On the Complexity of Partial Order Properties

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Abstract

The recognition complexity of ordered set properties is considered (in terms of how many questions must be put to an adversary to decide if an unknown partial order has the prescribed property). We prove a lower bound of $\Omega(n^2)$ for properties that are characterized by forbidden substructures of fixed size. For the properties being connected, and having exactly k comparable pairs we show that the recognition complexity is $\binom{n}{2}$; the complexity of interval orders is exactly $\binom{n}{2} - 1$. Non-trivial upper bounds are given for being a lattice, containing a chain of length $k \geq 3$ and having width k.

1 Introduction

A property $\mathcal{P}$ on a finite set S is defined to be a subset of the power set of S. We say that a subset X of S has the property $\mathcal{P}$ iff X is in the set $\mathcal{P}$.

Consider a two person game in which player B has an arbitrary set $X \subset S$ at his disposal that is unknown to player A and player A attempts to determine if X has the property $\mathcal{P}$ by asking questions of the form "Is $x \in X$?" where x is an element of S and B can answer "yes" or "no". The aim of A is to minimize the number of questions, while B tries to force A

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to ask as many questions as possible. The game ends when A can decide on the basis of the gathered information whether $X \in \mathcal{P}$.

The complexity $\mathbf{C}(\mathcal{P})$ of the property $\mathcal{P}$ is defined to be the number of questions that are necessary to complete the game assuming both A and the adversary B play optimally. The property $\mathcal{P}$ is called *evasive* if B can force A to ask all the $|S|$ possible questions. If $\mathcal{P}$ is considered as a Boolean function, the complexity of $\mathcal{P}$ is a lower bound for the time any algorithm recognizing $\mathcal{P}$ must take in the worst case on any model of sequential machine [8].

A well studied special case of this problem is, when the considered set S is the set of all possible edges of an n-vertex graph. The relation between this concept of recognition complexity of graph properties and the computer representation of graphs is discussed in [9]. In this context The main open problem is the "Evasiveness Conjecture for Monotone Graph properties". A celebrated partial solution using topological methods to this problem is given in [7].

In [3] Faigle and Turán suggest to play the game on properties of partial orders. Here player A asks for the comparability status of two elements a and b, and B answers "$a < b$", "$a > b$" or "a and b are incomparable."

Considering a property $\mathcal{P}$ of partial orders with n elements, $\mathcal{P}$ is evasive if B can force A to ask all possible $\binom{n}{2}$ questions. Obviously, the game for properties of partial orders does not fit into the concept of set properties discussed before, since there are three possible answers instead of two. Moreover, the transitivity of partial orders may lead to situations, where player A can infer the comparability status of two elements without asking it – independently from the considered property. (In such a case we will say that the pair ab has been "gained"). Note that the property of being a linear order is equivalent to the sorting problem and hence it's complexity is $O(n \log n)$.

In this paper we study the recognition complexity of several properties of partial orders. First we describe situations that induce the comparability status of an unasked pair of elements independently from the considered property. For properties that are characterized by forbidden substructures of fixed size the Erdős-Stone Theorem leads to a lower bound of $\Omega(n^2)$ for the recognition complexity. In Section 2 we prove evasiveness for connectedness and having exactly k comparable pairs, $k \leq n^2/4$ and non-evasiveness for larger values of k. For the class of interval orders we prove that $\binom{n}{2} - 1$ is the exact value of its recognition complexity. Finally we derive bounds for the recognition complexity of being a lattice, containing a chain of length k, for $k \geq 3$ and having width k, for k fixed. All these properties are non-evasive.

1.1 Notation and terminology

We first introduce some basic notations. A *partial order* $P = (V, <)$ consists of a finite ground set V and the order relation $<$, incomparability is denoted by $\|$. An element b *covers* a (denoted $a \prec b$) if $a < b$ and there is no $c \in V$ with $a < c < b$. Throughout this paper we illustrate partial orders by their *diagram*. The vertices of the diagram are the elements of V and b covers a in P iff a and b are connected by an edge going from a up to b. A *partial order property* $\mathcal{P}$ is a set of partial orders over the same ground set closed under isomorphism.

Consider the game introduced in Section 1 for a partial order property $\mathcal{P}$ over a n-element ground set V. The state of the game after $q \leq \binom{n}{2}$ questions can be interpreted as a triple $((C, <), I, N)$, where (C, I, N) is a partition of the set of all two-element subsets of V. The pairs in C are those which have been given comparable in one of the q steps and $<$ is the corresponding order relation. I is the set of pairs given incomparable and N is the set of pairs not yet asked for.

We call a triple $((C, <), I, N)$ *legal* if there exists a partial order $P = (V, <_P)$ *compatible* with the triple, i.e. satisfying

1. If $\{a, b\} \in C$ and $a < b$ then $a <_P b$.
2. If $\{a, b\} \in I$ then $a \| b$ in P.

An *algorithm* for player A is a mapping φ assigning to each legal triple $((C, <), I, N)$ a pair $\{a, b\} \in N$, i.e. φ prescribes the next question "$a : b$" at state $(((C, <), I, N)$.

A *strategy* for player B is a mapping ψ which assigns to a given legal triple $((C, <), I, N)$ and $\{a, b\} \in N$ a new legal triple which is one of the following two

$$\Big((C, <), I \cup \{a, b\}, N \setminus \{a, b\}\Big) \quad , \qquad \Big((C \cup \{a, b\}, <), I, N \setminus \{a, b\}\Big).$$

A *game is finished* at state $((C, <), I, N)$ if either all partial orders P compatible with the triple are in $\mathcal{P}$, or for all of them $P \notin \mathcal{P}$ holds.

The complexity of a property $\mathcal{P}$ for a fixed algorithm φ and a fixed strategy ψ is the minimum number of questions needed to finish a game if player A uses φ and player B uses ψ, i.e.

$$\mathbf{C}(\mathcal{P}; \varphi, \psi) = \min \Big\{ q \,|\, \text{game ends at state } ((C, <), I, N,) \text{ with } |C \cup I| = q \Big\}.$$

The *complexity* of a property $\mathcal{P}$ is the minimum number of questions needed to finish a game if both A and B play optimally, i.e.

$$\mathbf{C}(\mathcal{P}) = \min_{\varphi} \max_{\psi} \mathbf{C}(\mathcal{P};\varphi,\psi).$$

For a legal triple $((C,<),I,N)$ with $|C \cup I| = q$, the number of pairs of elements whose comparability status is known may be more than q, they have been gained.

1.2 Some general observations

We now give situations, where the comparability status of a pair $\{a,b\} \in N$ is induced by the comparability status of some other pairs independent from the partial order property under consideration.

Situation 1 If there exist elements a_1, a_2, a_3 with $a_1 < a_2$ and $a_2 < a_3$ then by transitivity $a_1 < a_3$ holds.

Situation 2 If there exist elements a_1, a_2 and b_1, b_2 with $a_1 < a_2$, $b_1 < b_2$, $a_1 \parallel b_2$ and $a_2 \parallel b_1$ then both of $a_1 \parallel b_1$ and $a_2 \parallel b_2$ hold.
Proof. With each of the 4 possible comparabilities $a_2 < b_2$, $b_2 < a_2$, $a_1 < b_1$ and $b_1 < a_1$ we would introduce as transitive edge either $a_1 < b_2$ or $b_1 < a_2$ contradicting the incomparability of this pair. (See Figure 1a). □

We always illustrate partial orders by their diagram with solid lines, incomparabilities are denoted by dashed edges, and dotted edges denote an unknown comparability status.

Situation 3 Consider a state $((C,<),I,N)$ of a game where there exists a 5-chain $a_1 < a_2 < a_3 < a_4 < a_5$ and an element $b \notin \{a_1, ..., a_5\}$ with $\{b, a_i\} \in N$. Then player A can deduce the comparability status of all five pairs $\{b, a_i\}$, $1 \leq i \leq 5$ by asking only four questions.
Proof. Player A asks for the comparability status of the pairs $b : a_2$ and $b : a_4$. If one of these pairs is comparable we gain a transitive edge. In case both pairs are given incomparable A concludes $b \parallel a_3$. (See Figure 1b). □

Situation 4 Consider a state $((C,<),I,N)$ of a game where there exist two 3-chains $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ and all the pairs $\{a_i, b_j\}$ are in N. Then player A can deduce the comparability status of all six pairs $a_i : b_j$ by asking only five questions.
Proof. Player A asks for the comparability status of $a_1 : b_2$ and $b_1 : a_2$. If both $a_1 \parallel b_2$ and $b_1 \parallel a_2$, then situation 2 applies, i.e. $a_1 \parallel b_1$ and $a_2 \parallel b_2$.

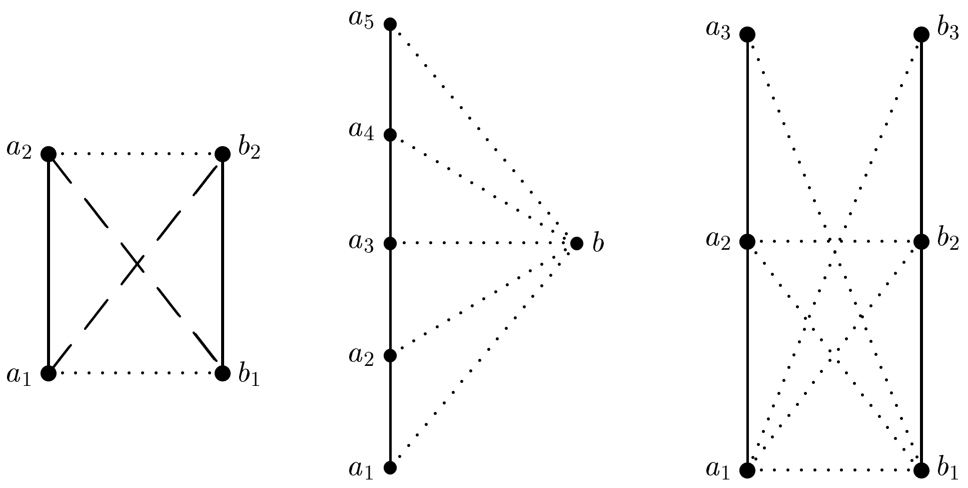

Figure 1: Three standard situations

Otherwise, i.e. if at least one of these pairs is given comparable, A gains a transitive edge. (See Figure 1c). □

As an application of the Erdős-Stone Theorem (see [2]) the next theorem shows a quadratic lower bound for the complexity of properties with a forbidden subposet. The *height* of an order is the number of elements in a longest chain minus one.

Theorem 1 *Let $\mathcal{P}$ be a partial order property such that*

1. *$\mathcal{P}$ contains the n-element antichain;*
2. *there exists a partial order $P_0 = (V_0, <_0)$ of height h such that a partial order P that contains P_0 as a suborder is not in $\mathcal{P}$.*

The complexity of $\mathcal{P}$ on n elements is at least $(1/2h - o(1))n^2$.

Proof. Player B can make use of the following 'greedy strategy'. As long as the graph of unasked edges contains a copy G of the comparability graph of P_0 the answer to a question $a : b$ is $a \parallel b$. The n-antichain and the orientation of G which is the order P_0 together with independent elements are both compatible orders, one in $\mathcal{P}$ the other not in $\mathcal{P}$.

The comparability graph of P_0 has chromatic number $h+1$. The Erdős-Stone Theorem ensures that as long as the graph of unasked edges contains more then $\frac{h-1}{2h}n^2$ edges it also contains a copy of the comparability graph of

P_0. Hence, player A has to ask at least $(1 - \frac{h-1}{h})\frac{n^2}{2}$ questions. Taking into account that the Erdős-Stone Theorem only gives assymptotic bounds this leads to the expression claimed in the theorem. □

This theorem applies to a lot of partial order properties, e. g. for being an interval order, being a lattice, having dimension at most 2 or containing a chain of 3 elements. In the next section we will give more accurate estimates for the complexity of some of these properties.

2 Specific properties

2.1 Connected orders

A partial order is said to be *connected* if its diagram considered as an undirected graph is connected.

Theorem 2 *The property $\mathcal{P}$ of all connected partial orders over set V is evasive.*

Proof. We describe a strategy ψ for player B such that $\mathbf{C}(\mathcal{P};\varphi,\psi) = \binom{n}{2}$ for all algorithms φ of player A.

Let the first question be $a : b$, player B answers $a < b$. For further questions $a : b$ he answers $a \parallel b$, except in case $\{a,b\}$ is the last possible edge between one of the elements, say a, that is not comparable to another element, and an element b comparable to some other element. Then the comparability is given according to the comparability status of b, such that b remains a minimal element or a maximal element. More precisely, given a legal state $((C,<),I,N)$ let $M = \bigcup_{\{x,y\}\in C} \{x,y\}$ then for $\{a,b\} \in N$ the answer of B is:

$a < b$ if $a \notin M$, $b \in M$ and for all $x \in M \setminus \{b\}$ we have $\{x,a\} \in I$, and $b > c$ for some $c \in M$;

$b < a$ if $a \notin M$, $b \in M$ and for all $x \in M \setminus \{b\}$ we have $\{x,a\} \in I$, and $b < c$ for some $c \in M$;

$a \parallel b$ else.

The strategy ψ obviously preserves the invariants:

(1) The partial order induced by $(C,<)$ over M is connected, and $\binom{M}{2} \subseteq C \cup I$.

(2) All $x \in M$ are either minimal or maximal with respect to $(C, <)$.

(3) For each $x \in V \setminus M$ there is a $y \in M$ such that $\{x, y\} \in N$.

Applying ψ, the game ends with a legal triple $((C, <), I, N)$. If $|C \cup I| < \binom{n}{2}$, then the partial orders compatible with $((C, <), I, N)$ would all be connected or all be disconnected. But invariants 1 and 2 contradict the assumption that all compatible partial orders are connected, while invariant 3 contradicts the case that they all are not connected. □

2.2 k comparable pairs

Theorem 3 *Consider the game being played on a ground set of* n *vertices. Let* $\mathcal{P}_k$ *denote the property of* G *having exactly* k *comparable pairs. Property* $\mathcal{P}_k$ *is not evasive for any* $k > \frac{n^2}{4}$.

Proof. We give an algorithm for player A for avoiding at least one question whenever $k > \frac{n^2}{4}$. Note that an order with k comparabilities has to contain a 3-chain; this is an elementary application of Turán's Theorem from graph theory. The basic idea for the algorithm is to go for the gain of a transitive edge of a 3-chain, if player B refutes to allow this kind of gain the game will end in a position where all compatible orders are of height one and hence have less than k comparabilities.

Player A selects a vertex $v \in W_0 = V$ and asks for the comparability status of v with all the vertices in W_0 if no comparable vertex is found insert v into the set of isolated vertices U_0. This is repeated until comparable pair $A - 1 < b_1$ is found.

Let $W_1 = W_0 \setminus (U_0 \cup \{a_1, b_1\})$. Repeat the same steps with W_1; i.e., select a vertex $v \in W_1$ and ask for the comparability status of v with all the vertices in W_1 if no comparable vertex is found insert v into U_1, do this until a comparable pair $a_2 < b_2$ is found. Repeat with $W_2 = W_1 \setminus (U_1 \cup \{a_2, b_2\})$ and so on. When all vertices are exhausted there is a sequence of sets which form a partition of V:

$$U_0, \{a_1, b_1\}, U_1, \{a_2, b_2\}, \ldots, U_{t-1}, \{a_t, b_t\}, U_t.$$

At this point any two elements $x \in U_i$ and $y \in U_j$, with $i = j$ possible, are known to be incomparable. In the next step the algorithm takes the elements of the sets U_i and asks for their status with respect to elements a_j and b_j which have not been asked before. When for an element $x \in U_i$ a comparability is discovered, say $x < b_j$ (if $x > b_j$ player A has gained a transitive edge) then x is moved into a set A_j associated with a_j, element a_j is itself a member of A_j. Conversely, when $x > a_j$ then x is moved into B_j which also contains b_j. Isolated elements end in U_0.

When this step is completed we have a set U_0 of isolated vertices and sets A_i and B_i for $i = 1, .., t$ where each element of A_i is below b_i and each element of B_i is above a_i.

Finally compare all elements of A_i with those of A_j and dually those of B_i with those of B_j for $i \neq j$. To avoid transitive edges player B has to give the answer "incomparable" to all these queries.

We claim that at this point the number of possible comparabilities in a compatible order is less than $\frac{n^2}{4} + 1$. The possible comparabilities are of three types:

1. Comparabilities between an a_i and a b_j; there are t^2 of this type.
2. Comparabilities between $x \in A_i \setminus \{a_i\}$ and a_i or $x \in B_i \setminus \{b_i\}$ and b_i; there are at most $n - 2t$ of these.
3. Comparabilities between $x \in A_i \setminus \{a_i\}$ and b_j or $x \in B_i \setminus \{b_i\}$ and a_j; there are at most $(n - 2t)t$ of these.

In total this makes at most $p_n(t) = t^2 + (t + 1)(n - 2t)$ comparabilities. It is easy to check that $p_n(t) > \frac{n^2}{4} + 1$ has no real solution t. Therefore, if $k > \frac{n^2}{4} + 1$ player A can stop the game at this point and claim that the order has less than k comparabilities. This completes the proof of this case.

Now let $k = \lfloor \frac{n^2}{4} \rfloor + 1$. The only integral solutions of $p_n(t) \geq \lfloor \frac{n^2}{4} \rfloor + 1$ are $t = n/2 - 1$ in the even case and $t = \lfloor n/2 \rfloor$ or $t = \lfloor n/2 \rfloor - 1$ in the odd case. Each of these solutions gives $p_n(t) = \lfloor \frac{n^2}{4} \rfloor + 1$, hence, all the possible comparabilities counted by $p_n(t)$ have to be realized by player B.

Let x be one of the elements that was in $\bigcup_{i=0}^{t} U_i$, actually x was in U_t, otherwise x is known to be incomparable to a_t and b_t. Assume w.l.o.g. that $x \in A_1$. Next ask the status of x and a_1. To realize the comparabilities counted by $p_n(t)$ player B has to answer $x < a_1$. The last question is $a_1 : b_2$. To reach the aspired number of comparabilities player B has to answer $a_1 < b_2$ but this allows the gain of the transitive edge $x < b_2$ and the proof is complete.

□

Theorem 4 *Property $\mathcal{P}_k$ is evasive for any $k \leq \frac{n^2}{4}$.*

We prepare for the proof of the theorem with a definition and two lemmas.

A partial order $P = (V, <)$ is an *interval order* iff there exists a collection $(I_x)_{x \in V}$ of intervals on the real line, such that $x < y$ iff I_x lies entirely to the left of I_y. Interval orders have several nice characterizations. We mention two of them, see Fishburn's book [5] for full proofs.

1. P is an interval order iff P does not contain a suborder **2+2**, where **2+2** $= (\{a, b, c, d\}, <)$ with $a < b, c < d$ and no further comparabilities.

2. P is an interval order iff the sets of successors $Suc(x) = \{y : y > x\}$ of elements of P are linearly ordered with respect to inclusion.

3. P is an interval order iff the sets of predecessors $Pred(x) = \{y : y < x\}$ of elements of P are linearly ordered with respect to inclusion.

Lemma 1 *For every integer $k \leq \frac{n^2}{4}$ there is a bipartite interval order on n vertices with k comparable pairs.*

Proof. Take a partition $(k_1, k_2, \ldots, k_{\lfloor n/2 \rfloor})$ of k, with $\lceil n/2 \rceil \geq k_1 \geq k_2 \geq \ldots \geq k_{\lfloor n/2 \rfloor} \geq 0$. Let $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$ and $y_1, y_2, \ldots, y_{\lfloor n/2 \rfloor}$ be the n vertices of the order. As relations take $x_i < y_j$ for $1 \leq i \leq k_j$ and all y. It is easily shown that the resulting order does not contain an induced **2+2**, hence, it is a bipartite interval order.

□

Lemma 2 *Let I be a bipartite interval order. Player A has to ask all $\binom{n}{2}$ pairs of vertices to verify that player B constructs I.*

Proof. Let (X, Y) be a partition of the vertex set of I such that X is the set of minimal elements and Y is the set of maximal elements (independent vertices can arbitrarily be considered as minimal or maximal). Suppose player B has answered the first $\binom{n}{2} - 1$ queries of player A complying to the status of the pair in I. Let $\{a, b\}$ be the remaining pair. With the following case discussion we show that irrespective of the pair $\{a, b\}$ player B always has two legal answers so that the final order is either I or non-isomorphic to I.

$a \in X$, $b \in Y$. Independent of the status of a, b in I the answers $a \parallel b$ and $a < b$ are legal answers for player B.

$a, b \in X$. From the second characterization of interval orders we know that $Suc(a)$ and $Suc(b)$ are comparable, say, $Suc(a) \subseteq Suc(b)$. In this case the answers $a \parallel b$ and $a > b$ are legal answers for player B.

$a, b \in Y$. Since the dual (all comparabilities reversed) of an interval order is an interval order this case reduces to the case $a, b \in X$. □

Note that the proof of the previous lemma shows that verification of a bipartite interval order is evasive even if the labeling of the vertices is fixed.

Proof. (Theorem 4) A strategy for player B proving the theorem is to choose a bipartite interval order I_k with k comparable pairs (Lemma 1). Player B answeres the first $\binom{n}{2} - 1$ queries complying to the status of the pair in I. By Lemma 2 the answer to the last question allows player B to construct either I_k or some order in $\mathcal{P}_{k-1} \cup \mathcal{P}_{k+1}$. □

An easy extension of the above argument leads to lower bounds for the complexity of $\mathcal{P}_k$ for $k > n^2/4$.

Player B chooses an appropriate l for the given value of k and constructs an order P with l ranks of about the same size n/l. Any two nonadjacent ranks induce a complete bipartite order and comparabilities between adjacent ranks are chosen so that the following properties hold:

- The order P is an interval order.
- P has exactly k comparabilities.
- If x, y are from the same rank and $Pred(x) \subset Pred(y)$ then $Suc(x) \supseteq Suc(y)$.
- If x, y are from the same rank and $Suc(x) \subset Suc(y)$ then $Pred(x) \supseteq Pred(y)$.

Using obvious extensions of the previous methods this can always be done.

For the above order P the recognition problem contains the sorting problem on the l sets. In addition every possible edge between 2 adjacent ranks and also those between elements of the same rank must be asked. This leads to the following bound :

Proposition 1 *Let l be so that $\frac{n^2}{2}(1-\frac{1}{l-1}) < k \leq \frac{n^2}{2}(1-\frac{1}{l})$. The recognition complexity of $\mathcal{P}_k$ is at least $l \log l + (l-1)(\frac{n^2}{l^2} - 1) + \frac{n(n-l)}{2l}$. And the bound is tight for both ends ($l = 2$ and $l = n$).*

2.3 Interval orders

Theorem 5 *The recognition complexity of the class $\mathcal{P}$ of interval orders is* $\binom{n}{2} - 1$.

Proof. We first prove the lower bound of $\binom{n}{2} - 1$, this is the easier part. To force all but at most one question to be asked player B can use a simple 'greedy-strategy': *For all states of the game and all questions $x : y$ player B answers $x \parallel y$ unless there is no compatible partial order containing a* **2+2**.

Consider the state of the game when player B answers to the question $\{a, b\}$ concedes the first comparable pair. Let G_N be the of all edges that have not yet been asked. It is easy to see that G_N is either a triangle or a star with $k \geq 1$ leaves. In Figure 2 we show the five possible situations of $G_N \cup \{a, b\}$.

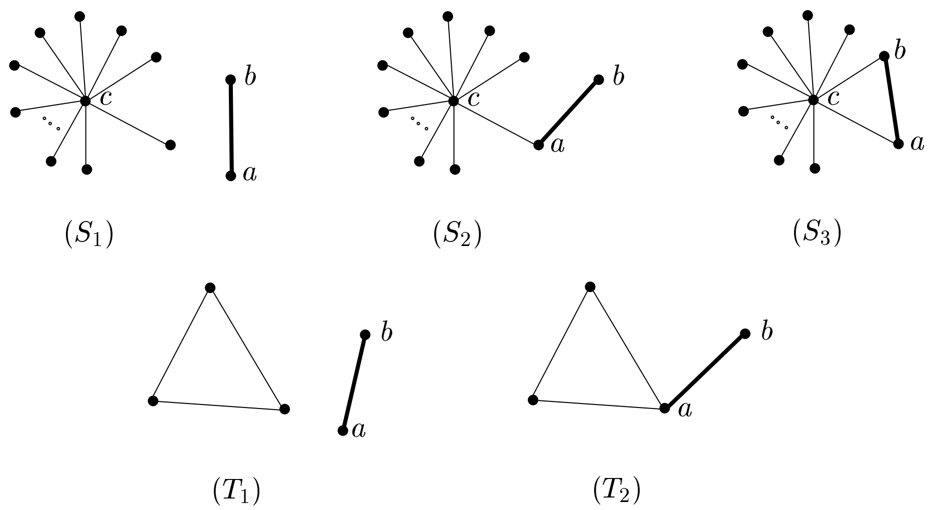

Figure 2: The bold edge $\{a, b\}$ and the possible ways to attach it to the star or the triangle formed by G_N.

In the cases S_1 and T_1 player B can force all the remaining questions by giving incomparabilities until the last edge is asked.

By adding incomparable pairs case S_2 is either transformed to case S_1 or a, c is the last unasked edge. In that case player B gives the second last pair comparable so that a, c is a diagonal of a **2+2**. This forces player A to ask this last edge.

By adding incomparable pairs case S_3 is either transformed to case S_2 or the three last remaining questions together with the edge a, b form a triangle with an edge sticking out. In this case as well as in case T_2 player A can gain a question by first asking the edge disjoint from a, b. Player B gives the edge comparable. At this point player A only has to ask the diagonal edge to decide the game.

We now turn to the upper bound. We have to describe an algorithm for player A which saves a question.

Let $V = \{x_1, \ldots, x_n\}$, player A takes the elements by increasing index and asks for their comparability status to all elements with higher index, until B gives the first comparability.

In case $x_i \parallel x_j$ for $i \leq n-3$ and all j, every compatible partial order is an interval order and A gains 3 questions. So, let $\{x_k, x_{k+l}\}$ be the first comparable pair, assume $x_k < x_{k+l}$ (the other case is dual).

Let $V' = \{x_{k+1}, \ldots, x_{k+l-1}, x_{k+l+1}, \ldots, x_n\}$, player A continues by asking all pairs of elements from V'. Let Q' be the resulting order. If Q' is an antichain then all compatible partial orders are in $\mathcal{P}$. We thus assume that at least one pair is comparable. Also Q' has to be an interval order, otherwise all compatible orders contain the **2+2** of Q' and are not in $\mathcal{P}$.

Let $a = x_k$, $b = x_{k+l}$ and z be a maximal element of Q' with a maximal set of predecessors. Player A asks all questions $b : x$ for elements elements $x \in V' \setminus \{z\}$. Every comparability given has to be of the form $b > x$, otherwise player A would gain $a < x$ as a transitive edge. Next all questions $a : x$ for $x \in V'$ are asked. The only pair that has not been asked at this point is the pair b, z. Let Q be the order obtained so far.

Claim: If there is no **2+2** in Q then making $\{b, z\}$ comparable can not cause one. Otherwise there would be two comparable elements $x < y$ which are both incomparable to b and z. Since $a < b$ we find $x, y \neq a$. Hence $x, y \in V'$ and $x < y$ implies $x < z$ by the choice of z.

If there is a **2+2** in Q then the diagonals have already been asked since both b and z are maximal elements of Q. Hence, the comparability status of b, z can not serve to destroy the **2+2**.

In conclusion, player A can save the question $b : z$. □

2.4 Lattices

In this section we prove that the property $\mathcal{P}_L$ of being a lattice is not evasive and then derive a lower bound on the complexity of $\mathcal{P}_L$.

Consider a partial order $P = (V, <)$, two elements $a, b \in V$ have the *minimum* $x \in V$, denoted $x = \min\{a, b\}$ if $x \leq a$ and $x \leq b$, and $z \leq a$ and $z \leq b$ implies $z \leq x$. The *maximum* is defined analogously. $P = (V, <)$ is a *lattice* iff $\min\{a, b\}$ and $\max\{a, b\}$ exist for all $a, b \in V$.

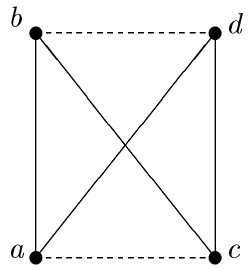

Figure 3: Forbidden subdiagram for lattices.

Theorem 6 *Let $\mathcal{P}_\mathcal{L}$ be the property of being a lattic on a set V, $|V| > 3$, then $\mathbf{C}(\mathcal{P}_\mathcal{L}) < \binom{n}{2}$.*

Proof. In the following we use the fact that $L = (V, <)$ is a lattice iff it its diagram does not contain an edge subdivision of the diagram shown in Figure 3 and it contains a unique minimum and a unique maximum, i.e. elements $x, z \in V$ such that $x \leq y$ and $y \leq z$ for all $y \in V$. (Denote the minimum resp. maximum of L by $\min(L)$ resp. $\max(L)$.)

An algorithm φ for player A with $\mathbf{C}(\mathcal{P}; \varphi, \psi) < \binom{n}{2}$ for all strategies ψ is first to ask all $\binom{n-1}{2}$ questions over $V \setminus \{x\}$ for a fixed $x \in V$. The state of the game after these $\binom{n-1}{2}$ questions is a legal triple $((C, <), I, N)$ with $N = \{\, \{x, y\} \mid y \in V\}$.

Case 1. The order induced by $((C, <), I)$ is not a lattice. The 'defect' of $((C, <), I)$ relative to lattices has to be so small that adding x in the right way leads to a lattice. The possible situations are:

1.1 The unique minimum or maximum is missing. W.l.o.g. let $((C, <), I)$ induce a partial order containing no minimum. It must contain a maximum z and all lattices compatible with $((C, <), I, N)$ contain x as its minimum. Let A ask $a : x$ for an arbitrary $a \in V \setminus \{x, z\}$. Player B has to answer $x < a$, else there is no compatible lattice, but with $x < a$ and $x < y$ player A gains the transitive edge $a < y$.

1.2 The partial order induced by $((C, <), I)$ contains a forbidden substructure on elements a, b, c, d. In this case A asks $b : x$ and $d : x$. Either B gives a transitive edge between x and the minimum or the maximum, or B answers $b \parallel x$ and $d \parallel x$, which implies that there exists no compatible lattice.

Case 2. The order induced by $((C, <), I)$ is a lattice.

2.1 If the lattice contains a 5-chain, then situation 3 from section 2 applies.

2.2 If the lattice has height 3, i.e. there is a 4 chain $\min < a_1 < b_1 < \max$, then the non-extremal elements are partitioned into $a_1, ..., a_k$, those covering the minimum, and the remaining elements $b_1, ..., b_l$. Note that all the b_i are covered by the maximum. Player A asks $x : u$ for all $u \in V \setminus \{x, \min, \max, a_1\}$ any comparability among the answers would induces a transitive edge. Hence we assume that B always answers $x \parallel u$. The next two questions are $x : \min$ and $x : \max$. To guarantee that there exists a compatible partial order that is a lattice, B has to answer $x < \max$ and $\min < x$. But now the comparability status of x and a_1 may be chosen arbitrarily, since all orders compatible with that state of the game do not contain the forbidden substructure, i.e. are lattices.

2.3 If the lattice has height 2, A first asks $x : \min$ and $x : \max$. If the answers of B are $\min \not< x$ or $x \not< \max$, player A either gains a transitive edge or there exist no compatible lattices.

Otherwise, if $\min < x < \max$, the comparability status of x and all other elements of V may be chosen arbitrarily, since there are no compatible partial order that contains a forbidden substructure, i.e. all compatible partial orders are lattices. □

We nextderive a lower bound for P_L based on the forbidden substructure. Recall the following result from extremal graph theory (see [2]). *If a graph G with n vertices contains more than $\frac{n}{4}(1 + \sqrt{4n-3})$ edges then G must contain a cycle of length 4.*

Theorem 7 *For recognition complexity of lattices is at least*

$$\binom{n}{2} - \frac{(n-2)}{4}(1 + \sqrt{4n-11}).$$

Proof. Player B answers the first question $x : z$ with $x < z$ and sets aside aside x and z to be the maximum and minimum vertices respectively. Further questions involving x and z are answered appropriately. For any question involving the other $n-2$ vertices the answer is incomparable.

For the above strategy it is clear that A must ask the status of x and z with all the other vertices. Furthermore, as long as there is a cycle of length 4 in the graph G_N of not asked edges on $V \setminus \{x, z\}$ there are compatible non-lattices. The above result implies that A has to ask at least $1 + 2(n-2) + (\binom{n-2}{2} - \frac{(n-2)}{4}(1 + \sqrt{4(n-2)-3})) = \binom{n}{2} - \frac{(n-2)}{4}(1 + \sqrt{4n-11})$ questions. □

2.5 Height and width

We first investigate the complexity of the property of containing a chain of at least k elements, i.e., the property of being of height at least $k-1$.

Containing a chain of length 2, i.e., a comparable pair is obviously evasive.

Theorem 8 *The property $\mathcal{P}$ of all partial orders over set V with $|V| = n \geq 4$, that contain a k-chain, $k \geq 4$ has complexity $\mathbf{C}(\mathcal{P}) < \binom{n}{2}$.*

Proof. Player A asks all possible questions over $V \setminus \{x\}$ for fixed x. To guarantee that for the state of the game after these $\binom{n-1}{2}$ questions there exists a compatible partial order containing a chain of length k, player B has to construct a chain of length $k-1$, say $a_1 < a_2 < ... < a_{k-1}$. Now A asks $x : a_2$. If B answers $x \parallel a_2$, then the comparability status of $x : a_1$ and $x : a_i$, $2 < i \leq k-1$, is not relevant for $\mathcal{P}$. Otherwise, if B answers $x < a_2$ or $a_2 < x$ there is an induced transitive edge. □

This argument does not apply to the case $k = 3$. In an extended abstract [6] Grőger claims that the exact complexity of this property is $\binom{n-1}{2} + 3$. With the theorem below we prove the much weaker statement that the property of containing a 3-chain is non-evasive. The lower bound of $\binom{n-1}{2} + 1$ is rather obvious, consider the order with one element x dominating all others and no further comparabilities. Let B give the first answer a comparability and thus decide the element x. Player A then has to ask all pairs in $V \setminus \{x\}$ to make a 3-chain impossible.

Theorem 9 *The property $\mathcal{P}$ of all partial orders over set V, $|V| = n \geq 5$, that contain a 3-chain has complexity $\mathbf{C}(\mathcal{P}) < \binom{n}{2}$.*

Proof. The proof is based on the following facts.

Fact 0. If for a strategy ψ there exists a state $((C, <), I, N)$ with two elements a, b with $\{a, b\} \in N$ and $\{x : x \parallel a \text{ or } x \parallel b\} = V \setminus \{a, b\}$ then there exists an algorithm φ such that $\mathbf{C}(\mathcal{P}; \varphi, \psi) < \binom{n}{2}$.

The pair a, b can not contribute to a 3-chain since the third element x of every triple a, b, x is incomparable to a or b.

Fact 1. If for a strategy ψ there exists a state $((C, <), I, N)$ and four elements a, b, c, d such that $\{a, b\}, \{c, d\} \in C$, $a < b, c < d$ and $\{a, d\} \in N$, $\{b, d\}, \{a, c\} \in N \cup I$ then there exists an algorithm φ with $\mathbf{C}(\mathcal{P}; \varphi, \psi) < \binom{n}{2}$.

Player A can obtain $\{a, c\} \in I$ and $\{d, b\} \in I$, otherwise there would be a 3-chain. Now A asks all remaining questions $\{a, x\}$, $x \neq d$ and all questions

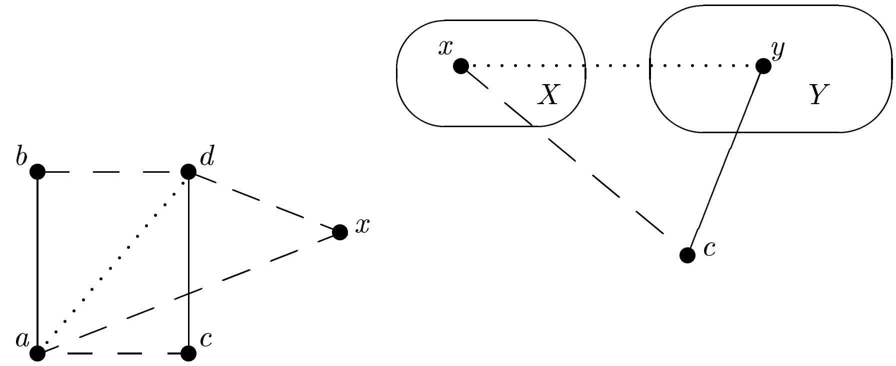

Figure 4: Illustrations for the two facts.

$\{d,x\}$, $x \neq a$. If $\{a,x\} \in C$ then $a < x$, otherwise we would have a 3-chain, symmetrically $\{d,x\} \in C$ implies $x < d$. From this we conclude that for all $x \neq a, d$ either $\{a,x\} \in I$ or $\{d,x\} \in I$, but now the comparability status of $a : d$ is not essential for $\mathcal{P}$, since this pair can not contribute to a 3-chain in a compatible order. (See left side of Figure 4).

Fact 2. If for a strategy ψ there exists a state $((C,<),I,N)$ such that

i) $\{c,v\} \in C \cup I$ for a fixed $c \in V$ and all $v \in V \setminus \{c\}$,

ii) there are $x, y \in V$ with $\{x,c\} \in I, \{y,c\} \in C$,

iii) for all $x, y \in V$ with $\{x,c\} \in I, \{y,c\} \in C$, we have $\{a,b\} \in N$

then there exists an algorithm φ such that $\mathbf{C}(\mathcal{P};\varphi,\psi) < \binom{n}{2}$.

W.l.o.g. $c < b$. Let $X = \{x \in V \setminus \{c\} : \{c,x\} \in I\}$ and $Y = \{y \in V \setminus \{c\} : \{c,y\} \in C\}$, then $c < y$ for all $y \in$ since otherwise there is a 3-chain. We next ask for all the remaining pairs $\{x,x'\}$ and $\{y,y'\}$. The pairs $\{x,x'\}$ are given incomparable, otherwise Fact 1 applies. Moreover, all pairs $\{y,y'\}$ are given incomparable to avoid a 3-chain. But then there exists no compatible partial order in $\mathcal{P}$, i.e. the comparability status of all $x : y$ for $x \in X$ and $y \in Y$ is not essential for property $\mathcal{P}$ since these pairs cannot cause a 3-chain. (See right side of Figure 4).

An algorithm φ with $\mathbf{C}(\mathcal{P};\varphi,\psi) < \binom{n}{2}$ for all strategies ψ is the following. Let $V = \{x_1,\ldots,x_n\}$. First, A asks $x_1 : x_2$. If the answer is $x_1 \parallel x_2$ then A asks $x_1 : x_i$ for $2 < i \leq n-1$. Player B answers $x_1 \parallel x_i$, otherwise Fact 2 applies. But then the question $x_1 : x_n$ can be saved by Fact 0.

Assume B answers $x_1 < x_2$. Now, A asks $x_1 : x_3$. Because of Fact 2, respectively to avoid a chain of length three, B always answers $x_1 < x_3$.

Next A asks $x_4 : x_2$. To avoid a 3-chain, respectively because of Fact 1 with $a = x_4$, $b = x_2$, $c = x_1$, $d = x_3$, player B answers $x_2 \parallel x_4$.

The next question is $x_4 : x_3$. To avoid a chain of length three, B will not answer $x_3 < x_4$. If $x_3 \parallel x_4$ player A continues with $x_4 : x_i$, $i = 5, \ldots, n$ If there is a y comparable to x_4 then Fact 1 applies with the four elements x_4, y, x_1, x_2. Otherwise, the question $x_1 : x_4$ can be saved by Fact 0.

Now assume $x_4 < x_3$. Player A asks all questions $x_i : x_j$ for $5 \leq i, j \leq n$. If one of the answers is a comparability Fact 1 applies. Now assume that all answers are incomparabilities.

Player A asks for $x_2 : x_3$ with answer $x_2 \parallel x_3$. The next question is $x_5 : x_1$, if $x_5 \parallel x_1$ then Fact 0 applies with x_2, x_5.

Now let $x_1 < x_5$, player A continues with $x_5 : x_2$, $x_5 : x_3$ and $x_1 : x_4$, all answers are incomparable, otherwise there is a 3-chain. But now Fact 0 applies with x_4 and x_5. $\square$

We have already seen that properties properties with a large recognition complexity must contain orders of low height. We now consider the *width* of partial orders, i.e. the maximal size of an antichain.

Theorem 10 *Let $\mathcal{P}$ be the property of all partial orders of width k over V, for a fixed k, then $\mathbf{C}(\mathcal{P}) \leq 2kn \log n$.*

Proof. The algorithm φ with $\mathbf{C}(\mathcal{P}; \varphi, \psi) \leq 2kn \log n$ is based on sorting. Let the ground set be indexed, i.e $V = \{x_1, \ldots, x_n\}$, then player A determines one after another the order on $\{x_1, \ldots, x_i\}$ for $1 \leq i \leq n$. Consider $P_i = (\{x_1, \ldots, x_i\}, <)$, if the width of P_i is more than k, then all compatible orders have this property and the game is over. Therefore we assume the width of P_i to be at most k and, by the theorem of Dilworth P_i can be partitioned into k chains $H_i^1, \ldots, H_i^k$.

Let H_i^j be a chain of the chain partition of P_i, say $H_i^j = c_1 < c_2 < \ldots < c_l$. A determines the comparability status of $\{x_{i+1}, c_j\}$ for $1 \leq j \leq l$ using binary search. First A asks for $x_{i+1} : c_{\lceil \frac{l}{2} \rceil}$. If $x_{i+1} < c_{\lceil \frac{l}{2} \rceil}$ (resp. $x_{i+1} > c_{\lceil \frac{l}{2} \rceil}$) then A recursively determines the comparability status of x_{i+1} with the elements of the remaining 'half-chain' $\{c_j \,|\, 1 \leq j < \lceil \frac{l}{2} \rceil\}$ (resp. $\lceil \frac{l}{2} \rceil < j \leq l$).

If $x_{i+1} \parallel c_{\lceil \frac{l}{2} \rceil}$, then A has to determine the indices $l_1 = \max\{j \mid j = 0 \text{ or } c_j < x_{i+1}\}$ and $l_2 = \min\{j \mid j = l+1 \text{ or } x_{i+1} < c_j\}$. This is done by binary search to both half-chains. The comparability status of x_{i+1} with all elements of H_i^j can thus be determined with $2 \log l$ questions. The comparability status of all pairs from $\{x_1, \ldots, x_i\}$ is known after at most

$2k \log n$ queries. Adding the n elements one by one we obtain the overall complexity of $2kn \log n$. □

Algorithm φ not only decides if an unknown partial order has width at most k, but also if it is isomorphic to a fixed partial order P_0 of width k. Thus Theorem 10 improves the upper bound given in [3] (which is $2kn \log n + 3kn$) for the P_0*-recognition problem.*

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