corresponding to a negative variable $x_f^{(i)}$ where f is an inner face of G and $i \in \{1, 2\}$.*

We can not prove that iterating this local modifications that are supposed to yield local progress (Thm. 5) can guarantee global progress. Therfore, a proof is still missing that this algorithm (or a variant of it) always terminates with a solution. However similar algorithms for the computation of contact representations by homothetic squares or triangle have been described in [5] and [6], these algorithms have been subject to extensive experiments [8, 9]. They have always been successful. We therefore conjecture that the algorithm for computing regular pentagon contact representations always terminates with a solution.

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