Empty Rectangles and Graph Dimension

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Abstract We consider rectangle graphs whose edges are defined by pairs of points in diagonally opposite corners of empty axis-aligned rectangles. The maximum number of edges of such a graph on n points is shown to be $\lfloor \frac{1}{4}n^2 + n - 2 \rfloor$. This number also has other interpretations:

- It is the maximum number of edges of a graph of dimension $[3 \updownarrow\updownarrow 4]$, i.e., of a graph with a realizer of the form $\pi_1, \pi_2, \overline{\pi_1}, \overline{\pi_2}$.
- It is the number of 1-faces in a special Scarf complex.

The last of these interpretations allows to deduce the maximum number of empty axis-aligned rectangles spanned by 4-element subsets of a set of n points. Moreover, it follows that the extremal point sets for the two problems coincide.

We investigate the maximum number of of edges of a graph of dimension $[3 \updownarrow 4]$, i.e., of a graph with a realizer of the form $\pi_1, \pi_2, \pi_3, \overline{\pi_3}$. This maximum is shown to be $\frac{1}{4}n^2 + O(n)$.

Box graphs are defined as the 3-dimensional analog of rectangle graphs. The maximum number of edges of such a graph on n points is shown to be $\frac{7}{16}n^2 + o(n^2)$.

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1 Introduction

A set of points in the plane can serve as the vertex set of various geometrically defined graphs. The most popular example is the Delaunay triangulation which has an edge between two points iff there is an empty circle through them. Graph theoretically Delaunay triangulations are a subclass of planar triangulations. Schnyder showed that the set of all planar triangulations can be obtained if we let two points form an edge iff they are on the boundary of an empty triangle with sides parallel to three given lines.

In this paper we consider rectangle graphs whose edges are defined by pairs of points on the boundary of empty axis-aligned rectangles and variants of it. An appealing aspect of the topic is that the objects dealt with have natural interpretations in different areas. The first three sections approach the theme from different directions and establish connections between them. These sections are

2. Empty Rectangles and Empty Boxes
3. Dimension of Graphs
4. Orthogonal Surfaces and Scarf's Theorem

The lengthy proofs of Theorems 1 and 4 are taken to the appendix.

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2 Empty Rectangles and Empty Boxes

Throughout this paper a *rectangle* is an axis-aligned rectangle in the plane $\mathbb{R}^2$, i.e., a set of the form

$$R = R(r_1, r_1', r_2, r_2') = \{s = (s_1, s_2) \in \mathbb{R}^2 : r_1 \le s_1 \le r_1' \text{ and } r_2 \le s_2 \le r_2'\}.$$

Given a finite set X of points a rectangle R is an *empty rectangle* if the open interior R^o of R contains no point of X. The rectangle R *spanned* by $A \subseteq X$ is the smallest rectangle containing all points of A,

$$R = R[A] = \{s \in \mathbb{R}^2 : \begin{array}{l} \min(x_1 : x \in A) \le s_1 \le \max(x_1 : x \in A) \text{ and} \\ \min(x_2 : x \in A) \le s_2 \le \max(x_2 : x \in A)\}. \end{array}$$

Define the *rectangle graph* $G_r(X) = (X, E_r)$ *of the point set* X such that two points $p, q \in X$ are an edge in E_r iff they span an empty rectangle $R[p, q]$. The edges of the rectangle graph $G_r(X)$ are pairs of points from X spanning an empty rectangle. Let $\mathsf{span}_2(X)$ be the number of edges of $G_r(X)$. In the degenerate situation where all points of X have one coordinate in common the graph $G_r(X)$ is a complete graph. A set X is called *generic* if no two points of X share a coordinate.

Theorem 1 *For every generic set X of n points in $\mathbb{R}^2$*

$$\mathsf{span}_2(X) \le \left\lfloor \frac{n^2}{4} + n - 2 \right\rfloor,$$

this bound is best possible.

The proof is in Section 5.

Generalizing notation we let $\mathsf{span}_k(X)$ be the number of k-subsets A of X such that $R[A]$ is an empty rectangle, that is the interior R^o is empty and all elements of A are on the boundary of R. If X is generic, then each side of a rectangle can contain at most one point, hence $\mathsf{span}_k(X) = 0$ for $k \ge 5$. The numbers $\mathsf{span}_3(X)$ and $\mathsf{span}_4(X)$ count the triangles and 4-cliques of $G_r(X)$.

A point p in X is *orthogonally exposed* if one of the four quadrants determined by p is empty, i.e., contains no points from X. Let $\mathsf{exposed}(X)$ be the number of orthogonally exposed points of X. There are linear relations between the quantities $\mathsf{span}_2(X)$, $\mathsf{span}_4(X)$ and $\mathsf{exposed}(X)$ as well as $\mathsf{span}_2(X)$, $\mathsf{span}_3(X)$ and $\mathsf{exposed}(X)$.

Theorem 2 *For every generic set X in $\mathbb{R}^2$*

$$\begin{aligned} \mathsf{span}_2(X) - \mathsf{span}_4(X) + \mathsf{exposed}(X) &= 3\,(|X| - 1), \\ 2\,\mathsf{span}_2(X) - \mathsf{span}_3(X) + \mathsf{exposed}(X) &= 4\,(|X| - 1). \end{aligned}$$

I include the following proposition because I think that the result is surprising, at least at the first glance. The proof is left to the reader as an exercise.

Proposition 1 *Let X be a set of n random points from the unit square. The expected number of edges of the rectangle graph $G_r(X)$ equals the expected number of comparisons performed by random quick-sort on an n element input, i.e., it is $\sum_{1 \le i < j \le n} \frac{2}{j-i+1}$.*

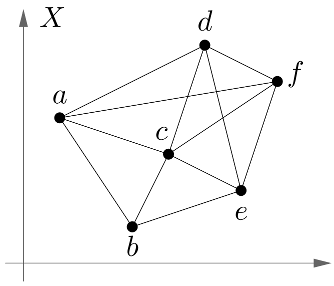

Figure 1: A point set X with its graph $G_r(X)$. The statistics are $|X| = 6$, $\mathsf{exposed}(X) = 5$, $\mathsf{span}_2(X) = 12$, $\mathsf{span}_3(X) = 9$ and $\mathsf{span}_4(X) = 2$.

2.1 Diagrams of 2-Dimensional Orders

An order $P = (X, <_P)$ is 2-dimensional if it can be represented as dominance order on a generic set of points in $\mathbb{R}^2$, i.e., $(x_1, x_2) <_P (y_1, y_2)$ iff $x_1 < y_1$ and $x_2 < y_2$. The edges in the diagram of P then correspond to spanned empty rectangles with one point in the lower-left and one point in the upper-right corner.

A conjugate P^c of P can be obtained from the representation by reverting the direction of one axis. If the first coordinate was reverted, then we have $(x_1, x_2) <_{P^c} (y_1, y_2)$ in P^c iff $x_1 > y_1$ and $x_2 < y_2$. The edges in the diagram of P^c correspond to spanned empty rectangles with one point in the upper-left and one point in the lower-right corner. Hence, the union of P and P^c is the rectangle graph $G_r(X)$

Since diagrams are triangle free graphs, they have at most $n^2/4$ edges, this bound is tight even for 2-dimensional orders. Theorem 1 shows that the diagrams of P and a conjugate P^c together never have much more edges.

The order theoretic point of view can yield insights about rectangle graphs. Kríz and Nešetřil [10] have shown that diagrams of 2-dimensional orders have unbounded chromatic number, hence, so do rectangle graphs. Rödel and Winkler showed that almost all 2-dimensional orders enjoy this property since their diagrams have independence number $o(n)$. The proof is lost and Peter Winkler says: "So we are obliged to tell people that the problem is again open!"

2.2 Box Graphs

Define box graphs as 3-dimensional analogs to rectangle graphs. That is, for a set Y of points in $\mathbb{R}^3$ we define the *box graph* $G_b(Y)$ such that two points $p, q \in Y$ form an edge iff they span an empty axis aligned box $B[p, q]$. We ask the question for the maximum number of edges of a box graph defined by n points.

Theorem 3 *Box graphs of sets of n point have at most $\frac{7}{8}n^2 + o(n^2)$ edges. Up to the error term this bound is best possible.*

Proof. In the next section we prove:

(1) K_{17} is a forbidden subgraph for the class of box graphs (Proposition 3).

(2) Every complete 8-partite graph is a subgraph of some box graph (Proposition 4).

(3) K_{16} is a box graph (Proposition 5).

(4) There are complete 9-partite graphs which are forbidden as subgraphs of box graphs. (Proposition 6).

With (2) we have box graphs with at least $\frac{7}{8}n^2$ edges. For the upper bound we need the Erdős-Stone Theorem which asserts that for any fixed graph H with chromatic number $\chi(H) = k$ the number of edges of graphs containing no subgraph isomorphic to H is of order $\frac{k-2}{2(k-1)}n^2 + o(n^2)$. Since the forbidden subgraphs from (4) have chromatic number 9 we obtain the bound. $\square$

3 Dimension of Graphs

Let $G = (V, E)$ be a finite simple graph. A nonempty family $\mathcal{R}$ of permutations (linear orders) of the vertices of a graph G is called a *realizer* of G provided

($*$) For every edge $S \in E$ and every vertex $x \in V \setminus S$, there is some $\pi \in \mathcal{R}$ so that $x > y$ in π for every $y \in S$.

The *dimension* of G, denoted $\dim(G)$, is defined as the least positive integer t for which G has a realizer of cardinality t.

In order to avoid trivial complications when the condition ($*$) is vacuous, we restrict our attention to connected graphs with three or more vertices.

The definition of dimension has a natural generalization to hypergraphs: Just replace 'edge' by 'hyperedge' in condition ($*$). We review two, by now classical, facts about the dimension of graphs and hypergraphs:

- A graph G is planar if and only if its dimension is at most 3.
  (Schnyder [11])
- The hypergraph $H = (V, F)$ of vertices V versus edges and faces of a 3-polytope has dimension 4 but the dimension drops to 3 when a face f is removed from F.
  (Brightwell-Trotter [3] – see [5], [6] and [8] for simplified proofs)

In [7] a refined concept for dimension of graphs was investigated. For an integer $t \geq 2$, define the dimension of a graph to be $[t \updownarrow t+1]$ if it has dimension greater than t yet has a realizer of the form $\{\pi_1, \pi_2, \ldots, \pi_t, \pi_{t+1}\}$ with $\pi_{t+1} = \overline{\pi_t}$, where $\overline{\pi}$ is the *reverse* of π, i.e., $x < y$ in π if and only if $x > y$ in $\overline{\pi}$ for all $x, y \in X$. The following facts about this 'intermediate' dimension are known (Felsner-Trotter [7]):

- A graph G is outerplanar if and only if its dimension is at most $[2 \updownarrow 3]$.
- A graph G with dimension at most $[3 \updownarrow 4]$ has at most $\frac{1}{4}n^2 + o(n^2)$ edges.

Generalizing the notation we say that the dimension of a graph is at most $[t-1 \updownarrow\!\updownarrow t]$ if it has a realizer of the form $\{\pi_1, \pi_2, \ldots, \pi_t\}$ with $\pi_t = \overline{\pi_{t-2}}$ and $\pi_{t-1} = \overline{\pi_{t-3}}$.

Proposition 2 *Graphs of dimension at most* $[3 \updownarrow\!\updownarrow 4]$ *are exactly the rectangle graphs.*

Proof. We just have to translate the edge-condition ($*$) for a realizer $\{\pi_1, \pi_2, \overline{\pi_1}, \overline{\pi_2}\}$ into the emptiness of a spanned rectangle and vice versa.

Given a set X of points in the plane $\mathbb{R}^2$ we obtain two linear orders $\pi_1(X)$ and $\pi_2(X)$ from projections to the coordinate axes. Conversely, the first two linear orders of a realizer determine a pair of integer coordinates in the range $[1, \ldots, n]$ and thus a set X of points with $|X| = n$.

Let $p, q \in X$ span an empty rectangle. This implies that for every $z \in X \setminus \{p, q\}$ at least one of the following four relations must be false: (1) $z_1 < \max(p_1, q_1)$ or (2) $z_2 < \max(p_2, q_2)$ or (3) $z_1 > \min(p_1, q_1)$ or (4) $z_2 > \min(p_2, q_2)$. If the ith of these relations is false, then $z > p, q$ in π_i, where $\pi_3 = \overline{\pi_1}$ and $\pi_4 = \overline{\pi_2}$. This in turn implies condition $(*)$ for the edge $\{p, q\}$. Since the argument can be reversed we obtain: p, q span an empty rectangle exactly if p, q satisfy condition $(*)$. $\square$

From the proposition and Theorem 1 we obtain:

Corollary 1 *A graph G of dimension at most $[3 \updownarrow\updownarrow 4]$ has at most $\lfloor \frac{1}{4}n^2 + n - 2 \rfloor$ edges.*

Geometric arguments enable us to improve upon the upper bound of order $\frac{1}{4}n^2 + o(n^2)$ (see [7]) for the number of edges of a graph of dimension $[3 \updownarrow 4]$.

Theorem 4 *A graph G of dimension at most $[3 \updownarrow 4]$ has at most $\frac{1}{4}n^2 + O(n)$ edges.*

From the proof of the theorem in Section 6 it follows that actually $\frac{1}{4}n^2 + 5n$ is a valid bound for all n. This is very much in contrast to the old bound which was obtained by combining the Product Ramsey Theorem and the Erdős-Stone Theorem. Therefore, the resulting bound was only asymptotic.

In the preprint preceeding the publication [7] a structural characterization of graphs with dimension at most $[3 \updownarrow 4]$ was conjectured. This conjecture would have implied that these graphs have at most $\frac{1}{4}n^2 + 2n - 6$ edges, a construction with that number of edges is known. Though the conjecture was disproved in [4] it is still possible that the bound on the number of edges is correct.

Problem 1 *Is it true that graphs of dimension at most $[3 \updownarrow 4]$ have at most $\frac{1}{4}n^2 + 2n - 6$ edges?*

In [1] it has been shown that a graph of dimension ≤ 4 (i.e., a graph admitting a realizer $\{\pi_1, \pi_2, \pi_3, \pi_4\}$) has at most $\frac{3}{8}n^2 + o(n^2)$ edges. The proof combines the Product Ramsey Theorem and the Erdős-Stone Theorem. We will use this technique in the next subsection when we bound the number of edges of box graphs.

It would be interesting to have a proof of the $\frac{3}{8}n^2$ bound which is elementary and geometric. Such a proof should also yield an improvement in the error term.

Problem 2 Show that a graph with a realizer $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ has at most $\frac{3}{8}n^2 + Cn$ edges for some reasonable constant C.

3.1 Realizer for Box Graphs

We now come back to box graphs. Just as in the proof of Proposition 2 it can be shown that box graphs are exactly the graphs with a realizer $\{\pi_1, \pi_2, \pi_3, \overline{\pi_1}, \overline{\pi_2}, \overline{\pi_3}\}$.

Proposition 3 *Box graphs contain no 17-cliques.*

Proof. Recall the Erdős-Szekeres Theorem: A permutation of n numbers contains a monotone subsequence of length $\lceil \sqrt{n} \rceil$.

Let G be a graph with vertices labeled 1 to n and a realizer $\{\pi_1, \pi_2, \pi_3, \overline{\pi_1}, \overline{\pi_2}, \overline{\pi_3}\}$. We may assume that π_1 is the identity permutation. Consider a set A of 17 vertices of G. Permutation π_2 contains a monotone subsequence of length at least 5 on the vertices from A, let B be such a set of 5 vertices. Permutation π_3 contains a monotone subsequence of length at least 3 on the vertices from B, let $\{i, j, k\}$ be such a set of 3 vertices. The vertices i, j, k are in this order or in the reverse in each of the six permutations of the realizer. Hence, j obstructs the edge i, k and the set A can't induce a clique. $\square$

Proposition 4 *Every complete 8-partite graph is a subgraph of some box graph.*

Proof. Consider pairwise disjoint sets X_i for $i = 1, .., 8$ and let σ_i be a permutation of X_i. Define permutations by concatenations of the σ_i's as follows:

$$\begin{aligned}
\pi_1 &= \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8 \\
\pi_2 &= \sigma_5 + \sigma_3 + \sigma_2 + \sigma_8 + \sigma_1 + \sigma_7 + \sigma_6 + \sigma_4 \\
\pi_3 &= \sigma_7 + \sigma_4 + \sigma_8 + \sigma_6 + \sigma_3 + \sigma_1 + \sigma_5 + \sigma_2
\end{aligned}$$

The claim is that these permutations together with the reversed form a realizer of a graph containing all edges between vertices from different sets X_i. Since each permutation comes with its reversed the condition $(*)$ for edges can be written as

$(**)$ For every x, y with $x \in X_i$, $y \in X_j$, $i \neq j$ and every vertex $z \neq x, y$, there is an π_a so that z is not between x and y in π_a.

If $z \in X_k$ with $k \neq i, j$, then $(**)$ is satisfied if σ_i and σ_j are consecutive in one of the permutations. This condition is fulfilled for all pairs $\{i, j\}$ except $1, 4$ and $1, 6$ and $2, 4$ and $2, 6$ and $2, 7$ and $3, 7$ and $3, 8$ and $5, 7$ and $5, 8$. For these pairs it is enough to look at π_1 and π_2 to verify that there is no σ_k which is between σ_i and σ_j in all three permutations.

Now consider $z \in X_i$. If we have σ_i before σ_j in π_a and σ_j before σ_i in π_b, then z is outside the interval x, y in either π_a or π_b. Again it is not hard to inspect all pairs i, j of indices and verify this criterion for them.

Proposition 5 *K_{16} is a box graph.*

Proof. Do the above construction from the proof of the previous proposition with sets X_i of cardinality 2 each. The pair of vertices from X_i stays together in all the permutations, hence, the edge condition $(**)$ is trivially satisfied for these pairs. $\square$

Proposition 6 *For sufficiently large n the complete 9-partite graph $\mathbf{T}(9n, 9)$ with parts of equal size n is a forbiden subgraph for box graphs.*

For the proof we need a Ramsey type theorem. This Theorem has been used in [1] and [7] in a similar context.

Product Ramsey Theorem *Given positive integers m, r and t, there exists an integer n_0 so that if $S_1, \ldots, S_t$ are sets with $|S_i| \geq n_0$ for all i, and f is any map which assigns to each transversal $(x_1, \ldots, x_t)$ with $x_i \in S_i$ a color from $[r]$, then there exist $H_1, \ldots, H_t$ with $H_i \subseteq S_i$ for all i and a color $\alpha \in [r]$ so that*

- *$f(g) = \alpha$ for every transversal $g = (x_1, \ldots, x_t)$ with $x_i \in H_i$, for all $i = 1, 2, \ldots, t$.*

Proof of Proposition 6. Let $\mathbf{T} = \mathbf{T}(9n, 9)$ be the complete 9-partite graph with parts of equal size n. Suppose that there is a box graph G containing $\mathbf{T}$ as a subgraph. Let $S_1, \ldots, S_9$ be the independent sets of $\mathbf{T}$. Define a coloring of the transversals as follows. Each transversal is just a 9-element set containing one point from each S_i. Then consider the order of these 9 points in the permutations π_1, π_2, π_3 of a realizer for G. In each π_a, the 9 points can occur in any of 9! orders. So taking the 3 orders altogether, there are at $r = (9!)^3$ patterns.

Applying the Product Ramsey Theorem, it follows that, if n was large enough, then for each $i \in [9]$, there is a 2-element subset H_i of S_i so that all the transversals with elements from these subsets receive the same color. This implies that the linear orders treat the sets H_1, H_2, H_3, H_4 and H_5 as *blocks*, i.e., if a point from one block is over a point from another block in π_k, then both points from the first block are over both points from the second block in π_k.

Now restrict the realizer to the vertex set $H_1 \cup H_2 \cup \cdots \cup H_9$. This gives a graph with all the edges of $\mathbf{T}(18, 9)$ but since the vertices of each H_i stay together in all the permutations they also satisfy the edge condition $(**)$. Therfore, the restricted realizer represents a complete graph K_{18}. This is a contradiction since K_{18} is not a box graph (Proposition 3). □

4 Orthogonal Surfaces and Scarf's Theorem

Let $V \subset \mathbb{R}^d$ be a (finite) antichain V in the dominance order on $\mathbb{R}^d$. The *orthogonal surface* S_V generated by V is the topological boundary of the filter $\langle V \rangle = \{x \in \mathbb{R}^d : \exists v \in V \text{ such that } x_i \geq v_i \text{ for } i = 1, .., n\}$.

An orthogonal surface S_V in $\mathbb{R}^d$ is *suspended* if V contains one element from each positive coordinate axis and the coordinates of all the other members of V are strictly positive. An orthogonal surface S_V is *generic* if no two points in V have a common coordinate, i.e., $v_i \neq v'_i$ for all $v, v' \in V$ and $i = 1, .., d$. In the case of a suspended surface the genericity condition is only applied to coordinates of positive value.

The *Scarf complex* Δ_V of a generic orthogonal surface S_V generated by V consists of all the subsets U of V with the property that $\bigvee\{v : v \in U\} \in S_V$, the join $\bigvee\{v : v \in U\}$ is the point u with coordinates $u_i = \max(v_i : v \in U)$. The property $u \in S_V$ is equivalent to either of i and ii:

(i) There is no $v \in V$ which is strictly dominated by u (a point q is strictly dominated by p if $q_i < p_i$ for $i = 1, .., d$).

(ii) Every $v \in V$ which is dominated by u has at least one coordinate in common with u.

It is a good exercise to show that the Scarf complex Δ_V of a generic V is a simplicial complex.

Theorem 5 (Scarf'73) *The Scarf complex Δ_V of a generic suspended orthogonal surface S_V in $\mathbb{R}^d$ is isomorphic to the face complex of a simplicial d-polytope with one facet removed.*

Figure 2 shows an example in dimension 3. A proof of the theorem is given in [2]. The dimension 3 case of Scarf's theorem was independently obtained by Schnyder [11].

It is known that not all simplicial d-polytopes have a corresponding Scarf complex. For example there are neighborly simplicial 4-polytopes, they have the complete graph as skeleton. From bounds for the dimension of complete graphs it can be concluded that for $n \geq 13$ these 4-polytopes are not realizable by an orthogonal surface in $\mathbb{R}^4$.

A more general criterion was developed by Agnarsson, Felsner and Trotter [1]. They show that the number of edges of a graph of dimension 4 can be at most $\frac{3}{8}n^2 + o(n^2)$.

It would be very interesting to know more criteria which have to be satisfied by simplicial d-polytopes which are Scarf, i.e., can be realized by an orthogonal surface.

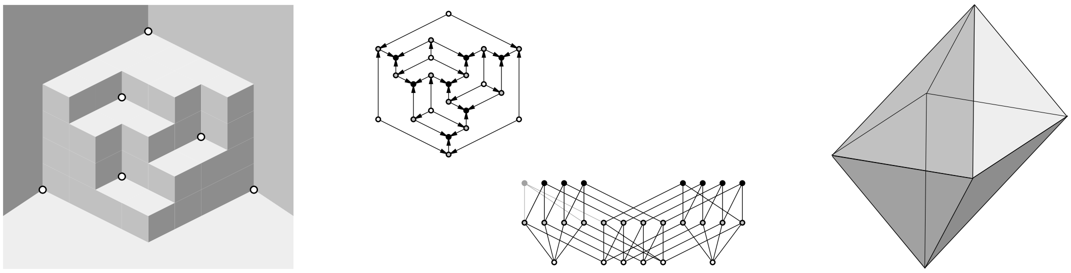

Figure 2: An orthogonal surfaces, two diagrams of its complex (save the **0** element) and the corresponding 3-polytope.

Scarf's Theorem is the tool for the proof of Theorem 2. Let $X \subset \mathbb{R}^2$ be a generic point set with $|X| = n$ and suppose that all coordinates of points in X are strictly between 0 and M. We map a point $x = (x_1, x_2)$ in X to the vector $v^x = (x_1, x_2, M - x_1, M - x_2) \in \mathbb{R}^4$ and let $V = \{v^x : x \in X\} \cup S$, where $S = \{(M, 0, 0, 0), (0, M, 0, 0), (0, 0, M, 0), (0, 0, 0, M)\}$ is the set of suspension vertices. The set V is suspended and generic, therefore, by Scarf's Theorem the Scarf complex corresponds to a 4-polytope. We are interested in the face numbers F_i of the polytope and consequently also in the face numbers f_i of the Scarf complex.

Proposition 7 *The Scarf complex Δ_V of the above V has face numbers*

- $f_0 = n + 4$,
- $f_1 = \mathsf{span}_2(X) + 4n + 6$,
- $f_2 = \mathsf{span}_3(X) + 10n - \mathsf{exposed}(X)$,
- $f_3 = \mathsf{span}_4(X) + 6n - \mathsf{exposed}(X) - 2$.

Proof. There are n vertices v^x and four suspension vertices in V. Therefore, $f_0 = n + 4$.

From the characterization (i) we know that $U \subset V$ is a face of this complex iff there is no vertex in V which is strictly dominated by $\bigvee\{v : v \in U\}$. Suppose $U \cap S = \emptyset$, i.e, $U = \{v^y : y \in Y\}$ for a certain subset Y of X, and let $u = (u_1, u_2, u_3, u_4)$ Since $u_i = \max(v_i^y : y \in Y)$ the definition of v^y implies that $u_i = \max(y_i : y \in Y)$ for $i = 1, 2$ and $u_i = \max(M - y_{i-2} : y \in Y) = M - \min(y_{i-2} : y \in Y)$ for $i = 3, 4$. It turns out that there is a strictly dominated vertex $v^x < u$ iff x is contained in the rectangle $R[Y]$ with first coordinate between $\max(y_1 : y \in Y)$ and $\min(y_1 : y \in Y)$ and second coordinate between $\max(y_2 : y \in Y)$ and $\min(y_2 : y \in Y)$, see Figure 3.

This correspondence explains the occurrences of $\mathsf{span}_2(X)$, $\mathsf{span}_3(X)$ and $\mathsf{span}_4(X)$ in the equations for f_1, f_2 and f_3 given in the proposition. The remaining terms in these expressions are needed for the contributions of the suspension vertices. In the case of f_1 this is rather easy. Each pair (v^x, s) with $x \in X$ and $s \in S$ is a 1-face, for example if s is the suspension for coordinate four, then $v^x \vee s = (x_1, x_2, M - x_1, M)$ and $v^y < v^x \vee s$ would require $y_1 < x_1$ and $M - y_1 < M - x_1$ which is impossible. In addition to these $4n$ 1-faces there are 6 which connect pairs of suspension vertices. Therefore, $f_1 = \mathsf{span}_2(X) + 4n + 6$.

We now turn to the 4-element subsets counted by f_3, as already shown there are $\mathsf{span}_4(X)$ such sets which contain no suspension vertex. To count those which contain suspension vertices it is convenient to adapt the plane visualization with empty rectangles. Recall that the rectangle $R[Y]$ has its right boundary at $u_1 = \max(y_1 : y \in Y)$ and this is just the 1st

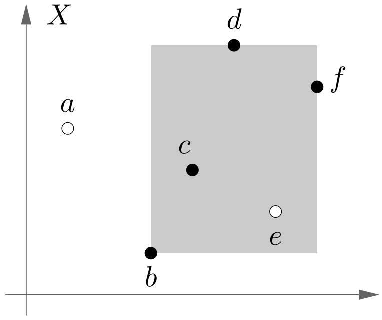

Figure 3: The rectangle corresponding to $Y = \{b, c, d, f\}$ contains c and e in the interior, therefore, $\{v^y : y \in Y\} \notin \Delta_V$.

coordinate of $\bigvee\{v^y : y \in Y\}$. Similarly the top, left and bottom of $R[Y]$ are at u_2, u_3 and u_4. Now suppose that the first coordinate of $\bigvee\{v : v \in U\}$ is M, this can be represented by allowing R an unbounded right side. Again R corresponds to a face of the Scarf complex iff it contains no points of X in the interior.

Now we can count these generalized rectangles. To begin with we find $4n$ generalized rectangles by starting at a point $x \in X$ and selecting one of the four directional rays starting from x. This ray is then widened to both sides until it hits a point from X (see Figure 4 for some widened rays) . If the widened ray doesn't hit a point there is another unbounded side. Note that this procedure counts all empty generalized rectangles with just one and the four with exactly three unbounded sides, see Figure 4. Those with exactly two unbounded sides which are generated are generated twice. There are as many of this kind as there are edges on the ortho-hull, this number in turn equals the number of points on this hull which is just the number $\mathsf{exposed}(X)$. Still uncounted are those rectangles which have exactly two unbounded sides opposite to each other, the figure shows an example. There are $2(n-1)$ rectangles of that type. Collecting the numbers we find $f_3 = \mathsf{span}_4(X) + 4n - \mathsf{exposed}(X) + 2n - 2$.

For the 3-element subsets counted by f_2, we know that $\mathsf{span}_3(X)$ such sets contain no suspension vertex. Each triple of suspension vertices contributes to f_2. For those triples containing one or two points from X we again consider generalized empty triangles. For each point x and each of the four directional rays, the ray can be widened to either side until it hits a point from X, if it doesn't hit a point there is another unbounded side. This gives a total of $8n$ but we have generated rectangles with two unbounded sides twice. There is one such generalized rectangle associated to each exposed point, the four points which have a extreme coordinate actually have two empty quadrants. Thus the corrected contribution is $8n - \mathsf{exposed} - 4$. Still uncounted are those rectangles which have exactly two unbounded sides opposite to each other. These are $2n$ since each point of X belongs to two of them . Summing up we obtain $f_2 = \mathsf{span}_3(X) + 4 + 8n - \mathsf{exposed}(X) - 4 + 2n$. □

Given this proposition the proof of Theorem 2 is easy.

Proof. (Theorem 2) Let F_i be the face numbers of the 4-polytope corresponding to the Scarf complex S_V. This polytope has one facet which is not represented in the Scarf complex, this is the facet of the four suspension vertices, i.e., $F_3 = f_3 + 1$ and $F_i = f_i$ for $i = 0, 1, 2$.

The Euler-Poincaré formula $1 - F_0 + F_1 - F_2 + F_3 - 1 = 0$ together with the double counting identity for simplicial 4-polytopes $2F_2 = 4F_3$ yields $F_3 = F_1 - F_0 = f_1 - f_0 = \mathsf{span}_2 + 3n + 2$. On the other hand $F_3 = f_3 + 1 = \mathsf{span}_4(X) + 6n - \mathsf{exposed}(X) - 1$. Combining the two

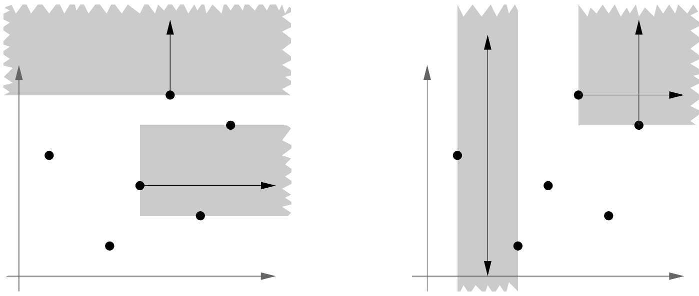

Figure 4: Generalized empty rectangles, left types with one or three unbounded sides, right the two types with two unbounded sides.

expressions completes the proof of the first relation.

For the second relation of the theorem use Euler-Poincaré to obtain $F_2 = 2\,(F_1 - F_0) = 2\,\mathsf{span}_2 + 6n + 4$ and combine it with the expression for f_2 from the proposition. □

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5 Appendix: Proof of Theorem 1

Let X be a set of n points in $\mathbb{R}^2$ with a maximal number of empty spanned rectangles. Let $G_r(X)$ be the rectangle graph on X. We have to prove that $|E_r(X)| \le \lfloor \frac{n^2}{4} + n - 2 \rfloor$.

Let m and M be the points with minimal and maximal first coordinate. If the number of edges incident to these two points is at most $(n-2)+3$, then we can remove them and do with induction: $|E_r(X)| \le |E_r(X - \{m, M\})| + (n-2) + 3 \le \lfloor \frac{(n-2)^2}{4} + (n-2) - 2 \rfloor + (n-2) + 3 = \lfloor \frac{n^2}{4} + n - 2 \rfloor$.

Now suppose that m and M are incident to at least $(n-2)+4$ edges. This implies that the two points have at least 3 common neighbors. The following lemma is useful to localize the common neighbors.

Lemma 1 *Let (x, y) be an edge and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then there is at most one common neighbor of x and y in each of the four region $A^{\uparrow}(x, y)$, $A^{\rightarrow}(x, y)$, $A^{\downarrow}(x, y)$ and $A^{\leftarrow}(x, y)$ shown in Figure 5.*

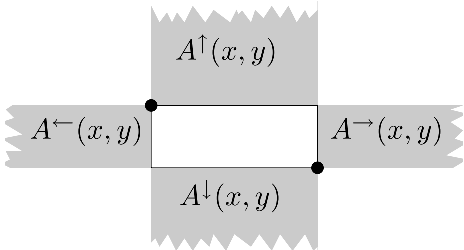

Figure 5: Areas in which x and y can only have one common neighbor.

Proof. We prove the statement for the region $A^{\uparrow}(x, y)$. Let $z = (z_1, z_2) \in A^{\uparrow}(x, y)$ i.e., $\min(x_1, y_1) < z_1 < \max(x_1, y_1)$ and $z_2 > \max(x_1, y_1)$. Suppose z is a neighbor of x and y and consider z' with $z'_2 > z_2$. Either $z \in R[x, z']$ or $z \in R[y, z']$, i.e., one of rectangles is non-empty. Therefore, z' can not be a common neighbor of x and y. □

Since $A^{\uparrow}(m, M)$ and $A^{\downarrow}(m, M)$ contain at most one common neighbor of m and M and by the choice of the points $A^{\rightarrow}(m, M) = \emptyset$ and $A^{\leftarrow}(m, M) = \emptyset$ it follows that $R[m, M]$ can not be empty. This implies that m, M is not an edge and there must be at least two common neighbors of m and M in $R[m, M]$. Let a and b be two such points. Without loss of generality we may assume $M_2 > m_2$ and $a_1 < b_1$.

If the number of edges incident to the four points $\{m, M, a, b\}$ is at most $2(n-4)+8$, then we can remove these points and complete with induction. The four points already induce at least 4 edges, if $R_0 = R[a, b]$ is empty they even induce 5. Figure 6 shall help us analyze where we can expect points which are common neighbors to more than two from the set $\{m, M, a, b\}$. Given a 3-subset of $\{m, M, a, b\}$ the region where common neighbors of the three points can live are restricted by the emptiness of the spanned rectangles.

Claim 1 $\{m, a, b\}$ can have at most two common neighbors, one in R_0 and one in $R_1 \cup R_2$.
Proof. Regions R_3 and R_4 are obstructed for a by b. Regions R_5 and R_6 are obstructed for b by a and Regions R_7 and R_8 are obstructed for m by a. Since $R_0 = A^{\rightarrow}(m, a)$ and $R_1 \cup R_2 = A^{\uparrow}(m, b)$ Lemma 1 implies the statement. △

The proofs of the following claims are similar and omitted.

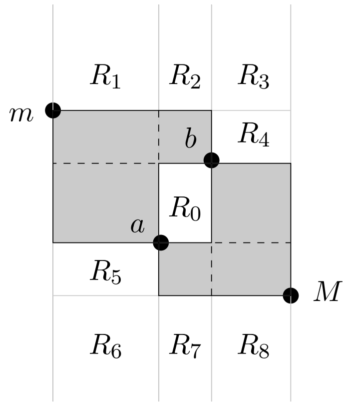

Figure 6: Regions defined by position relative to points m, M, a, b.

Claim 2 $\{m, a, M\}$ can have at most two common neighbors, one in R_0 and one in $R_5 \cup R_6$.

Claim 3 $\{m, b, M\}$ can have at most two common neighbors, one in R_0 and one in $R_3 \cup R_4$.

Claim 4 $\{a, b, M\}$ can have at most two common neighbors, one in R_0 and one in $R_7 \cup R_8$.

This makes a total contribution of 8 edges from the points of high degree. Together with the 4 or 5 edges induced by the four selected points this is 12, way too much. The following claim reduces the contribution of R_0 to 2 and, hence, the contribution from high degree points to 6.

Claim A if R_0 contains r_0 points, then these points can share at most $2r_0 + 2$ edges with $\{m, a, b, M\}$.

Let $z \in R_0$ be a point with more than two of these edges, suppose a, m, b are neighbors of z. It follows that z is the only point which spans an empty rectangle with the middle of the three, in our case this is m. The only remaining triple is a, M, b but again, this triple can only contribute 1. $\triangle$

If the contribution from high degree points is 6, then there are points in R_0 as well as in $R_3 \cup R_4$ and in $R_5 \cup R_6$. When we remove a and b these points will span at least two new empty rectangles which are not present in the original. These two additional edges may be subtracted from the inductive estimate. This allows us to get around with a total of $2(n-4) + 4 + 6$ edges incident to points in $\{m, a, b, M\}$.

The situation where the contribution from the high degree points is 5 is similar. We can, as before, take advantage of empty rectangles that appear when removing a and b.

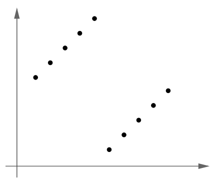

Figure 7: A point set with 10 points maximizing span_2.

The example shown in Figure 7 indicates a construction which yields a set of points with exactly $\lfloor \frac{n^2}{4} + n - 2 \rfloor$ edges for every $n \geq 2$. $\square$

6 Appendix: Proof of Theorem 4

As in the previous proof we use induction. We remove some vertices such that the number of edges incident to them is small enough. The next lemma quantifies the term 'small enough'.

Lemma 2 *Let $\mathcal{G}$ be a class of graphs such that for every member G of $\mathcal{G}$ with n vertices it is possible to remove a set of $2k$ vertices which is incident to at most $k(n-2k)+2ks$ edges, then the number of edges of graphs in $\mathcal{G}$ is at most $\frac{1}{4}n^2 + sn$.*

Proof. Let e_n be the maximal number of edges of a graph with n vertices in $\mathcal{G}$. By induction $e_n \leq k(n-2k) + 2ks + e_{n-2k} \leq k(n-2k) + 2ks + \frac{1}{4}(n-2k)^2 + s(n-2k)$. This yields $e_n \leq \frac{1}{4}n^2 + sn - k^2$. △

Let $G = (V, E)$ be a graph with dimension at most $[3 \updownarrow 4]$ and let $\pi_1, \pi_2, \pi_3, \pi_4 = \overline{\pi_3}$ be a realizer of G. For a vertex v let v_i be the position of v in π_i. Condition $(*)$ is that for every edge u, v and $w \neq u, v$ there is an i with $w_i > \max(u_i, v_i)$. Writing $[s, t]$ for the interval of integers between s and t we can restate the edge condition as follows: u, v is admissible for an edge if every $w \neq u, v$ fulfills $w_1 > \max(u_1, v_1)$ or $w_2 > \max(u_2, v_2)$ or $w_3 \notin [u_3, v_3]$.

For points $p, q \in \mathbb{R}^2$ let $R_{\mathbf{0}}[p, q] = R[0, p, q]$ be the rectangle spanned by the two points together with the origin. The first two 'coordinates' of a vertex $v \in V$ give a point in the plane $P_{1,2}$ (we introduce no new notation and simply call the point v). Since we are interested in maximizing the number of edges we can assume that every pair $u, v \in V$ which is admissible for an edge by condition $(*)$ is actually in E. Here is a geometric description of the edges of G:

$(\star)$ A pair $u, v \in V$ is an edge of G iff every $w \in R_{\mathbf{0}}[u, v]$ satisfies $w_3 \notin [u_3, v_3]$.
Given a vertex v we classify the neighbors of v according to the quadrant of v containing them in the plane $P_{1,2}$. Formally:

$$
\begin{aligned}
N^{\llcorner}(v) &= \{u \in N(v) : u_1 > v_1 \text{ and } u_2 > v_2\} \\
N^{\lrcorner}(v) &= \{u \in N(v) : u_1 < v_1 \text{ and } u_2 > v_2\} \\
N^{\urcorner}(v) &= \{u \in N(v) : u_1 < v_1 \text{ and } u_2 < v_2\} \\
N^{\ulcorner}(v) &= \{u \in N(v) : u_1 > v_1 \text{ and } u_2 < v_2\}
\end{aligned}
$$

Fact 1 A vertex v has at most 2 neighbors in $N^{\urcorner}(v)$

Proof. The two candidates for $u \in N^{\urcorner}(v)$ are the vertices with u_3 maximal subject to the condition $u_3 < v_3$ and with u_3 minimal subject to $u_3 > v_3$. These vertices obstruct the $\star$ property for all the other vertices in this quadrant of v. △

To simplify the analysis we will disregard all the edges $\{u, v\}$ with $u \in N^{\urcorner}(v)$. Thereby we loose at most $2n$ edges which is insignificant since the bound stated in the theorem allows imprecision of order $O(n)$. Let $G' = (V, E')$ be the remaining graph.

All neighbors of a vertex v in G' are either north-west or south-east of v. It is useful to consider a finer partition of the neighborhood $N(v)$ of v:

$$
N^{\lrcorner}_{+}(v) = \{u \in N^{\lrcorner}(v) : u_3 > v_3\} \qquad N^{\ulcorner}_{+}(v) = \{u \in N^{\ulcorner}(v) : u_3 > v_3\}
$$
$$
N^{\lrcorner}_{-}(v) = \{u \in N^{\lrcorner}(v) : u_3 < v_3\} \qquad N^{\ulcorner}_{-}(v) = \{u \in N^{\ulcorner}(v) : u_3 < v_3\}
$$

Each of these sets has the nice property that among the members of the set two of the coordinates are bound together:

Fact 2 (a) $w, w' \in N^{\lrcorner}_{+}(v)$, then $w_2 < w'_2 \iff w_3 > w'_3$,

(b) $w, w' \in N^{\ulcorner}_{+}(v)$, then $w_1 < w'_1 \iff w_3 > w'_3$,

(c) $w, w' \in N^{\lrcorner}_{-}(v)$, then $w_2 < w'_2 \iff w_3 < w'_3$,

(d) $w, w' \in N^{\ulcorner}_{-}(v)$, then $w_1 < w'_1 \iff w_3 < w'_3$.

Proof. We only prove (a), the other statements are obtained by permutations in coordinates and signs. Let $w, w' \in N^{\lrcorner}_{+}(v)$ with $w_2 < w'_2$ this implies that $w \in R_{\mathbf{0}}[w', v]$, hence, $w_3 \notin [v_3, w'_3]$. This together with $w \in N_+(v)$, i.e., $w_3 > v_3$, implies $w_3 > w'_3$. Conversely, $w, w' \in N^{\lrcorner}_{+}(v)$ and $w_3 > w'_3$ implies $v_3 < w'_3 < w_3$. Since $w \in N(v)$ this requires $w' \notin R_{\mathbf{0}}[w, v]$ which forces $w_2 < w'_2$. △

The common neighbors of a pair u, v of vertices are located north-west or south-east of both or they lie in the rectangle spanned by u and v:

$$N^{\lrcorner}(u,v) = N^{\lrcorner}(u) \cap N^{\lrcorner}(v) \qquad N^{\ulcorner}(u,v) = N^{\ulcorner}(u) \cap N^{\ulcorner}(v)$$
$$N^{\square}(u,v) = \{w \in N(u) \cap N(v) : w \in R[u,v]\}$$

In the following analysis we say that a set A is *essentially contained* in B, denoted by $A \subseteq^* B$ if this is true with at most four exceptional elements, i.e., $(A \setminus A') \subseteq B$ for some A' with $|A'| \leq 4$.

Fact 3 (a) If $u_1 < v_1$ and $u_3 < v_3$, then $N^{\lrcorner}(u,v) \subseteq^* N^{\lrcorner}_{+}(v)$.

(b) If $u_1 < v_1$ and $u_3 > v_3$, then $N^{\lrcorner}(u,v) \subseteq^* N^{\lrcorner}_{-}(v)$.

(c) If $u_2 > v_2$ and $u_3 > v_3$, then $N^{\ulcorner}(u,v) \subseteq^* N^{\ulcorner}_{+}(u)$.

(d) If $u_2 > v_2$ and $u_3 < v_3$, then $N^{\ulcorner}(u,v) \subseteq^* N^{\ulcorner}_{-}(u)$.

Proof. We only prove (a), essentially the same argument, with appropriate permutations of coordinates shows (b), (c) and (d).

If $w \in N^{\lrcorner}(u,v) = N^{\lrcorner}(u) \cap N^{\lrcorner}(v)$ then $u \in R_{\mathbf{0}}[w, v]$. Therefore, an edge w, v requires $w_3 > u_3$. There can only be one vertex w in $N^{\lrcorner}(u,v)$ with $u_3 < w_3 < v_3$ since w obstructs either the edge with u or the edge with v for every w' with larger 2nd coordinate. Hence, all but at most one element from $N^{\lrcorner}(u,v)$ have 3rd coordinate larger than v, i.e., are members of $N^{\lrcorner}_{+}(v)$. △

Let m and M be the elements of V with maximal and minimal 3rd coordinate. If m and M are incident to at most $(n-2)+6$ edges in E', then we can apply Lemma 2 with $k=1$ and $s=3$.

From Fact 3 it follows that the essential portion of common neighbors of m and M must be in $N^{\square}(m,M)$. In particular, if there are more than two common neighbors of m and M in G', then they are the south-east and the north-west corners of their rectangle $R[m,M]$. By symmetry in the first two coordinates we may assume that m is east, i.e., $m_1 > M_1$ and $m_2 < M_2$.

Combining part (a) and (d) of Fact 2 it follows that for $w, w' \in N^{\square}(m,M)$ the coordinates are bound by

$$w_1 < w'_1 \iff w_2 > w'_2 \iff w_3 < w'_3. \tag{1}$$

Let a and A be the elements of $N^{\square}(m,M)$ with minimal and maximal 3rd coordinate. Figure 8 shows the relative positions of M, a, A, m in the planes $P_{1,2}$ and $P_{1,3}$. Now consider the edges incident to these four vertices in E'. If their number is at most $2(n-4)+12$, then we apply Lemma 2 ($k=2$, $s=3$).

The next goal is to identify points which are neighbors to at least three vertices from M, a, A, m. The common neighbors of m and M have been localized before. From the way their coordinates are coupled (see 1) it follows that both of a and A can have at most one common neighbor with m and M. Therefore, we have to look for common neighbors of a, A, m or a, A, M. Appealing to symmetry we concentrate on the first of these cases.

Fact 3(b) implies that $N^{\lrcorner}(A, m) \subseteq^* N^{\lrcorner}_{-}(m)$ but $N^{\lrcorner}_{-}(m) = \emptyset$ by the choice of m. Fact 3(c) implies that $N^{\ulcorner}(A, m) \subseteq^* N^{\ulcorner}_{+}(A)$ but $N^{\ulcorner}(a, A) \subseteq^* N^{\ulcorner}_{-}(a)$ by 3(d). That is, all but at most one of the common neighbors of a, A and m are in $N^{\square}(A, m)$. These elements are in $N^{\ulcorner}(a, A) \subseteq^* N^{\ulcorner}_{-}(a)$ and in $N^{\lrcorner}_{+}(m)$. From part (a) and (d) of Fact 2 it follows that for two common neighbors w, w' of a, A, m the coordinates are bound by $w_1 < w'_1 \iff w_2 > w'_2 \iff w_3 < w'_3$, i.e., as in 1.

Let b and B be the common neighbors with minimal and maximal 3rd coordinate (see Figure 8). Now consider the edges incident to the vertices M, a, A, b, B, m in E'. If their number is at most $3(n-6)+18$, then we apply Lemma 2 ($k = 3$, $s = 3$).

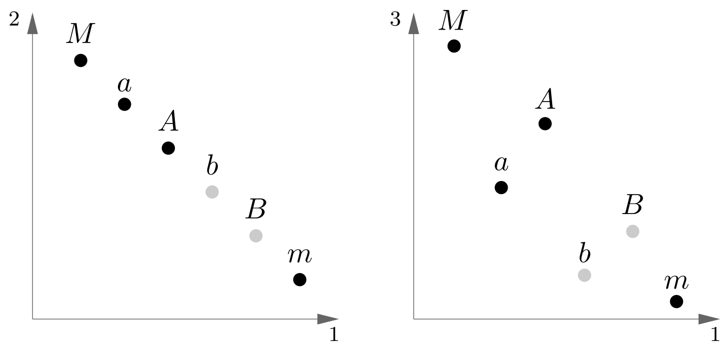

Figure 8: The relative positions of M, a, A, b, B, m in the planes $P_{1,2}$ and $P_{1,3}$.

If there are too many incident edges, then there are 'many' vertices which are adjacent to at least four out of the six vertices. As before m and M can not both belong to such a four set. Consider four sets containing a and M, an argument as before shows that all but at most one of the common neighbors of a and M are in $N^{\square}(M, a)$ and they have 3rd coordinate larger than a_3. Therefore, a obstructs edges from these vertices to b, B and m. Similarly, the pairs A, M, b, m and B, m can not be contained in a four set with many common neighbors. The only remaining candidate for such a four set is a, A, b, B. These elements may indeed have a large common neighborhood. These neighbors must be contained in $N^{\square}(A, b)$ because they are in $N^{\ulcorner}(a, A) \subseteq^* N^{\ulcorner}_{-}(a)$ and in $N^{\lrcorner}(b, B) \subseteq^* N^{\lrcorner}_{+}(B)$. Hence, we again know that for any two w, w' of them the coordinates are again bound as in 1: $w_1 < w'_1 \iff w_2 > w'_2 \iff w_3 < w'_3$.

Let c and C be the common neighbors of a, A, b, B with minimal and maximal 3rd coordinate. Now consider the edges incident to the vertices M, a, A, c, C, b, B, m in E'. If their number is at most $4(n-8)+24$, then we are done by Lemma 2 ($k = 4$, $s = 3$).

If there were more edges incident to these eight vertices, then there would have to be 'many' vertices which are adjacent to at least five out of the eight vertices. Along the lines of the previous analysis, however, it can be shown that there is no five element subset of vertices from M, a, A, c, C, b, B, m which has more than four common neighbors. The set itself induces only 14 edges, so that $4(n-8)+24$ is truly an upper bound for the number of edges incident to vertices from M, a, A, c, C, b, B, m. $\square$