Interval Orders: Combinatorial Structure and Algorithms

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Dipl. Math. Stefan Felsner
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(D 83)

Preface

This thesis is based on my research on partially ordered sets and specially interval orders, that began when I came to Berlin in 1988. Professor R.H. Möhring introduced me to the field, I like to thank him for stimulations and guidance over the years. Beside the introduction, the thesis combines five chapters that have in common the central role played by interval orders. On the other hand, they are only loosely connected and so I decided to make the chapters self-contained. To emphasize the independency of the chapters references are given at the end of each one. Articles and books, that are of significance to the general theme and have been consulted without being cited directly, are collected in the references of the introduction. An outline of the contents of the thesis can be found in the preview at the beginning of the introduction. All further chapters are opened by a section called 'Introduction and Overview'. That special section may serve as an extended abstract for the contents of the chapter, it also gives the relationship to the existing literature.

I am indebted to many people for encouragement and discussions. Special thanks go to Tom Trotter, Lorenz Wernisch and the members of our group 'discrete algorithmical mathematics' which provided an ageeable and creative atmosphere.

Stefan Felsner
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Contents

Chapter 1

Introduction

1.1 Preview

Here we only give a very brief survey of the contents of this thesis. For more detailed information on the subject of a single chapter we refer to introductory sections, called 'Introduction and Overview', there we also relate our own results to the existing literature.

A family $(I_x)_{x \in X}$ of intervals on the real line may be used to define a partial ordering on X. We put $x < y$ if $a \in I_x$ and $b \in I_y$ implies $a < b$. A partial order $I = (X, <)$ thus obtained is called an *interval order*. In this chapter we introduce basic terminology and facts about partial orders, linear extensions and interval orders that are used throughout all parts. The next three chapters then deal with problems for interval orders that are known to be NP-complete for general partial orders.

In the next chapter we consider the *jump number* problem. That is, we are looking for a linear extension of I which has a minimal number of adjacent pairs that are incomparable in I. We introduce parameters α and β of an interval order and prove a close relationship with the jump number.

(1) The jump number of an interval order is at least $\max(\alpha, \alpha + \frac{\beta - \alpha}{3})$.

(2) A linear extension with at most $\alpha + \frac{\beta}{2}$ jumps can always be found. This is an approximation ratio of $\frac{3}{2}$.

(3) There is a polynomial algorithm which decides whether the jump number is exactly α.

(4) It is NP-complete to decide whether the jump number is exactly $\alpha + \frac{\beta - \alpha}{3}$.

Chapter 3 deals with the *dimension* of interval orders. The dimension of a partial order is the minimal dimension of an euklidean space that admits an embedding of the order. We first use the concept of marking intervals to obtain easy proofs for some logarithmic bounds. Afterwards, we introduce the step-graph of an interval order. A partition of the arcs of the step-graph into semi-transitive classes leads to a realizer. This observation is used to give more bounds, particularly, a doubly logarithmic bound. Motivated by the proof of the lower bound for the dimension of interval orders we, finally, study colorings and arc-colorings of digraphs and step-graphs.

In Chapter 4 we directly deal with colorings, namely with the chromatic number of the diagram of an interval order. It is shown that $2 + \log n$ colors always suffice for diagrams of hight n. On the other hand, there are interval orders of this hight, such that the diagram requires $1 + \log n$ colors. The construction of good colorings relies on the existence of certain sequences of sets of colors (α-sequences). An upper bound for the length of α-sequences is given and we show that α-sequences of this length correspond to level accurate Hamiltonian paths in the Boolean lattice. In Boolean lattices of order ≤ 9 we could construct such paths.

Chapter 5 deals with the interplay of *interval dimension* and dimension. We define a transformation $P \to Q$ between partial orders, such that the dimension of Q and the interval dimension of P agree. We provide two interpretations of this transformation, a combinatorial one and a geometrical one. These two interpretations are used for several consequences: We relate the interval dimension of subdivisions of an order P to the dimension of P. We give a new proof for the comparability invariance of interval dimension.

Finally, in the last chapter, we deal with *tolerance graphs*. The complement of a bounded tolerance graph has an orientation as partial order of interval dimension 2. This order admits a square representation while general orders of interval dimension 2 require non-square rectangels in their representation. This observation is the starting point for our invertigations about the relations between several classes of graphs. Some of our results are: If the complement of a tolerance graph admits an orientation as partial order, then this order has interval dimension 2. We give an example of an alternatingly orientable graph that is the complement of an order of interval dimension 2 but is not a tolerance graph. We also characterize the tolerance graphs among the complements of trees.

1.2 Partially Ordered Sets

In this and the next section we introduce necessary terminology and basic results about partially ordered sets, linear extensions and interval orders.

A *strict partial ordering* R on a (finite) set X is a binary relation on X such that

(1) (x, x) is not in R for any $x \in X$.

(2) if (x, y) and (y, z) are in R then (x, z) is in R.

Condition 1 is the *irreflexivity* of R and Condition 2 says that R is *transitive*. Note, that as a consequence of the two conditions we obtain that R is *antisymmetric*, i.e., if $(x, y) \in R$ and $x \neq y$, then (y, x) is not in R.

In some situations it is appropriate to use the idea of a *reflexive partial ordering*. This is a relation R which is *reflexive* (i.e., (x, x) is in R for each $x \in X$), antisymmetric and transitive. The difference between a strict and a reflexive ordering is normally clear from the context, and the word ordering may refer to either kind of ordering.

A set X together with a partial ordering R on X is called a *partially ordered set* and denoted by (X, R). We often abbreviate 'partially ordered set' as either *ordered set*, *order* or *poset*. The commonly used notation for a partially ordered set P is $P = (X, <)$ in the case of a strict ordering, and $P = (X, \leq)$ if the ordering is reflexive. We then write $x < y$ instead of $(x, y) \in <$ and $y > x$ to mean $x < y$. There is a further abuse of notation which should be mentioned. If $P = (X, <)$ is a poset we sometimes write $x \in P$ or $(x, y) \in P$ instead of $x \in X$ and $x < y$. The *comparability graph* of a poset $P = (X, <)$, denoted $\mathsf{Comp}(P)$, is a graph on X, its edges are the *comparabilities* of P, i.e., $\{x, y\}$ is an edge if either $x < y$ or $y < x$.

A binary relation R on a set X can be represented graphically by a directed graph (digraph for short). We represent the elements of X as points and use arrows (arcs) to represent the ordered pairs in R. When the binary relation is a partial ordering the graphical representation can be simplified. Since the relation is understood to be transitive, we can omit arrows between points that are connected by a sequence of arrows. When the graphical representation is oriented such that all arrowheads point upwards, we can even omit the arrowheads as the example in Figure 1.1 shows. Such a graphical representation of a poset in which all arrowheads are understood to point upwards is also known as the *diagram* (or *Hasse diagram*) of the poset.

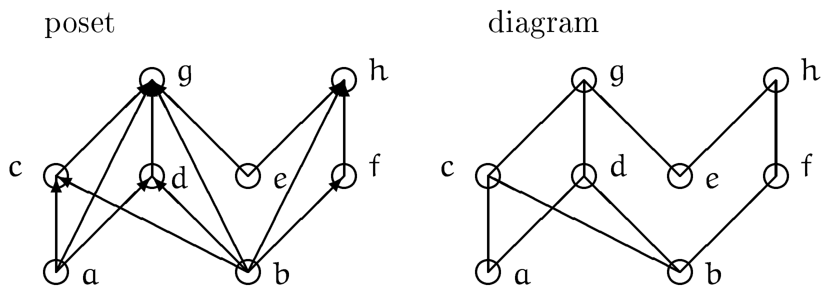

Figure 1.1: A poset and its diagram.

Let $P = (X, <)$ be a partially ordered set. A subset of X is called a *chain* if every two elements in the subset are comparable. Note that if $\{x_1, x_2, \dots, x_k\}$ is a chain in P, then there is some rearrangement of indices, such that $x_{i_1} < x_{i_2} < \dots < x_{i_k}$. We refer to the number of elements in a chain as the *length* of the chain. The *height* of an element $x \in P$, denoted by $\mathsf{height}(x)$, is the length of the longest chain $x_1 < x_2 < \dots < x$, i.e., of the longest chain ending in x. The *height* of a poset P, denoted $\mathsf{height}(P)$, is the length of a longest chain in P. A subset of X is called an *antichain* in the poset $P = (X, <)$ if no two distinct elements in the subset are related. A pair x, y of unrelated elements is also called an *incomparable* pair and denoted by $x||y$. The *width* of a partially ordered set $P = (X, <)$, denoted $\mathsf{width}(P)$, is the maximal size of an antichain in P. For example in the poset P of Figure 1.1, $\{a, c, g\}, \{b, d, g\}$ and $\{a, d\}$ are chains, and $\{c, d, e, f\}, \{a, f\}$ and $\{b, e\}$ are antichains, the height of c is 2, while $\mathsf{height}(P) = 3$ and $\mathsf{width}(P) = 4$.

Let $P = (X, <)$ be a partially ordered set. An element x in P is called a *minimal element* if there is no y with $y < x$, the set of all minimal elements of P is denoted by $\mathsf{Min}(P)$. An element x in P is called a *maximal element* if there is no y with $y > x$, the set of all maximal elements is denoted by $\mathsf{Max}(P)$. If P is the poset of Figure 1.1 then $\mathsf{Min}(P) = \{a, b, e\}$ and $\mathsf{Max}(P) = \{g, h\}$. An element x is said to *cover* another element y, denoted $x \succ y$, if $x > y$ and there is no element z with $x > z > y$. Note that the covering pairs of an order P are exactly the edges in the diagram of P.

As an illustration of the concepts of chains and antichains in partially ordered sets we present two theorems that show a close relationship between them.

Theorem 1.1 *Let $P = (X, <)$ be a partially ordered set. The height of P equals the minimum number of antichains required to cover the elements of P.*

Proof. Let C be a chain of length $\mathsf{height}(P)$. Since an antichain may contain at most one element of C, a covering of P by antichains requires at least $\mathsf{height}(P)$ antichains.

For the converse we use induction on the height of P. Let $\mathsf{height}(P) = n$ and $M = \mathsf{Min}(P)$. Clearly, M is a nonempty antichain. Consider now the poset $P' = (X \setminus M, <)$ and note that $\mathsf{height}(P') = n-1$. By induction, P' can be covered by $n-1$ antichains. Thus P can be covered by n antichains. □

The dual of this theorem is given next. It is known as Dilworth's theorem.

Theorem 1.2 (Dilworth) *Let $P = (X, <)$ be a partially ordered set. The width of P equals the minimum number of chains needed to cover the elements of P.*

Proof. Let A be an antichain of size $\mathsf{width}(P)$. Since any chain may contain at most one element of A, a covering of P by chains requires at least $\mathsf{width}(P)$ chains.

For the converse we use induction on $|X|$, the number of elements in P. Let C be a maximal chain of P. If C nontrivially intersects every maximum antichain, then we remove C and use induction. Otherwise, there is a maximum antichain A disjoint from C. Set $A^U = \{x \in X \mid x \geq a \text{ for some } a \in A\}$ and $A^D = \{x \in X \mid x \leq a \text{ for some } a \in A\}$. Note that $C \subseteq A^U$ would imply, by maximality of C, that the minimal element of C is in A. Similarly, $C \not\subseteq A^D$. Hence $|A^U| < |X|$ and $|A^D| < |X|$. By induction A^U and A^D both can be covered with $|A|$ chains, which have the elements of A as maximal respectively minimal elements. At these points the chains of A^U and A^D can be put together. Since A is maximal, $A^U \cup A^D = X$ and we obtain a covering of P by $|A|$ chains. □

The probably most important family of partial orders are the Boolean lattices. The *Boolean lattice* $\mathcal{B}_n$ is the set of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion, i.e., $A < B$ iff $A \subset B$. As a nice application of chain coverings we will now derive another classical theorem of poset theory. It is known as Sperner's theorem.

Theorem 1.3 (Sperner) *Let A be an antichain of subsets of an n-set. Then*

$$|A| \leq \binom{n}{\lceil \frac{n}{2} \rceil}.$$

Proof. We use induction on n. For $n = 1$ the claim is trivially true.

If $C = (S_1 < S_2 < \ldots < S_\ell)$ is a chain of length $\ell > 1$ in $\mathcal{B}_{n-1}$. Then $C^1 = (S_1 < S_2 < \ldots < S_\ell < S_\ell \cup \{n\})$ and $C^2 = (S_1 \cup \{n\} < S_2 \cup \{n\} < \ldots < S_{\ell-1} \cup \{n\})$ are chains in $\mathcal{B}_n$. Now, let $\{C_1, C_2, \ldots, C_t\}$ be a minimal family of chains covering $\mathcal{B}_{n-1}$. By induction $t = \binom{n-1}{\lceil \frac{n-1}{2} \rceil}$. Note that every chain C_i has to contain a set from the antichain of $\lceil \frac{n-1}{2} \rceil$-element sets.

Now consider the collection $\mathbf{C}$ consisting of the chains C_i^1, C_i^2, if the length of C_i is at least 2, together with the chains C_i^1, if the length of C_i is one. It can be seen that $\mathbf{C}$ is a chain covering of $\mathcal{B}_n$. Moreover, every chain in $\mathbf{C}$ contains a set of size $\lceil \frac{n}{2} \rceil$. Therefore, $\mathbf{C}$ consists of $\binom{n}{\lceil \frac{n}{2} \rceil}$ chains and and every antichain in $\mathcal{B}_n$ has at most this size. □

A poset $P = (X, <)$ which is a chain is called a *linear order*. Let $L = (X, <_L)$ be a linear order, we then write $L = x_1, x_2, \ldots, x_n$ as an abbreviation for $x_1 <_L x_2 <_L \ldots <_L x_n$. An *extension* Q of a poset $P = (X, <_P)$ is a poset on the same elements with $x <_Q y$ whenever $x <_P y$. Of special importance are those extensions of P which are linear orders, they are called *linear extensions*.

A linear extension $L = x_1, x_2, \ldots, x_n$ of $P = (X, <)$ induces a partition $L = C_1, C_2, \ldots, C_m$ of P into chains. The chains in this partition are maximal segments of L such that the elements in the segment are pairwise comparable in P, consequently $\mathsf{max} C_i \not< \mathsf{min} C_{i+1}$ for each i. The problem of minimizing the number of chains in a partition induced by a linear extension is the *jump number* problem. In the context of this problem it is useful to view linear extensions as the result of an algorithmic process. A generic algorithm for linear extensions is Algorithm 1.1.

Using appropriate specifications of the subroutine **choose** we may obtain every linear extension of P as output of the algorithm. With the proof of the next lemma we give an application of this freedom of specification.

Lemma 1.1 *Let $P = (X, <)$ be a poset and $x||y$. Then there is a linear extension L of P which takes x before y, i.e., $x <_L y$.*

Algorithm 1.1:

```
LINEAR EXTENSION
L = [ ]        (* the empty list *)
for i = 1 to n do
    choose( x_i ∈ Min(P) )
    L = L + x_i
    P = P \ {x_i}
output L
```

Proof. Specify the choice in the following way: Choose any element distinct from y, as long as $\mathsf{Min}(P) \neq \{y\}$. Using this rule y is chosen exactly when $P = \{z \in X \mid y < z\}$. Since x is not in this set we will find x somewhere before y in L. □

Note that the intersection of two partial orderings is again irreflexive and transitive, that is a partial ordering. With this in mind we can state another classical theorem.

Theorem 1.4 (Dushnik and Miller)
Every poset P is the intersection of its linear extensions.

Proof. This is an immediate consequence of Lemma 1.1. □

A family of linear extensions of P such that P is their intersection is called a *realizer* of P. Note that for each incomparable pair $x||y$ in P there must be two linear extensions L_1, L_2 in a realizer with $x <_{L_1} y$ and $y <_{L_2} x$, we then say the pair x, y is *realized* by L_1, L_2. The size of the smallest realizer of $P = (X, <)$ is called the *dimension* of P and abbreviated $\mathsf{dim}(P)$. Some authors prefer the more precise names *order dimension* or *Dushnik–Miller dimension*. Nowadays, dimension theory is a strong branch in the theory of partially ordered sets. This is documented by the recent book of Trotter [Tr], which gives a comprehensive survey.

Let $\{L_1, \ldots, L_k\}$ be a realizer of an order P. With every $x \in P$ we associate the vector $(x^1, \ldots, x^k) \in \mathbb{R}^k$, where x^i gives the position (coordinate) of x in L_i. This mapping of the points of P to points of $\mathbb{R}^k$ embeds P into the componentwise ordering of $\mathbb{R}^k$. Ore defined $\mathsf{dim}(P)$ as the minimum k such that P embeds into $\mathbb{R}^k$ in this way. Since the projections of such an embedding on each coordinate yield a realizer, the two definitions are equivalent (see Figure 1.2).

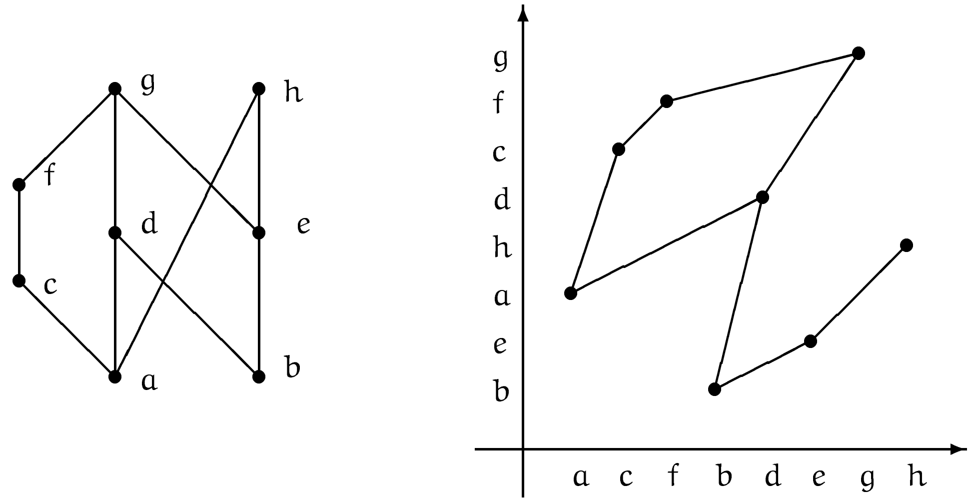

Figure 1.2: A 2-dimensional poset P and an embedding of P in the plane.

Lemma 1.2 *A poset $P = (X, <)$ is two dimensional iff there is a mapping of the elements $x \in X$ to intervals I_x on the real line, such that $x < y$ iff $I_x \subset I_y$.*

Proof. We first claim that there is no loss of generality if we only deal with families $(I_x)_{x \in X}$ of intervals containing a common point. To see this, note that for any positive number M, we may increase each interval in length by M symmetrically about its center, without affecting the containment relation.

A geometric argument then proves the lemma. Let $r \in \mathbb{R}$ be a point with $r \in \bigcap_{x \in X} I_x$. Take r as pivot and turn the left side of the axis up, thus transforming the intervals into hooks. The construction is illustrated in Figur 1.3. It should be clear how to carry out the converse construction. □

1.3 Interval Orders

Let $(I_x)_{x \in X}$ be a family of intervals on the real line. This family may be used to define a graph and two partial orderings on X.

intersection graph: Edges correspond to pairs of intersecting intervals. That is $G = (X, E)$ is the intersection graph of the family if $E = \{ \{x, y\} : I_x \cap I_y \neq \emptyset \}$. A graph obtained in this way is called an *interval graph*.

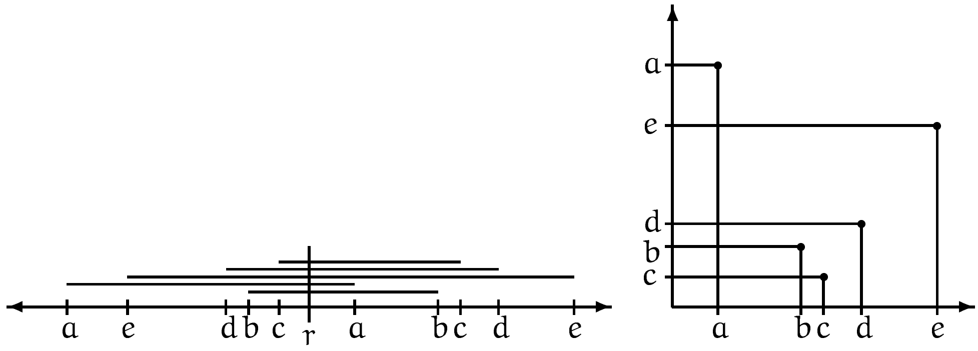

Figure 1.3: A two dimensional order in two representations.

containment order: The ordering on X is given by proper containment, i.e., $x < y$ iff $I_x \subset I_y$. As we have seen in Lemma 1.2, the class of posets obtained in this way is exactly the class of 2-dimensional orders.

visibility order: Here we put $x < y$ if, when looking to the right, every point of I_x can see every point of I_y. In other words $x < y$ iff I_x is entirely to the left of I_y. A partial order obtained in this way is called an *interval order*.

In each of the three cases the collection $(I_x)_{x \in X}$ is called an *interval representation*. In many arguments we will have to refer to the *endpoints* of the intervals corresponding to the elements $x \in X$. To simplify these references let us adopt the convention that $[a_x, b_x]$ is the interval of x, i.e., a_x is the left and b_x is the right endpoint.

From the definitions, we immediately obtain.

Lemma 1.3 *$G = (X, E)$ is an interval graph iff it is the cocomparability graph of an interval order. That is, there is an interval order $I = (X, <)$ such that $\{x, y\} \in E$ iff $x||y$ in P.*

With $\mathsf{Pred}(x) = \{y \in X : y < x\}$ we denote the set of all *predecessors* of x in $P = (X, <)$. Dually, $\mathsf{Succ}(x) = \{y \in X : x < y\}$ denotes the set of all *successors* of x.

There are several important characterizations of interval orders, we combine some of them in the next theorem.

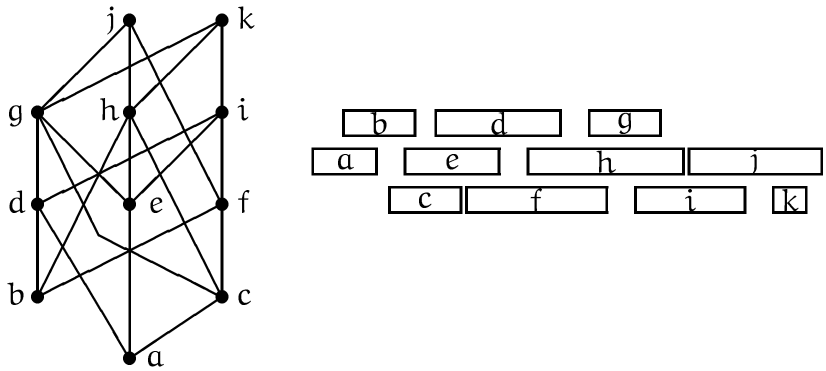

Figure 1.4: An example of an interval order with an interval representation. In most of our figures we indicate the intervals by rectangles.

Theorem 1.5 (Interval Order Characterization)
Let $P = (X, <)$ be a partial order. The following statements are equivalent:

(1) *P is an interval order.*

(2) *P does not contain a **2+2** as induced suborder, that is, a subset $\{u, v, x, y\}$ of X with $u < v$, $x < y$ and no more comparabilities (see Figure 1.5).*

(3) *For all $x, y \in X$ the sets of predecessors are related by $\mathsf{Pred}(x) \subseteq \mathsf{Pred}(y)$ or by $\mathsf{Pred}(x) \supseteq \mathsf{Pred}(y)$, i.e., the sets of predecessors are linearly ordered with respect to inclusion.*

(4) *The sets of successors are linearly ordered with respect to inclusion.*

Proof. First, let us inspect that an interval order can not contain a **2+2**. Note that in an interval representation of a 2-chain $x < y$ we necessarily have $b_x < a_y$. Now let $u < v$ and $x < y$ be two 2-chains and consider the ordering of the numbers b_u, a_v, b_x and a_y. From $b_u < a_v$ and $b_x < a_y$ we conclude that either $b_u < a_y$ or $b_x < a_v$ and hence $u < y$ or $x < v$. In any case the set $\{u, v, x, y\}$ can not induce a **2+2**.

To see that (2) implies (3) suppose that (3) does not hold. Then we find v and y such that there are elements $u \in \mathsf{Pred}(v)$ and $x \in \mathsf{Pred}(y)$ with $x \notin \mathsf{Pred}(v)$ and $u \notin \mathsf{Pred}(y)$. The 4 element set $\{u, v, x, y\}$ is thus recognized as a **2+2**.

Essentially the same argument can be used to show that (4) implies (3). If (3) does not hold we find a **2+2** with chains $u < v$ and $x < y$. The sets $\mathsf{Succ}(u)$

and $\mathsf{Succ}(x)$ then are not related by inclusion. The dual argument shows that (3) implies (4).

We finally prove that (3) implies (1). Let $P_1 \subset P_2 \subset \ldots \subset P_\mu$ be the chain of sets of predecessors. For $x \in X$ let $a_x \in \{1, \ldots, \mu\}$ be such that $\mathsf{Pred}(x) = P_{a_x}$ and let b_x be the least index with $x \in P_{b_x}$, if such an index does not exist, i.e., if $x \in \mathsf{Max}(P)$, we define $b_x = \mu + 1$. Note that since $x \notin \mathsf{Pred}(x)$ we always have $a_x < b_x$. We claim that the open intervals (a_x, b_x) for $x \in X$ represent P. If $x < y$ then $x \in \mathsf{Pred}(y)$, hence $b_x \leq a_y$ and the interval of x precedes the interval of y. On the other hand, if $x||y$ then $x \notin \mathsf{Pred}(y)$ implies $a_y < b_x$ and $y \notin \mathsf{Pred}(x)$ implies $a_x < b_y$. Therefore, the intervals of x and y intersect. This representation of an interval order by open intervals with integer endpoints is called the *canonical representation*. □

Of course, if we would have applied the dual construction to prove that (4) implies (1), the resulting interval representation would be the same, i.e., the canonical representation. As a consequence we obtain that the number of different sets of predecessors agrees with the number of different sets of successors, this number, denoted by μ, is called the *magnitude* of the interval order.

If we replace the open intervals (a_x, b_x) of the canonical representation by the (possibly degenerate) intervals $[a_x, b_x - 1]$ we obtain a closed representation with endpoints in $\{1, \ldots, \mu\}$. Therefore, every finite interval order has open and closed representations. In fact the magnitude μ of an interval order I is the least positive integer n for which I has a closed representation with integer endpoints in $\{1, \ldots, n\}$.

With an interval order $I = (X, <)$ we can associate two special linear orderings. In the (*increasing*) Pred–*order* we have $x < y$ iff $\mathsf{Pred}(x) \subset \mathsf{Pred}(y)$ or $\mathsf{Pred}(x) = \mathsf{Pred}(y)$ and $\mathsf{Succ}(x) \subset \mathsf{Succ}(y)$, elements with equal holdings, i.e., elements with identical sets of predecessors and of successors, are ordered arbitrarily. In the (*decreasing*) Succ–*order* we have $x < y$ if iff $\mathsf{Succ}(y) \subset \mathsf{Succ}(x)$ or $\mathsf{Succ}(y) = \mathsf{Succ}(x)$ and $\mathsf{Pred}(y) \subset \mathsf{Pred}(x)$, elements with equal holding are again ordered arbitrarily. Note that Pred–order and Succ–order are linear extensions. In general, the Pred–order of I and the Succ–order of I need not be the same.

An interval order $I = (X, <)$ is called a *semi-order* iff I has an interval representation $([a_x, b_x])_{x \in X}$ such that $b_x - a_x = 1$ for all $x \in X$. We close the introduction with mentioning the characterization theorem for semi-orders.

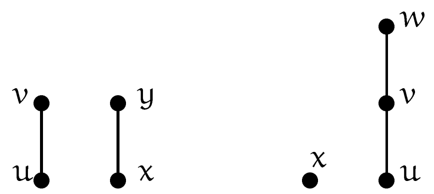

Figure 1.5: The partial orders **2+2** and **1+3**

Theorem 1.6 (Semi-Order Characterization)
Let $I = (X, <)$ *be an interval order. The following statements are equivalent:*

(1) *I is a semi-order.*

(2) *I does not contain a* **1+3** *as induced suborder, that is, a subset* $\{u, v, w, x\}$ *of X with* $u < v < w$ *and no more comparabilities (see Figure 1.5).*

(3) *The* Pred*–order and the* Succ*–order of I are identical.*

1.4 References for Chapter 1

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Chapter 2

The Jump Number of Interval Orders

2.1 Introduction and Overview

Let $L = x_1, x_2, \ldots, x_n$ be a linear extension of P. Two consecutive elements x_i, x_{i+1} of L are separated by a *jump* iff x_i is incomparable with x_{i+1} in P; sometimes we call this the 'jump after x_i'. If $x_i < x_{i+1}$ the pair x_i, x_{i+1} is called a *bump*. The total number of jumps of L is denoted by $s_P(L)$ or $s(L)$. The *jump number* $s(P)$ *of* P is the minimum number of jumps in any linear extension, i.e.

$$s(P) = \min\{s_P(L) : L \text{ is a linear extension of } P\}$$

A linear extension L of P with $s_P(L) = s(P)$ will be called *optimal*.

The jump number has been introduced by Chein and Martin [CM]. The minimization problem, 'determine $s(P)$ and find an optimal linear extension', has been shown to be NP-hard even for bipartite orders (Pulleyblank [Pu], Müller [Mü]). Nevertheless, efficient methods for jump-minimization have been found for large classes of partially ordered sets such as N-free orders (Rival [Ri]), cycle-free orders (Duffus, Rival, Winkler [DRW]) or orders with bounded decomposition width (Steiner [St]). In this chapter we consider the jump number problem on interval orders.

In the next section it is shown that the jump number problem can be reduced to a choice problem: *find an appropriate first element for an optimal linear extension*. We then give some algorithms using different choice rules. In particular the greedy rule and some derivates of the greedy rule are investigated.

The third section deals with lower bounds for the jump number of interval orders. These bounds are derived from a so-called auxiliary list, which is a representation of interval orders which is close to a linear extension.

The auxiliary list is used in section four for the analysis of two algorithms. First we study the T-greedy algorithm (Felsner [Fe1]), which has a performance ratio of 3/2. That is, if P is an interval order and L is the linear extension of P generated by the T-greedy algorithm, then $s(P) \leq \frac{3}{2}s_P(L)$. Approximation algorithms with the same performance bound have recently been proposed by Sysło [Sy] and Mitas [Mi2].

We then give an algorithm based on bipartite matching which recognizes defect optimal interval orders. Mitas [Mi1] obtained the first polynomial recognition of defect optimal interval orders. Earlier, a subclass of defect optimal interval orders has been characterized by Faigle and Schrader [FS2].

In general, approximation algorithms are the best we can hope for. This results from the NP-competeness of the jump number problem on interval orders (Mitas [Mi2]). In section five we give a modified proof of her result, which nicely fits into the theory developed before.

A different light is shed on the complexity of the jump number problem on interval orders by a recent result of de la Higuera [Hi]. He shows, that the jump number problem is polynomial on the class of semi-orders.

2.2 Greedy Linear Extensions and Starting Elements

In the context of the jump number problem it is useful to view linear extensions as being constructed by the generic algorithm, (Algorithm 1.1 on page 11). Note, that the choice made for the i^{th} element determines whether the pair x_{i-1}, x_i is a jump or a bump. This suggests the idea of guiding the choices, such that, elements producing bumps are preferred. This idea leads to the greedy algorithm, Algorithm 2.1.

Linear extensions constructed by the greedy algorithm are called *greedy* linear extensions. Part of the interest in greedy linear extensions has its origin in the following theorem.

Algorithm 2.1:

GREEDY ALGORITHM

```
L = [ ]
for i = 1 to n do
    if Min(P) ∩ Succ(x_{i-1}) ≠ ∅ do
        choose( x_i ∈ Min(P) ∩ Succ(x_{i-1}) )
    else
        choose( x_i ∈ Min(P) )
    L = L + x_i
    P = P \ {x_i}
output L
```

Theorem 2.1 (Rival, Zaguia)
Every poset P has a greedy linear extension which is optimal.

Proof. Call an element x_i *nongreedy* in $L = x_1, x_2, \ldots, x_n$, if for $Q = P \setminus \{x_1, x_2, \ldots, x_{i-1}\}$ we have $\mathsf{Min}(Q) \cap \mathsf{Succ}(x_{i-1}) \neq \emptyset$ and $x_i \notin \mathsf{Min}(Q) \cap \mathsf{Succ}(x_{i-1})$. Note, that linear extensions without nongreedy elements are exactly the greedy linear extensions.

Now assume that P has no optimal greedy linear extensions. For a linear extension L of P let $ng(L)$ be the least index of a nongreedy element in L (if L is greedy let $ng(L) = n + 1$). Our assumption implies

$$r = \max\{ng(L) : L \text{ optimal } \} \leq n.$$

Now let $L = y_1, y_2, \ldots, y_n$ be an optimal linear extension with $ng(L) = r$. Define $L' = y'_1, y'_2, \ldots, y'_n$ as follows.

- $y'_i = y_i$ for $i < r$
- Choose $z \in \mathsf{Min}(P \setminus \{y_1, \ldots, y_{r-1}\}) \cap \mathsf{Succ}(y_{r-1})$ and let $y'_r = z$
- If $z = y_k$ then $y'_i = y_{i-1}$ for $r < i \leq k$
- $y'_i = y_i$ for $i > k$

The linear extensions L and L' only differ in three consecutive pairs, namely, (y_{r-1}, y_r), (y_{k-1}, y_k) and (y_k, y_{k+1}) have been replaced by (y_{r-1}, z), (z, y_r) and (y_{k-1}, y_{k+1}). At least two of the pairs in L are jumps, the pair (y_{r-1}, y_r) and since y_k is minimal in $P \setminus \{y_1, \ldots, y_{r-1}\}$ also the pair (y_{k-1}, y_k). On the other hand at least the pair (y_{r-1}, z) in L' is a bump, therefore, $s(L') \leq s(L)$, and L'

is optimal. But $ng(L') > ng(L) = r$, contradicting our assumption and, hence, proving the theorem. □

Rival [Ri] proved that any greedy linear extension of P is optimal if P is an N–free order. Since then a great amount of work has been invested into investigations of the interplay between greedy and optimal linear extensions. Ghazal et al. [GSZ], for example, characterized greedy interval orders, i.e., interval orders with the property that any optimal linear extension is greedy.

From the algorithms of this section and Theorem 2.1 we conclude that the optimality of a linear extension only depends on a good choice of $x \in \mathsf{Min}(P)$. Faigle and Schrader [FS1] defined the set $\mathsf{Start}(P) \subseteq \mathsf{Min}(P)$ of *starting elements* of P as the set of good choices, i.e., $\mathsf{Start}(P) = \{x \in P : x$ is the first element in some optimal linear extension of $P\}$.

The following two lemmas are valid for the starting elements of arbitrary partial orders. If $P = (X, <)$, we will use the abbreviation P_a to denote the induced order on the set $X \setminus \{a\}$.

Lemma 2.1 *If $a \in \mathsf{Min}(P)$ and $s(P) = s(P_a) + 1$, then $a \in \mathsf{Start}(P)$.*

Proof. Take any optimal extension L' of P_a. Since a is a minimal element of P the concatenation $L = a + L'$ is a linear extension of P. Counting jumps proves the optimality of L. □

Lemma 2.2 *If $a \in \mathsf{Start}(P)$ and $s(P) = s(P_a)$, then a has a successor which is a starting element of P_a, i.e, $\mathsf{Succ}(a) \cap \mathsf{Start}(P_a) \neq \emptyset$.*

Proof. Let $L = a + b + L'$ be any optimal linear extension of P starting with a. If $b \notin \mathsf{Succ}(a)$, or equivalently if a, b is a jump, then $s(P_a) \leq s_{P_a}(b + L') = s_P(L) - 1 = s(P) - 1$. This would contradict $s(P) = s(P_a)$, so $b \in \mathsf{Succ}(a)$. The equalities $s(P_a) = s(P) = s(L) = s(b + L')$ show the optimality of $b + L'$ for P_a, so $b \in \mathsf{Start}(P_a)$. □

The next algorithm (Algorithm 2.2) is very similar to the greedy algorithm. The starty algorithm differs, however, from the former in that it never fails in generating optimal linear extensions.

Lemma 2.3 *If L is the linear extension of P, which is generated by the starty algorithm, then $s_P(L) = s(P)$, i.e., L is optimal.*

Algorithm 2.2:

STARTY ALGORITHM

$\mathsf{L} = [\ ]$
for $i = 1$ **to** n **do**
  if $\mathsf{Start}(P) \cap \mathsf{Succ}(x_{i-1}) \neq \emptyset$ **do**
    choose($x_i \in \mathsf{Start}(P) \cap \mathsf{Succ}(x_{i-1})$)
  else
    choose($x_i \in \mathsf{Start}(P)$)
  $L = L + x_i$
  $P = P \setminus \{x_i\}$
output L

Proof. The proof is by induction on $|P|$. First note that if $L = a + L'$ is generated by the starty algorithm with input P, then with input P_a the starty algorithm produces L', w.r.t appropriate choices. Comparing the jump number of L and L' we find two possibilities:

- If $s(L) = s(L')$, then the trivial inequality $s(P) \geq s(P_a)$ and the optimality of L' for P_a imply the optimality of L.
- Otherwise, i.e., if $s(L) = s(L') + 1$, there is a jump after a, hence $\mathsf{Succ}(a) \cap \mathsf{Start}(P_a) = \emptyset$ and, by Lemma 2, $s(P) = s(P_a) + 1$. Again the optimality of L is deduced from the optimality of L'. □

Corollary. The NP-hardness of the jump number problem implies that, in general, the identification of starting elements is NP-hard, too.

We now turn to interval orders. Here we have slightly more information on starting elements than given in Lemmas 2.1 and 2.2. From the characterization theorem for interval orders (Theorem 1.5), we know, that in an interval order $I = (X, <)$ the sets of successors of any two elements $x, y \in X$ are related by either $\mathsf{Succ}(x) \subseteq \mathsf{Succ}(y)$ or $\mathsf{Succ}(x) \supseteq \mathsf{Succ}(y)$. In the sequel we will often refer to the decreasing Succ–order and especially to the minimal elements in the decreasing Succ–order, therefore, we introduce a new notation $\mathsf{SMin}(I) = \{x \in X : \mathsf{Succ}(x) \supseteq \mathsf{Succ}(y) \text{ for all } y \in X\}$.

Lemma 2.4 (Faigle, Schrader) *Let $I = (X, <)$ be an interval order. If $a \in \mathsf{Min}(I)$, then $a \in \mathsf{Start}(I)$ iff $a \in \mathsf{SMin}(I)$ or $s(I) = s(I_a) + 1$.*

Proof. In an interval order there can be at most one starting element with

$s(I) = s(I_a)$, since only for the Succ-maximal element, which has to be unique in this case, there may exist $b \in \mathsf{Succ}(a) \cap \mathsf{Min}(I_a)$ as required (see Lemma 2.2). Any other $a \in \mathsf{Start}(I)$ thus fulfills the equation $s(I) = s(I_a) + 1$.

So it only remains to prove that every Succ-maximal element a is a starting element. First note that if $L = x_1, x_2, \ldots, x_n$ is optimal and $a = x_i$, then the set $\{x_1, \ldots, x_i\}$ is an antichain in I, since $\mathsf{Succ}(x_j) \subseteq \mathsf{Succ}(a)$ for $j \leq i$. Let x_k be the first element of L with $x_k > x_1$, then $k > i$ and $g : L \mapsto g(L) = x_2 \ldots x_{k-1} x_1 x_k \ldots x_n$ is an operator, which does not increase the number of jumps. The linear extension $g^{i-1}(L)$ is optimal and starts with a. □

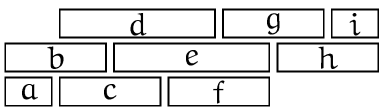
The preceding lemma motivated Faigle and Schrader [FS1] to offer the following greedy-heuristic for jump minimization in interval orders: Construct a greedy linear extension L extending the decreasing Succ-order that omits jumps whenever possible. We again present an algorithmic version of this approach (Algorithm 2.3).

Algorithm 2.3:

Faigle Schrader Heuristic

$L = [\ ]$
for $i = 1$ **to** n **do**
  if $\mathsf{Succ}(x_{i-1}) \cap \mathsf{SMin}(I) \neq \emptyset$ **do**
    choose($x_i \in \mathsf{Succ}(x_{i-1}) \cap \mathsf{SMin}(I)$ **)**
  else
    choose($x_i \in \mathsf{SMin}(I)$ **)**
  $L = L + x_i$
  $I = I \setminus \{x_i\}$
output L

Let L be constructed by the previous algorithm. Faigle and Schrader [FS1] asserted $s_I(L) \leq 2s(I)$. The factor of two, however, is not really correct. Consider this interval order:



The linear extension of I constructed by the previous algorithm has 7 jumps, it is $a|b|c|d|e|f|gi|h$. The optimal linear extension of I only has 3 jumps, it is $ad|be|cfh|gi$. However, with the methods introduced in the next two sections the real factor of their heuristic is easily seen to be less than 3.

2.3 Bounds for the Jump Number of Interval Orders

The main 'tool' in this section will be the *auxiliary list* $\mathbf{L}(I)$ of an interval order $I = (X, <)$. This list is an almost representation of I, i.e., the list encodes almost all the information required to generate the canonical representation of I. In fact, sometimes we can not recover the left endpoint of an interval. For the jump number problem, however, the list $\mathbf{L}(I)$ contains all the relevant information. That is, if two interval orders I and J have isomorphic lists then their jump number is the same. We build up the list $\mathbf{L}(I)$ of I in two steps:

1. Take the elements of I in decreasing Succ–order (see page 15). We thus obtain a linear extension $\Lambda_L = x_1, x_2, \ldots, x_n$.

2. We now include some additional information in this list. First observe that for every pair x_i, x_{i+1} in the list one of the following three cases applies.

   (a) $\mathsf{Succ}(x_i) = \mathsf{Succ}(x_{i+1})$.
   Call this an α-jump.

   (b) $\mathsf{Succ}(x_i) \neq \mathsf{Succ}(x_{i+1})$ and $x_{i+1} \in \mathsf{Succ}(x_i)$.
   This are the bumps of Λ_L.

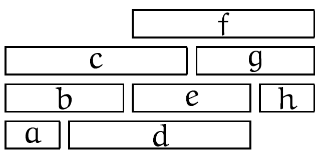
   (c) $\mathsf{Succ}(x_i) \neq \mathsf{Succ}(x_{i+1})$ and $x_{i+1} \notin \mathsf{Succ}(x_{i+1})$.
   Call this a β-jump.

   At every β-jump (x_i, x_{i+1}) of Λ_L we now fix 'a box containing' the set $N = \mathsf{Succ}(x_i) \cap \mathsf{Min}(\{x_{i+1}, \ldots, x_n\})$. This gives the list $\mathbf{L}(I)$.

The auxiliary list then looks like

$$\mathbf{L}(I) = L^1 + \square_{N_1} + L^2 + \square_{N_2} + \ldots + L^\beta + \square_{N_\beta} + L^{\beta+1}$$

Example.



Given this interval order I, we obtain $\mathbf{L}(I) = a\Box_d|b\Box_{ef}|c\Box_g|d|eh|f|g$ and $\Lambda_{\mathbf{L}}(I) = a|b|c|d|eh|f|g$. As indicated by the rules there are 6 jumps, 3 of them are counted by β.

Remarks.

1. Note that if x_i, x_{i+1} is a β-jump and we fix $x_1, x_2, \ldots, x_i$ then the greedy algorithm would choose as the $i+1^{st}$ element some member of $N = \mathsf{Succ}(x_i) \cap \mathsf{Min}(\{x_{i+1}, \ldots, x_n\})$.

2. The List $\mathbf{L}$ is uniquely defined up to interchanges of elements having the same sets of predecessors and successors.

3. The linear extension $\Lambda_{\mathbf{L}}$ could also be generated by the Faigle–Schrader heuristic.

4. The N_i are pairwise disjoint, i.e., an element y may appear at most once in one of the sets $\mathsf{Succ}(x_i) \cap \mathsf{Min}(\{x_{i+1}, \ldots, x_n\})$.

5. If $y \in N_i$, for some i, then y is a minimal element of the remaining poset, hence in $\Lambda_{\mathbf{L}}$ there must be a jump just before y.

6. There is another characterization of the I-invariant α of α-jumps in $\Lambda_{\mathbf{L}}$. Namely $\alpha = n - \mu$, where μ denutes the magnitude of the interval order I. An equality involving α and β is $\beta = n - \alpha - \mathsf{bumps}(\Lambda_{\mathbf{L}})$.

To investigate the performance of algorithms for our problem, it turned out to be useful, to investigate the effect of the choices made on α and β. More precisely, if a minimal element $x \in I$ has been chosen then the α and β of the lists $\mathbf{L}(I)$ and $\mathbf{L}(I_x)$, respectively, are to be compared. In a first application of this technique we will derive lower bounds for the jump number from the list version of the starty algorithm (Algorithm 2.4). We will also refer to this algorithm as 'list starty algorithm'.

Lemma 2.5 *The linear extension $\Lambda_S = \Lambda_S(\mathbf{L}(I))$ generated by the list starty algorithm is an optimal linear extension of I.*

Proof. With reference to Lemma 2.3, it suffices to show that there is a run of the starty algorithm with input I that generates $\Lambda_S(\mathbf{L}(I))$. Let $\Lambda_S = x_1x_2\ldots x_n$,

Algorithm 2.4:

STARTY ALGORITHM (LIST VERSION)

$\mathbf{L} = \mathbf{L}(I)$
while *there is a first* $\square$ *in* $\mathbf{L}$ **do**
  $\mathbf{L} = L + \square_N + \mathbf{L}_1$ (* decompose $\mathbf{L}$ *)
  $J = \{ x : x \in \mathbf{L}_1 \}$
  if $N \cap \mathsf{Start}(J) \neq \emptyset$ **do**
    choose($n \in N \cap \mathsf{Start}(J)$ **)**
    $\mathbf{L} = L + n + \mathbf{L}(J_n)$
  else
    $\mathbf{L} = L + \mathbf{L}_1$
$\Lambda_S = \mathbf{L}$
output Λ_S

the element x_1 is in $\mathsf{SMin}(I)$, so by Lemma 2.4 we know $x_1 \in \mathsf{Start}(I)$, and x_1 is a suitable first choice for the starty algorithm.

Suppose the starty algorithm has chosen $x_1 \dots x_i$, the first i elements of Λ_S. Let $J = I \setminus \{x_1, \dots, x_i\}$. If x_{i+1} is in $\mathsf{SMin}(J)$, then the starty algorithm may choose x_{i+1}. This is due to Lemma 2.4. Otherwise x_{i+1} will be an element of $N = \mathsf{Succ}(x_i) \cap \mathsf{Min}(J)$. In this case our algorithm found the decomposition $x_1 \dots x_i + \square_N + \mathbf{L}(J)$ and selected x_{i+1} as a starting element. Hence, again, x_{i+1} may be chosen by the starty algorithm. $\square$

As announced before, we now look for an expression of $s(\Lambda_S)$ in terms of α and β. We begin with the introduction of some new variables. Let c count the number of times the while loop is repeated in a run of the list starty algorithm, i.e., c is the number of boxes the algorithm finds on its way through the list $\mathbf{L}$. Let c_0 count the number of times the condition $N \cap \mathsf{Start}(J) \neq \emptyset$ appears false, i.e., c_0 is the number of boxes kept empty.

We now turn to the remaining $\mathsf{c} - \mathsf{c}_0$ boxes, i.e., to the boxes filled with some starting element $n \in N$. In each of these cases the tail of $\mathbf{L}$, i.e., $\mathbf{L}' = \mathbf{L}(J)$ is replaced by the new list $\mathbf{L}(J_n)$. A close look at the way auxiliary lists are built enables us to characterize the *transition* $\mathbf{L}(J) \to \mathbf{L}(J_n)$. Since $n \in \mathsf{Min}(J)$, but $n \notin \mathsf{SMin}(J)$, we know that in $\mathbf{L}'$ the element n is preceded by an element x that is incomparable with n. If $\mathsf{Succ}(x) \neq \mathsf{Succ}(n)$, i.e. $\mathsf{Succ}(x) \supset \mathsf{Succ}(n)$, then in $\mathbf{L}'$ we find a box between x and n, and possible patterns for the transition are:

(1)	$x\square_{N_i} \mid n\square_{N_{i+1}} \mid y$	$\to$	xy
(2)	$x\square_{N_i} \mid n\square_{N_{i+1}} \mid y$	$\to$	$x\square_{N_i \cup N_{i+1}} \mid y$
(3)	$x\square_N \mid ny$	$\to$	xy
(4)	$x\square_N \mid n \mid y$	$\to$	$x\square_N \mid y$

Remark. These transitions correspond to:

(1) $\mathsf{Succ}(n) \supset \mathsf{Succ}(y), \quad y \notin \mathsf{Succ}(n), \quad y \in \mathsf{Succ}(x).$

(2) $\mathsf{Succ}(n) \supset \mathsf{Succ}(y), \quad y \notin \mathsf{Succ}(n), \quad y \notin \mathsf{Succ}(x).$

(3) $\mathsf{Succ}(n) \supset \mathsf{Succ}(y), \quad y \in \mathsf{Succ}(n).$

(4) $\mathsf{Succ}(n) = \mathsf{Succ}(y).$

Note that in the auxiliary list we gave preference to the element with maximal set of predecessors in SMin. In the last case above we thus obtain $\mathsf{Pred}(n) \supseteq \mathsf{Pred}(y)$ from the fact that n precedes y in $\mathbf{L}$. This excludes the transition

$$x\square_N \mid n \mid y \quad \to \quad xy$$

Now let α_1, β_1 be the jump counters for $\Lambda_{\mathbf{L}}(L + \mathbf{L}(J))$, and let α_2, β_2 be those for $\Lambda_{\mathbf{L}}(L + n + \mathbf{L}(J_n))$. Depending on the pattern of the transition $\mathbf{L}(J) \to \mathbf{L}(J_n)$ we find

$\alpha_2 = \alpha_1$	$\beta_2 = \beta_1 - 2$	in case (1)
$\alpha_2 = \alpha_1$	$\beta_2 = \beta_1 - 1$	in cases (2) and (3)
$\alpha_2 = \alpha_1 - 1$	$\beta_2 = \beta_1$	in case (4).

If $\mathsf{Succ}(n) = \mathsf{Succ}(x)$ then the jump $x|n$ between x and n is counted by α. When n is pulled forward, then, with respect to the rest of the list, x takes the role of n. Thus the new jump counters are

$$\alpha_2 = \alpha_1 - 1 \qquad \beta_2 = \beta_1 \qquad \text{in case } \mathsf{Succ}(n) = \mathsf{Succ}(x).$$

Now partition the $c - c_0$ starting elements n, which have been pulled forward by the algorithm, according to the type of transition $\mathbf{L}(J) \to \mathbf{L}(J_n)$.

Let c_1 count the transitions of type (1)
c_2 count the transitions of type (2) and (3)
c_3 count the remaining transitions.

With these definitions the validity of the following equations is obvious.

$$c = c_0 + c_1 + c_2 + c_3 \tag{2.1}$$

$$\beta = c + 2c_1 + c_2 \tag{2.2}$$

$$= c_0 + 3c_1 + 2c_2 + c_3 \tag{2.3}$$

At this point we can express $s(I) = s(\Lambda_S)$ in terms of α and the c_i.

Lemma 2.6 *The jump number of I is* $s(\Lambda_S) = \alpha + c_0 + c_1 + c_2$.

Proof. In $\mathbf{L}(I)$ we distinguish between the jumps counted by α and those counted by β. If we transform $\mathbf{L}(I)$ step by step to Λ_S, i.e., for each of the c repetitions of the while loop in the list starty algorithm we consider Λ_L and observe, which jumps disappear. We see that disappearance only happens to jumps on the right side of the current box. Since number and type of disappearing jumps depend on the transition $\mathbf{L}(J) \to \mathbf{L}(J_n)$ only, they are counted by the c_i. Altogether we note the disappearance of c_3 jumps counted by α and $2c_1 + c_2$ jumps counted by β.

Hence $s(\Lambda_S) = \alpha + \beta - 2c_1 - c_2 - c_3$, insert the expression of β given in Formula 2.3 to proof the claim. □

We are ready for the bounds now.

Theorem 2.2 *If I is an interval order then* $s(I) \geq \mathsf{max}\left\{ \alpha\,,\ \alpha + \frac{\beta-\alpha}{3} \right\}$.

Proof. All the c_i are nonnegative, therefore, the first bound $s(I) \geq \alpha$ is an obvious consequence of Lemma 2.6.

Since $c_0 \geq 0$ and $c_2 \geq 0$ we may relax Equation 2.3 to $\beta \leq 3(c_0+c_1+c_2)+c_3$. With $c_3 \leq \alpha$ we obtain $\beta \leq 3(c_0+c_1+c_2)+\alpha$, which is equivalent to $c_0+c_1+c_2 \geq (\beta - \alpha)/3$. This together with Lemma 2.6 yields $s(I) \geq \alpha + \frac{\beta-\alpha}{3}$. □

2.4 An Approximation Algorithm

We start this section at the same point where the last one did end, with some calculations involving α and β. Our first aim will be an algorithm with a 3/2 approximation factor. Therefore, it will be useful to have a nicely expressible lower bound for $\frac{3}{2}s(I)$.

Lemma 2.7 $\frac{3}{2}s(I) \geq \frac{3}{2}\mathsf{max}\{\alpha\,,\ \alpha + \frac{\beta-\alpha}{3}\} \geq \alpha + \frac{\beta}{2}$

Proof. In view of Theorem 2.2 we only have to consider the second inequality. We distinguish two cases.
If $\alpha \leq \beta$ then

$$\frac{3}{2}s(I) \geq \frac{3}{2}\left(\alpha + \frac{\beta-\alpha}{3}\right) \geq \alpha + \frac{\beta}{2}$$

If $\beta < \alpha$ then

$$\frac{3}{2}s(I) \geq \frac{3}{2}\alpha > \alpha + \frac{\beta}{2}$$

□

For reasons of an easy analysis, we first state the next algorithm in its list version (Algorithm 2.5).

Algorithm 2.5:

T–Greedy Algorithm (List Version)

$\mathbf{L} = \mathbf{L}(I)$
while *there is a first* $\square$ *in* $\mathbf{L}$ **do**
  $\mathbf{L} = L + \square_N + \mathbf{L}_1$ (* decompose $\mathbf{L}$ *)
  $J = \{ x : x \in \mathbf{L}_1 \}$
  choose($n \in N$ **)**
  $\mathbf{L} = L + n + \mathbf{L}(J_n)$
$\Lambda_{TG} = \mathbf{L}$
output Λ_{TG}

When comparing this algorithm with the list starty algorithm we note the following.

- Both algorithms start with $\mathbf{L}(I)$ and transform this list step by step into their output.
- In each iteration of both algorithms some tail $\mathbf{L}(J)$ of $\mathbf{L}$ is replaced by $\mathbf{L}(J_n)$.
- The possible transitions $\mathbf{L}(J) \to \mathbf{L}(J_n)$ are the same in both algorithms, i.e. those of the previous section.

Introducing the transition counters c_i we can thus express $s(\Lambda_{TG})$ in terms of α and some c_i.

Lemma 2.8 *The jump number of* Λ_{TG} *is* $\alpha + c_1 + c_2$.

Proof. The proof is almost the same as for Lemma 2.6. The only difference is that now each box found is also filled up, hence $c_0 = 0$. $\square$

Theorem 2.3 $s(\Lambda_{TG}) \leq \frac{3}{2}s(I)$.

Proof. In view of Lemma 2.7 and the previous lemma we only have to show that $\alpha + c_1 + c_2 \leq \alpha + \frac{\beta}{2}$. Since $\beta = 3c_1 + 2c_2 + c_3$ (see Equation 2.3) and the c_i are nonnegative this is obvious. $\square$

It should be evident that the T-greedy algorithm can as well be stated without reference to the list $\mathbf{L}(I)$. For sake of completeness we also state this version (Algorithm 2.6).

Algorithm 2.6:

T–Greedy Algorithm

$L = [\,]$
for $i = 1$ **to** n **do**
  if $\mathsf{Min}(I) \cap \mathsf{Succ}(x_{i-1}) \neq \emptyset$ **do**
    if $\mathsf{SMin}(I) \cap \mathsf{Succ}(x_{i-1}) \neq \emptyset$ **do**
      choose($x_i \in \mathsf{SMin}(I) \cap \mathsf{Succ}(x_{i-1})$ **)**
    else
      choose($x_i \in \mathsf{Min}(I) \cap \mathsf{Succ}(x_{i-1})$ **)**
  else
    choose($x_i \in \mathsf{SMin}(I)$ **)**
  $L = L + x_i$
  $I = I \setminus \{x_i\}$
output L

2.5 Defect Optimal Interval Orders

The *defect* of a partially ordered set has been introduced in (Giertz, Poguntke [GP]) as $\mathsf{def}(P) = |P| - \mathsf{rank}(M_P)$ where M_P is the incidence matrix of P, i.e., $M_P(x, y) = 1$ iff $x <_P y$, otherwise $M_P(x, y) = 0$. The main result of [GP] is

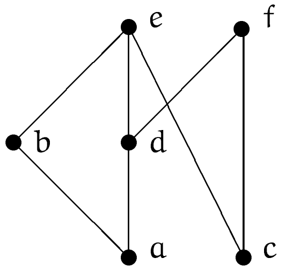
Theorem 2.4

$$s(P) + 1 \leq \mathsf{def}(P)$$

Remark. Orders with equality, i.e., with $s(P) + 1 = \mathsf{def}(P)$, are called *defect optimal*. We now give a short and simple proof for this bound (see [Fe2]).

Proof. Let L be a jump optimal linear extension of P, i.e., $s_L(P) = s(P)$. Order the rows and columns of M_P according to the order of L. Then M_P is an upper diagonal matrix with a zero diagonal. On the super-diagonal of M_P we find a 1 for each bump and a 0 for each jump (x_i, x_{i+1}) of L (see Figure 2.1). Delete the first column and the last row of M_P (both have all entries 0) as well as all the rows and columns corresponding to a 0 on the super-diagonal. The matrix M_P^* thus obtained is an upper diagonal. All the entries on the diagonal of M_P^* are 1, hence, $\mathsf{rank}(M_P^*) = \mathsf{size}(M_P^*) = |P| - s(P) - 1$. With $\mathsf{rank}(M_P) \geq \mathsf{rank}(M_P^*)$ we have finished the proof. □

If I is an interval order then we may arrange the rows of M_I in Succ–order and the columns in Pred–order. This leads to a staircase shape of M_I (see Figure



	a	d	c	f	b	e
a	0	1	0	1	1	1
d	0	0	0	1	0	1
c	0	0	0	1	0	1
f	0	0	0	0	0	0
b	0	0	0	0	0	1
e	0	0	0	0	0	0

Figure 2.1: An interval order I with its incidence matrix M_I. Rows and columns of M_I are arranged according to an optimal linear extension.

2.2). In this representation M_I is easily seen to be of rank $\mu - 1$, where μ denotes the magnitude of I. Therefore, the defect optimal interval orders are exactly the interval orders with $s(I) = |I| - \mu = \alpha$ (the second equality can be found in a remark on page 24).

	a	c	d	b	f	e
a	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	1
b	0	0	0	0	1	1
f	0	0	0	0	1	1
e	0	0	1	1	1	1

Figure 2.2: The 'staircase' shaped M_I of the order given in Figure 2.1.

From Lemma 2.6 we know that for $s(I) = \alpha$ we need $c_0 = 0$, $c_1 = 0$ and $c_2 = 0$ in a run of the list starty algorithm. This means that

1. All the boxes found are filled.

2. All the transitions $\mathbf{L}(J) \to \mathbf{L}(J_n)$ are counted by c_3.

Lemma 2.9 *A transition* $\mathbf{L}(J) \to \mathbf{L}(J_n)$ *is counted by* c_3 *iff* $\mu(J_n) = \mu(J)$.

Proof. Let the elements x, n, y appear consecutively in $\mathbf{L}(J)$. With the transition table it is easy to verify that a necessary and sufficient condition for the transition $\mathbf{L}(J) \to \mathbf{L}(J_n)$ to be counted by c_3 is that either $\mathsf{Succ}(n) = \mathsf{Succ}(x)$ or $\mathsf{Succ}(n) = \mathsf{Succ}(y)$. In both cases $\mu(J_n) = \mu(J)$.

If the transition $\mathbf{L}(J) \to \mathbf{L}(J_n)$ is not counted by c_3, then, $\mathsf{Succ}(x) \neq \mathsf{Succ}(n) \neq \mathsf{Succ}(y)$. In $\mathbf{L}(I)$ the elements are in decreasing Succ–order, so the element n contributes an indentation in the staircase shape of the incidence matrix. We conclude that in this case $\mu(J_n) < \mu(J)$. □

We now reformulate the above conditions. In a run of the list starty algorithm we arrive at $s(\Lambda_S) = \alpha$ exactly if the following holds.

(1) All the boxes found are filled.

(2) For $i = 1, \ldots, \mu$ let S_i be the i^{th} largest set of successors of elements of I. In each set $T_i = \{\, x \,:\, \mathsf{Succ}(x) = S_i \}$ there is at least one element that is not the n of any transition made by the algorithm.

We now construct a bipartite graph $G = (A, B; E)$ as follows:

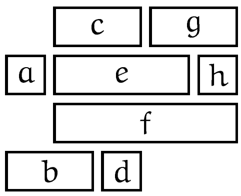
- As the vertices of A take the T_i for $i = 1, \ldots, \mu$.
- There are two kinds of vertices in B.
  Firstly, there are $|N_j| - 1$ copies of each set N_j associated to a box.
  Secondly, we take those elements of I which do not occur in any N_j.
- Let (a, b) be an edge of G
  if $a = T_i$ and either b is a copy of some N_j with $T_i \cap N_j \neq \emptyset$
  or b is a vertex of P and $b \in T_i$.

This construction is illustrated in Figure 2.3.

We now show how to reduce the question whether an interval order I is defect optimal onto a matching problem in G.

Theorem 2.5 *The jump number of I is α iff G has a matching of size μ.*

Proof. Suppose $s(I) = \alpha$ then there is a run of the list starty algorithm satisfying conditions (1) and (2) above. By condition (2) we find an x_i in each T_i which has not been the n of any transition. If x_i is a vertex of the second kind in B then take the edge (T_i, x_i) into the matching. Otherwise x_i is an element of some N_j use (T_i, b) with a still unmatched copy b of N_j for the matching. This never causes trouble since exactly one $n \in N_j$ gave rise to a transition, for the remaining elements of N_j the $|N_j| - 1$ copies of N_j suffice. Each of the μ many T_i is finally matched.



$$\mathbf{L}(I) = a\Box_{c,e,f}|bd|c\Box_g|eh|f|g$$

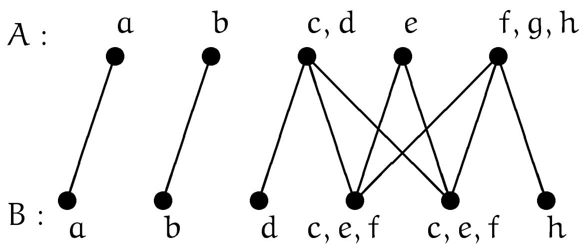


Figure 2.3: An interval order and the associated bipartite graph

Now assume that we have found a matching $M \subseteq E$ of size μ in G. We use the matching to construct a reduced list $\mathbf{L}_R$. First we label the edges of the matching with elements of P. The label $\lambda(a,b)$ of an edge (a,b) is

(i) b if b is a vertex of P

(ii)some element of $T_i \cap N_j$ if $a = T_i$ and b is a copy of N_j.

Define

$$\overline{N_j} = N_j \setminus \{\lambda(a,b) : (a,b) \text{ is a matched edge and } b \text{ is a copy of } N_j\}, \qquad .$$

Now let $\mathbf{L}_R$ be obtained from $\mathbf{L}$ by exchanging each box $\Box_N$ by $\Box_{\overline{N}}$. If we enter the list T–greedy algorithm with $\mathbf{L}_R$ then condition (2) is satisfied since each set T_i contains a element x_i which does not appear in any $\overline{N}$ of $\mathbf{L}_R$. It remains to show that we can fill all the boxes to satisfy condition (1). This can be done since $|\overline{N_j}| \geq 1$ for all j. □

2.6 The *NP*-Completeness Proof

In Theorem 2.2 we have established two lower bounds for $s(I)$. In the last section we saw that the question '$s(I) = \alpha$?' can be decided in polynomial time. In contrast we will now prove that the decision '$s(I) = \alpha + \frac{\beta-\alpha}{3}$?' is NP-complete. Again, we start with extracting necessary and sufficient conditions for equality from the proof of Theorem 2.2.

Lemma 2.10 *If* $s(\Lambda) = \alpha + \frac{\beta-\alpha}{3}$ *then* $c_0 = c_2 = 0$ *and* $c_3 = \alpha$.

Proof. See the proof of Theorem 2.2. In addition we obtain $c_1 = \frac{\beta-\alpha}{3}$. □

The result will be proved by a transformation from the NP-complete problem (X3C) *exact cover by 3-sets*. For background information on the theory of NP-completeness and the X3C Problem, we refer to Garey and Johnson [GJ]. The transformation is due to Mitas [Mi2]. We start with a presentation of the X3C Problem.

EXACT COVER BY 3-SETS

Instance: A set Y of size $3q$ for some positive integer q together with a family F of 3-element subsets of Y.

Question: Does F contain an exact cover of Y, that is a subset $F' \subseteq F$ of q pairwise disjoint subsets of F?

We now introduce a construction which associates an interval order $I_{(Y,F)}$ with a given instance (Y, F) of X3C. Let (Y, F) consist of

$Y = \{y'_1, \ldots, y'_n\}$, $n = 3q$,

$F = \{T_1, \ldots, T_m\}$, $T_i \subset Y$, $|T_i| = 3$.

The order $I_{(Y,F)}$ will consist of a *basis* representing the base-set Y together with a T-*segment* for each $T \in F$. The interval representation of $I_{(Y,F)}$ is an open representation, i.e, a representation by open intervals.

The basis B is a width 3 interval order consisting of 6 frame intervals $a_1, \ldots, a_6$ and an interval y_k for each $y'_k \in Y$. The intervals are:

$$a_1 = [0,1] \qquad a_2 = [0,2] \qquad a_3 = [0,3]$$

$$y_k = [k, k+3]$$

$$a_4 = [n+1, n+4] \qquad a_5 = [n+2, n+4] \qquad a_6 = [n+3, n+4]$$

The T-segment TS_i for $T_i \in F$ consists of 9 intervals. The body of the T-segment consists of 6 short intervals $b^i_0, \ldots, b^i_4$ and c^i, namely,

$$b^i_j = [n + 5i + j - 2, n + 5i + j] \quad \text{and} \quad c^i = [n + 5i + 3, n + 5i + 4].$$

The body of the T-segment is connected with the basis by three interval, t^i_1, t^i_2 and t^i_3, representing the elements of T_i. If $T_i = \{t_{i,1}, t_{i,2}, t_{i,3}\}$ and $t_{i,j} = y'_{k_j}$ then the starting points of t^i_j and y_{k_j} coincide as well as the ending points of t^i_j and b^i_j, that is

$$t_j^i = [k_j, n + 5i + j].$$

Example. For $Y = \{y_1', y_2', y_3', y_4', y_5', y_6'\}$ and $F = \{\{y_1', y_5', y_6'\}, \{y_2', y_3', y_5'\}\}$, the associated interval order is shown in Figure 2.4.

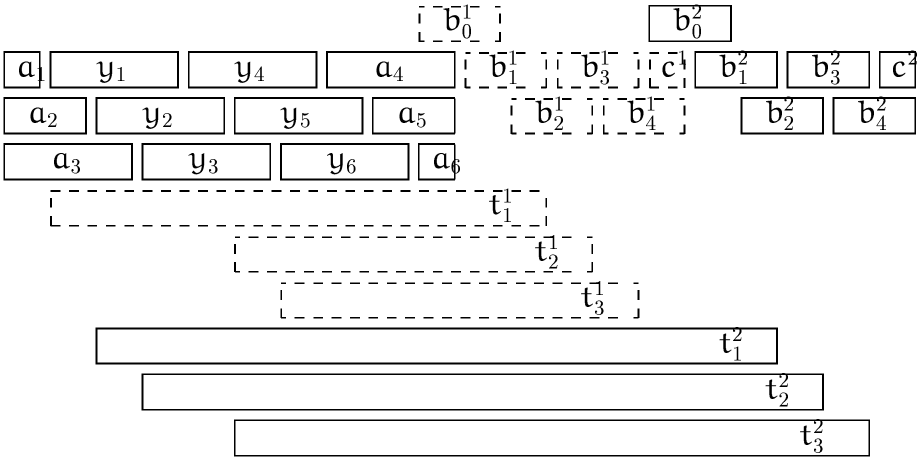


Figure 2.4: The intervals of the first T–segment are 'dashed'

As a useful consequence of this construction, we can obtain the list $\mathbf{L}(I_{(Y,F)})$ by concatenating the list $\mathbf{L}(B)$ with the lists $\mathbf{L}(TS_i)$. To give $\mathbf{L}(B)$ in a closed form we need a further definition. Let

$$N_k = \{y_k\} \cup \{\, t_j^i \,:\, t_{i,j} = y_k' \,\}$$

We then have

$$\mathbf{L}(B) \;=\; a_1\Box_{N_1}|a_2\Box_{N_2}|a_3\Box_{N_3}|y_1\Box_{N_4}|y_2\Box_{N_5}|\ldots$$
$$\ldots y_{n-3}\Box_{N_n}|y_{n-2}\Box_{\{a_4\}}|y_{n-1}\Box_{\{a_5\}}|y_n a_6|a_5|a_4 \qquad (2.4)$$
$$\mathbf{L}(TS_i) \;=\; \Box_{\{b_1^i\}}|b_0^i\Box_{\{b_2^i\}}|b_1^i|t_1^i\Box_{\{b_3^i\}}|b_2^i|t_2^i\Box_{\{b_4^i\}}|b_3^i|t_3^i c^i|b_4^i \qquad (2.5)$$

We count $\alpha(B) = 2$ and $\beta(B) = n+2$. Each T-segment TS_i contributes $\alpha(TS_i) = 4$ and $\beta(TS_i) = 4$. In the concatenated list $\mathbf{L}(I_{(Y,F)})$ we thus have $\alpha = 2 + 4m$ and $\beta = n + 2 + 4m$. The lower bound of Theorem 2.2 gives $s(I) \geq 2 + 4m + \frac{n}{3} = 2 + 4m + q$. With this we are able to state the main theorem of this section.

Theorem 2.6 *The jump number of $I_{(Y,F)}$ is $2 + 4m + q$ exactly if the X3C instance (Y, F) has a solution.*

Proof. Assume that the X3C instance has a solution F'. Let $i(k)$ and $j(k)$ be such that $y'_k = t_{j(k),i(k)} \in T_{i(k)} \in F'$. In a run of the T-greedy algorithm choose $t_{i(k)}^{j(k)} \in N_k$ as the element to be pulled into the box, for all other boxes choose the unique element contained in the box. This leads to

$$\begin{aligned} \Lambda_{TG}(B) &= a_1 t_{i(1)}^{j(1)} | a_2 t_{i(2)}^{j(2)} | a_3 t_{i(3)}^{j(3)} | y_1 t_{i(4)}^{j(4)} | y_2 t_{i(5)}^{j(5)} | \dots \\ & \quad \dots y_{n-3} t_{i(n)}^{j(n)} | y_{n-2} a_4 | y_{n-1} a_5 | y_n a_6 \qquad (2.6) \\ \Lambda_{TG}(TS_i) &= b_1^i | b_0^i b_2^i | t_1^i b_3^i | t_2^i b_4^i | t_3^i c^i \qquad (2.7) \\ \Lambda_{TG}(TS_i) &= b_1^i | b_0^i b_2^i b_4^i | b_3^i c^i \qquad (2.8) \end{aligned}$$

The case 2.7 applies if T_i is not used for the solution, i.e if $T_i \notin F'$, otherwise the T-segment of T_i is transformed to 2.8.

The basis contributes $n+2 = 3q+2$ jumps in $\Lambda_{TG}(B)$. If $T_i \notin F'$ then $\Lambda_{TG}(TS_i)$ contains 4 jumps. In the q T-segments corresponding to $T_i \in F'$ only 2 jumps are in $\Lambda_{TG}(TS_i)$. Altogether, we have $s(\Lambda_{TG}) = 3q + 2 + 4m - 2q = 2 + 4m + q$ and since Λ_{TG} achieves the bound of Theorem 2.2 it is optimal.

For the converse direction we have to show how to derive a solution of (Y, F) from the fact that $s(I_{(Y,F)}) = \alpha + \frac{\beta-\alpha}{3}$. We may assume that an optimal linear extension is produced by the list starty algorithm. Here are properties of an optimal run of this algorithm:

1. While scanning the list the algorithm fills up each box it finds. This because from Lemma 2.10 we know that $c_0 = 0$.

2. The algorithm may never choose $y_k \in N_k$ to fill a box $\square_{N_k}$. The corresponding transition would be counted by c_2, but $c_2 = 0$ by Lemma 2.10. Therefore some $t_{j(k)}^{i(k)} \in N_k$ is chosen.

The claim that proves the theorem is the following: *In an optimal run of the list starty algorithm from each set $T_i \in F$ either all three elements t_1^i, t_2^i and t_3^i are pulled into their boxes in the base or all three remain in the corresponding T-segment.* The elements chosen from the N_k therefore belong to only q different T_i which constitute an exact cover of Y.

To prove the claim we repeatedly apply the following argument:

> From Lemma 2.10 we know that $c_3 = \alpha$. Since each transition counted by c_3 removes an α-jump we may look at α-jumps and decide which of the two elements is pulled out.

Remember that

$$\mathsf{L}(\mathsf{TS}_i) = \square_{\{b_1^i\}}|b_0^i\square_{\{b_2^i\}}|b_1^i|t_1^i\square_{\{b_3^i\}}|b_2^i|t_2^i\square_{\{b_4^i\}}|b_3^i|t_3^i c^i|b_4^i$$

Assume that t_1^i has been pulled into the base. When the algorithm finds $\square_{\{b_1^i\}}$ he has to use the element b_1^i to fill this box. We then have the following situation (overbraced elements may have been pulled out).

$$b_1^i|b_0^i b_2^i \overbrace{|t_2^i} \square_{\{b_4^i\}}|b_3^i \overbrace{|t_3^i} c^i|b_4^i$$

Consider the α-jump $b_2^i|t_2^i$, for this jump, which is removed from the final list, the element t_2^i has to be pulled out. From the α-jump $b_3^i|t_3^i$ we can only remove t_3^i, and this has to be done. Hence, if t_1^i then also t_2^i and t_3^i have been pulled into the base.

Now assume, that t_1^i did remain in the T-segment but t_2^i has been pulled out. After filling the first two boxes, with the unique choice, we remain with

$$b_1^i|b_0^i b_2^i|b_1^i|t_1^i b_3^i \overbrace{|t_3^i} c^i|b_4^i$$

Now the α-jump $c^i|b_4^i$ can't be removed, contradicting the optimality condition. Therefore, if t_1^i remains then t_2^i also remains. Finally, assume that t_3^i has been pulled out. Taking b_3^i forward into the box after t_1^i we would generate a transition counted by c_2 contradicting the conditions of Lemma 2.10. Hence, with t_1^i all three t_j^i remain in the T-segment. □

This NP-completeness result together with the existence of $\frac{3}{2}$–approximation algorithms motivates the following problem.

Problem 2.1 *For which $\frac{3}{2} \geq \epsilon > 1$ does an ϵ–approximation algorithm for the jump number problem on interval orders exist?*

2.7 References for Chapter 2

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Chapter 3

The Dimension of Interval Orders

3.1 Introduction and Overview

A set $L_1, \ldots, L_r$ of linear extensions of P is a *realizer* of P if the intersection of the L_i equals P. This requires that all incomparable pairs $x||y$ are realized in the following sense, there are $i, j \in \{1, \ldots, r\}$ with $x <_{L_i} y$ and $y <_{L_j} x$. The *dimension*, $\mathsf{dim}(P)$, of P is defined as the minimum size of a realizer (for a survey on dimension see [KT]). By a theorem of Yannakakis [Ya] it is in general NP-hard to compute $\mathsf{dim}(P)$. For interval orders, however, the complexity of determining the dimension is open. In this chapter we try to give a comprehensive survey of the knowledge on 'dimension of interval orders'.

Although interval orders have a somewhat one-dimensional nature, their dimension can be arbitrarily high. The first proof of this [BRT] was essentially Ramsey-Theoretic. Here we give the more recent argument of [FHRT].

For an integer n, let $I_n = (X_n, <)$ denote the interval order whose elements are all the closed intervals with integer endpoints from $[n]$ with $[i_1, i_2] < [j_1, j_2]$ iff $i_2 < j_1$. We call the posets in the family $\{I_n : n \in \mathbb{N}\}$ *canonical interval orders*.

For integers n, k with $n > k$, Erdös and Hajnal [EH] defined the *shift-graph* $\mathbf{G}(n, k)$ as the (directed) graph whose vertex set is $\binom{[n]}{k}$ and whose arc set consists of all pairs $(\{x_1, x_2, \ldots, x_k\}, \{x_2, x_3, \ldots, x_{k+1}\})$ where $\{x_1, x_2, \ldots, x_{k+1}\} \in \binom{[n]}{k+1}$. In this context we always require the elements of a set $\{x_1, x_2, \ldots, x_t\}$ to be labeled such that $x_i < x_j$ whenever $i < j$. In Section 3 we will develop the chromatic number theory of shift-graphs. Here we need the following result (proved as Theorem 3.11 below).

Result *The chromatic number of the double shift-graph* $\mathbf{G}(n,3)$ *satisfies*

$$\chi(\mathbf{G}(n,3)) = \log\log n + (\frac{1}{2} + o(1))\log\log\log n$$

We are now able to give the lower bound for the dimension of interval orders.

Theorem 3.1 *Let* $n \geq 4$, *let* $I_n = (X_n, <)$ *be the canonical interval order and* $\mathbf{G}(n,3)$ *be the double shift graph. Then* $\mathsf{dim}(I_n) \geq \chi(\mathbf{G}(n,3))$.

Proof. Suppose that $\mathsf{dim}(I_n) = t$, and let $L_1, \ldots, L_t$ be a realizer of I_n. Now define a coloring $\psi : \binom{[n]}{3} \to [t]$ as follows. For each $\{i_1, i_2, i_3\}$ choose $c \in [t]$ such that $[i_1, i_2] > [i_2, i_3]$ in L_c. We claim that ψ is a proper coloring of $\mathbf{G}(n,3)$. Suppose, on the contrary, that $\psi(\{i_1, i_2, i_3\}) = \psi(\{i_2, i_3, i_4\}) = c_0$, i.e., the edge induced by the four element set $\{i_1, i_2, i_3, i_4\}$ is incident with vertices of the same color. Then in L_{c_0} we have $[i_1, i_2] > [i_2, i_3] > [i_3, i_4]$. Since $[i_1, i_2] < [i_3, i_4]$ in I_n this contradicts that L_{c_0} is a linear extension. We conclude that ψ is a proper coloring of $\mathbf{G}(n,3)$, so $\mathsf{dim}(I_n) \geq \chi(\mathbf{G}(n,3))$. □

In the next section we will study upper bounds for the dimension of interval orders. First, we introduce the concept of marking intervals. This is then used to give common access to logarithmic bounds in terms of the height [Ra2] and the width [FM] of interval orders.

Then we investigate the step graph of an interval order and derive bounds for the dimension from arc-colorings of the step graph.

Finally, we sketch the fascinating new construction of Füredi, Hajnal, Rödl and Trotter [FHRT]. They prove that the dimension of a height n interval order is bounded by a function $f(n)$ which is asymptotically equal to $\chi(\mathbf{G}(n,3))$.

In Section 3.4 we discuss chromatic and arc-chromatic numbers of directed graphs and their line graphs. The results are then applied to shift-graphs, giving estimates of their chromatic numbers. A combinatorial interpretation of the chromatic numbers of shift graphs follows.

3.2 Some Logarithmic Bounds

In this section we are going to develop upper bounds for the dimension of interval orders. These upper bounds are proved by giving a rule which generates a realizer

$L_1, \ldots, L_r$ of an interval order I, such that the size r of the realizer is bounded by a function $f(p(I))$, where p is some parameter of interval orders. The parameters will be the width, the height and the staircase length (see page 45) of I. The main tool in proving the first two bounds will be an alternative definition of the dimension of interval orders which relies on the following lemma.

Lemma 3.1 *Let* $I = (X, <)$ *be an interval order with a closed representation* $[a_x, b_x]$ *and* L *be a linear extension of I. Then there is a function* $m : X \to \mathbb{R}$ *such that*

(1) $m(x) \in [a_x, b_x]$.

(2) *if* $x <_L y$ *then* $m(x) \leq m(y)$.

A function m respecting (1) is called a *marking function*. A marking function having property (2) with respect to L is called an L–*marking*.

Proof. For $L = x_1, \ldots, x_n$ an L–marking is defined by

$$m(x_i) := \max(\ a_{x_i}\ ,\ m(x_{i-1})\ ).$$

It is obvious that m respects property (2). By definition, $a_{x_i} \leq m(x_i)$. To verify property (1) it remains to show that $m(x_i) \leq b_{x_i}$. Note, that for each i we find some l such that $m(x_i) = a_{x_{i-l}}$. Since L is a linear extension we get $x_{i-l} \not> x_i$, hence $a_{x_{i-l}} \not> b_{x_i}$. □

Conversely, if m is a marking function of I then with $x <_Q y$ iff $m(x) < m(y)$ we obtain an extension $Q = (X, <_Q)$ of I. Moreover, m is a L-marking with respect to every linear extension L of Q.

3.2.1 Marking Functions and Bounds

A realizer $L_1, \ldots, L_r$ of $I = (X, <)$ corresponds to a family $m_1, \ldots, m_r$ of marking functions, such that for every incomparable pair $x||y$ there are different $i, j \in \{1, \ldots, r\}$ with $m_i(x) \leq m_i(y)$ and $m_j(x) \geq m_j(y)$.

A special class of linear extensions is obtained if we concentrate on marking functions with $m(x) \in \{a_x, b_x\}$, i.e., we choose one of the endpoints of the interval for the mark. Such a marking function can be transformed into a Boolean vector $f \in \{0, 1\}^{|X|}$ by $f(x) = 0$ iff $m(x) = a_x$. A family $f_1, \ldots, f_r$ of Boolean vectors gives rise to a realizer of I if for $x||y$ there are i, j with $f_i(x) = 0$, $f_i(y) = 1$ and $f_j(x) = 1$, $f_j(y) = 0$. We now give two constructions for such a family of Boolean vectors.

The first construction is essentially the construction of Rabinovich [Ra2]. Let I be an interval order of height h. For $0 \leq k < h$ define H_k as the set of elements of I with $\mathsf{height}(x) = k$. Note that each H_k is an antichain. Moreover, if $k < l$ and $x \in H_k$, $y \in H_l$, then $a_x < a_y$.

We now look for linear extensions $L_1, \ldots, L_s$ of I with the following properties

(1) Each L_i corresponds to a Boolean vector f_i in the sense given above.

(2) If $x, y \in H_k$ then $f_i(x) = f_i(y)$ for $1 \leq i \leq s$, i.e., all the elements of H_k are treated in the same manner.

(3) For an incomparable pair $x||y$ with $x \in H_k$, $y \in H_l$ and $k < l$ there is some f_i with $f_i(x) = 1$ and $f_i(y) = 0$, i.e., y precedes x in L_i.

A family $L_1, \ldots, L_s$ of linear extensions with property 3 can be extended to a realizer with a single additional linear extension L^*. An appropriate L^* is obtained if the elements of H_k precede the elements of H_l in L^* for $k < l$ and the order of elements of the antichain H_k in L^* is exactly the inverse of the order of these elements in L_1.

By property (2) we only have to investigate families of vectors with $f_i \in \{0,1\}^h$. Property (3) then is:

(3′) for $k < l$ there is some i with $f_i^k = 1$ and $f_i^l = 0$.

Therefore, if we arrange the vectors f_i as the rows of a matrix F then every pair f^k, f^l of columns of F differs in some component. Since F is an $s \times h$ matrix we conclude that $2^s \geq h$.

Now let $f^1, \ldots, f^h$ be different vectors of length $s = \lceil \log h \rceil$. We assume that these vectors have been arranged in reversed lexicographic order to form the columns of a matrix F, i.e., for $1 \leq k < l \leq h$, if i is the least component with $f_i^k \neq f_i^l$, then $f_i^k = 1$ and $f_i^l = 0$. This is property (3′), so the columns of F give a family of linear extensions with properties (1)–(3) and we have proved Rabinovich's theorem.

Theorem 3.2 *The dimension of an interval order I of height h is bounded by $\lceil \log h \rceil + 1$.*

Suppose Y and Z are any two disjoint subsets of X. An *injection* of Y by Z is a linear extension L such that z precedes y in L if $y \in Y$, $z \in Z$ and $y||z$. Rabinovich's proof of Theorem 3.2 made essential use of the following characterization of interval orders.

Theorem 3.3 *$P = (X, <)$ is an interval order iff there is an injection of Y by Z for every two disjoint subsets Y and Z of X.*

Proof. If P is not an interval order then there is a **2+2** in P, let $\{y_1, z_1\}$ and $\{y_2, z_2\}$ be the two chains. The injection of $\{y_1, y_2\}$ by $\{z_1, z_2\}$ will not exist.

For the converse consider a marking function m with $m(y) = b_y$ and $m(z) = a_z$ for all $y \in Y$ and $z \in Z$. □

In the previous construction we partitioned the elements of I into the antichains H_k. This time we will make use of a partition into chains. If C and C' are disjoint chains in an interval order I, then all the incomparabilities $x||y$ with $x \in C$ and $y \in C'$ can be realized with two linear extensions $L_{0,1}$ and $L_{1,0}$. The order of $L_{0,1}$ corresponds to the Boolean function $f(x) = 0$ if $x \in C$ and $f(y) = 1$ if $y \in C'$. For $L_{1,0}$ interchange the roles of C and C'.

Let $C_1, \dots, C_w$ be a partition of I into disjoint chains. By Dilworth's theorem we may assume that $w = \mathsf{width}(I)$. If we decide to treat all the elements of C_i the same, a Boolean vector $f \in \{0,1\}^w$ corresponds to a linear extension of I. A family $f_1, \dots, f_r$ of Boolean vectors $f_i \in \{0,1\}^w$ gives rise to a realizer of I if

(4) for every two components k and l there are i and j, such that $f_i^k = 0$, $f_i^l = 1$ and $f_j^k = 1$, $f_j^l = 0$.

Let $f_1, \dots, f_r$ be a realizer of I in the above sense and arrange the vectors f_i as the rows of a matrix F of size $r \times w$. By (4) every pair f^k, f^l of columns of F is an incomparable pair of elements in the Boolean lattice $\mathcal{B}_r$, i.e., the columns are an antichain in $\mathcal{B}_r$. With Sperner's theorem we obtain $\binom{r}{\lceil \frac{r}{2} \rceil} \geq w$.

Now let $f^1, \dots, f^w$ be an antichain in the Boolean lattice $\mathcal{B}_r$ and arrange the f^k as the columns of a matrix F. The rows $f_1, \dots, f_r$ of F then correspond to a realizer of I. This proves the theorem.

Theorem 3.4 *The dimension of an interval order I is bounded by r, if r is the least integer with $\binom{r}{\lceil \frac{r}{2} \rceil} \geq w$.*

Remark. We will frequently need this function in the sequel, therefore, we introduce a name for it. Let

$$N(w) = \min\{r \in \mathbb{N} \ : \ \binom{r}{\lceil \frac{r}{2} \rceil} \geq w\}$$

Combining the ideas of the previous constructions we can prove another theorem.

Theorem 3.5 *The dimension of an interval order* I *which does not contain a* **1+t** *is bounded by* $N(t-1)+1$.

Proof. Let H_k be the set of elements x of I with $\text{height}(x) = k$. The forbidden suborder guarantees that $x < y$ if $x \in H_k$ and $y \in H_l$ with $l \geq k+t-1$. For $1 \leq k < t$ define H_k^* as the union of the levels $H_{k+\ell(t-1)}$ with $\ell \geq 0$. We will treat all the elements of each H_k^* in the same manner. All inequalities in I can be realized if we take the linear extensions corresponding to a set $f_1, \ldots, f_r$ of vectors in $\{0,1\}^{t-1}$ such that

(5) for $k \neq l$ there are i, j with $f_i^k = 0$, $f_i^l = 1$ and $f_j^k = 1$, $f_j^l = 0$,

together with an appropriate L^* which reverses the order in in each H_k relative to L_1. Since properties (5) and (4) are equivalent, a copy of arguments yields the condition we have to put on r, namely $\binom{r}{\lceil \frac{r}{2} \rceil} \geq t-1$, i.e., $r \geq N(t-1)$. □

Remark. As a consequence of this theorem we obtain another result of Rabinovich [Ra1]: *The dimension of a semi-order is at most 3.*

3.2.2 The Step-Graph and More Bounds

Since we will be concerned with bounds for the dimension of interval orders in this section too, we may start spending a fixed number of linear extensions and then investigate, how this set can be augmented to give a realizer. Our initial set consists of two linear extensions. Let (a_x, b_x) be the canonical representation of $I = (X, <)$, then

L_{up} takes x before y if $b_x < b_y$ or $b_x = b_y$ and $a_x > a_y$.

L_{down} takes x before y if $a_x < a_y$ or $a_x = a_y$ and $b_x > b_y$ [1].

Let $x||y$ be an incomparability which is not realized by L_{up}, L_{down}, if L_{down} takes x before y then $a_x < a_y < b_x < b_y$. A pair of elements x, y of I with this ordering of endpoints in the canonical representation will be called a *step* in I.

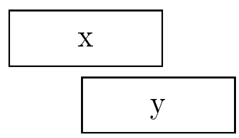

Figure 3.1: An example for a step

[1] if x and y have identical intervals then if x is before y in L_{up} iff y is before x in L_{down}.

A family $L_1, \ldots, L_s$ of linear extensions of I extends L_{up}, L_{down} to a realizer of I if every step is reversed, i.e., if there is an L_i taking y before x.

The *step graph* $S(I) = (X, U)$ of an interval order $I = (X, <)$ is the directed graph with arcs $y \to x$ if x, y is a step of I. A (directed) path in $S(I)$ corresponds to a sequence of steps, i.e., a *staircase*. Let $x_0 \to x_1 \to \ldots \to x_l$ be a path in $S(I)$, then the left endpoints of the intervals of the x_i are ordered by $a_0 > a_1 > \ldots > a_l$. This proves the next lemma.

Lemma 3.2 *The step graph $S(I)$ of an interval order I is acyclic.*

Let $F_1, \ldots, F_s$ be a partition of the arcs of $S(I)$, such that $I \cup F_i$ is acyclic for every i. We then can take linear extensions L_i of $I \cup F_i$ and obtain a realizer $L_{up}, L_{down}, L_1, \ldots, L_s$ of I. We now give a necessary and sufficient condition on a class F, such that $I \cup F$ is acyclic. A class F is called *semi-transitive* if for every path $x_1 \to x_2 \to \ldots \to x_l$ in F the (transitive) arc $x_1 \to x_l$ is in $S(I)$.

Lemma 3.3 *$I \cup F$ is acyclic exactly if F is semi-transitive.*

Proof. Let $x_1 \to x_2 \to \ldots \to x_l$ be a path in $S(I)$ with $x_1 \not\to x_l$. Along the path we have decreasing endpoints $a_1 > a_2 > \ldots > a_l$ and $b_1 > b_2 > \ldots > b_l$. Therefore, it is impossible that one of the intervals (a_1, b_1) and (a_l, b_l) is contained in the other. The reason for $x_1 \not\to x_l$ then has to be a comparability. From $a_1 > a_l$ we conclude $x_l < x_1$. If F is a class containing the above path then $F \cup I$ contains the cycle $[x_0, x_1, \ldots, x_l, x_0]$.

Now, let F be a semi-transitive class and assume that $[x_0, \ldots, x_l, x_0]$ is a cycle in $I \cup F$. In this cycle let $x_i, \ldots, x_{i+j+1}$ be a sequence of F-arcs together with the closing comparability, i.e., $x_{i+j} < x_{i+j+1}$ and $x_i \to \ldots \to x_{i+j}$. By the semi-transitivity of F we have $x_i \to x_{i+j}$ in $S(I)$, hence $a_{x_i} < b_{x_{i+j}}$. The comparability gives $b_{x_{i+j}} \leq a_{x_{i+j+1}}$, hence $a_{x_i} < a_{x_{i+j+1}}$. Let $[y_0, \ldots, y_t, y_0]$ be the subsequence of elements of $[x_0, \ldots, x_l, x_0]$ which appear as the right hand side of a comparability $x_j < x_{j+1}$. The starting points of the intervals in this sequence are strictly increasing, i.e., $a_{y_0} < a_{y_1} < \ldots < a_{y_t} < a_{y_0}$. This contradicts the assumption. □

Note that, if F is semi-transitive and $x \to y$ is the transitive arc of some path in F, then every linear extension of $I \cup F$ will take x before y.

Remark. Coverings of the arcs of a digraph $D = (X, A)$ by semi-transitive classes and by transitive classes, i.e., partial orders, are equivalent problems. The minimal number of partial orders on X whose union covers all arcs of D gives a notion of digraph dimension. Unfortunately, little seems to be known about this concept of dimension which gives a measure for the intransitivity of D.

An *arc-coloring* of a directed graph is an assignment of colors to arcs such that consecutive arcs obtain different colors.

Lemma 3.4 *An arc-coloring of* $S(I)$ *with* k *colors leads to a realizer of* I *with* $k+2$ *linear extensions.*

Proof. Let $F_1, \dots, F_k$ be the color classes of a proper arc-coloring. No F_i does contain consecutive arcs, hence, F_i is semi-transitive, and by the previous lemma $I \cup F_i$ is acyclic. Let L_i, for $i = 1, \dots, k$, be a linear extension of $I \cup F_i$. L_{up}, L_{down} and the k linear extensions L_i together are a realizer of I. □

We now discuss two methods leading to arc-colorings of $S(I)$. They lead us to two new bounds for the dimension of interval orders.

In Section 3.4 Lemma 3.11 we will prove that the arc-chromatic number of a digraph D is at most $N(\chi(D))$. Therefore, we can bound the arc-chromatic number of $S(I)$ by estimating the chromatic number.

Let $\mathsf{ind}(x)$ denote the indegree of the vertex x in $S(I)$ and let $\mathsf{ind}(S(I)) = \mathsf{max}_{x \in X} \mathsf{ind}(x)$. Using the well known list coloring algorithm with a list given by a topological ordering of the vertices of $S(I)$ we obtain a coloring with at most $\mathsf{ind}(S(I)) + 1$ colors. With Lemma 3.11 we then obtain.

Theorem 3.6 *The dimension of an interval order* I *is bounded by the expression* $N(\mathsf{ind}(S(I)) + 1) + 2$.

Here is a second strategy leading to an arc-coloring of $S(I)$. Label an element $x \in X$ with the length of the longest directed path in $S(I)$ which ends in x. Let $h(x)$ be the label of x and $h = \mathsf{max}_{x \in X} h(x)$. Now assume that an arc-coloring of the transitive tournament T_h on h points is given. The arc-set of T_h is $\{(i, j) : i, j \in \{1, \dots, h\}, i < j\}$. Assigning to $(x, y) \in S(I)$ the color of $(h(x), h(y)) \in T_h$, we will obtain a legal coloring of $S(I)$. Therefore, the arc-chromatic number of T_h is an upper bound for the arc-chromatic number of $S(I)$.

Arc-chromatic numbers will be discussed later in this chapter (Lemma 3.9), there we will prove that the arc-chromatic number of T_h is $\lceil \log h \rceil$.

Since h is just the maximal staircase length of I we obtain

Theorem 3.7 *The dimension of an interval order I is bounded by the expression*

$$\lceil \log(\text{maximal staircase length of I}) \rceil + 2.$$

3.2.3 Open Problems

We have obtained bounds for the dimension of an interval order I in terms of the height of I, the width of I, the minimum t such that I does not contain an $\mathbf{1{+}t}$ and in terms of parameters of the step graph. For each of this bounds there are interval orders for which the bound dominates the others. Unfortunately, there remain interval orders for which all these bounds are poorly bad. We now present such an example.

Example. Let $I(n;m)$ be the order defined by all the open intervals of length m with endpoints in $[n]$. For this order only the bound referring to $\mathbf{1{+}m}$ is good, it gives the true value, $\mathsf{dim}(I(n;m)) = 3$. If we add all the open length 1 intervals to obtain $I(n;m,1)$ this bound also goes up to m. If $C = (Y, <_C)$ is a chain, $P = (X, <_P)$ is an arbitrary poset and $Q = (X \cup Y, <_Q)$ induces P on X and C on Y, then the dimension of Q does not exceed the dimension of P by more than 2. Therefore, $\mathsf{dim}(I(n;m,1)) \leq 5$. In fact $\mathsf{dim}(I(n;m,1)) = 3$, we leave this as an exercise.

This example shall serve as a motivation for the following open problems (see [FHRT])

Problem 3.1 *Given an interval order I, is it* NP*-complete to determine the dimension of I?*

Problem 3.2 *Is it true, that for every $n \in \mathbb{N}$ there exists some $t_n \in \mathbb{N}$ so that if I is an interval order with $\mathsf{dim}(I) \geq t_n$ then I contains a subposet Q which is isomorphic to the canonical interval order I_n?*

3.3 The Doubly Logarithmic Bound

We are now ready for a sketch of the most ingenious construction in the field. It is due to Füredi, Hajnal, Rödl and Trotter (see [FHRT]).

Their work splits into two parts. First they proof an estimate for the dimension of canonical interval orders. Their bound is asymptotically optimal, i.e., coincides with the lower bound (Theorem 3.1). In the second part they proof that the dimension of a height n interval order is bounded by the size of the realizer constructed in the first part for I_{n+1}.

Although I did not contribute to the results of this section, proofs are included, since they are a very surprising combination of ideas we have seen earlier in this chapter.

3.3.1 Canonical Interval Orders

Let $I_n = (X_n, <)$ be a canonical interval order. We again start with two special linear extensions L_{up} and L_{down}. (Note that this time we have to change the definitions slightly, since we defined them for the case of open intervals. But I_n consists of closed intervals). We remain with the job of reverting the steps, i.e., the pairs x, y with $a_x < a_y \leq b_x < b_y$. Now let $t \geq N(\log n)$, i.e., if $s = \binom{t}{\lceil \frac{t}{2} \rceil}$ then $n \leq 2^s$. We will show that semi-transitive classes $F_0, F_1, \ldots, F_t$ suffice to cover the arcs of $S(I_n)$. This will prove the theorem

Theorem 3.8 $\mathsf{dim}(I_n) \leq N(\log n) + 3$

Let $M_1, M_2, \ldots, M_s$ be the antichain of subsets on level $\left\lceil \frac{t}{2} \right\rceil$ of $\mathcal{B}_t$ and let $f^1, f^2, \ldots, f^n$ be different column vectors in $\{0,1\}^s$, we assume that these functions are in lexicographic order, i.e., for $k < l$ if i is the least component (row) with $f^k_i \neq f^l_i$ then $f^k_i = 0$ and $f^l_i = 1$.

Lemma 3.5 (Lexicographic Property) *If $S \subseteq [n]$ with $|S| \geq 2$ and i is the least component in which the vectors $\{f^k : k \in S\}$ are not identical, then there is a $k_0 \in S$ such that for $l \in S$, $f^l_i = 0$ for $l < k_0$ and $f^l_i = 1$ for $l \geq k_0$.*

Proof. This is an easy consequence of the lexicographic ordering. □

Let $y \to x$ be an arc in $S(I_n)$, i.e., $a_x < a_y \le b_x < b_y$. The *distinguishing row* of $y \to x$ is the least component i in which the vectors $f^{a_x}, f^{a_y}, f^{b_x}, f^{b_y}$ are not all identical, we will denote this as $d(y \to x) = i$. It is an immediate consequence of the lemma that we may classify an arc $y \to x$ as *balanced*, *zero-dominant* or *one-dominant* by the following scheme:

- $y \to x$ is *balanced* if $i = d(y \to x)$ and $(f_i^{a_x}, f_i^{a_y}, f_i^{b_x}, f_i^{b_y}) = (0, 0, 1, 1)$.
- $y \to x$ is *zero-dominant* if $i = d(y \to x)$ and $(f_i^{a_x}, f_i^{a_y}, f_i^{b_x}, f_i^{b_y}) = (0, 0, 0, 1)$.
- $y \to x$ is *one-dominant* if $i = d(y \to x)$ and $(f_i^{a_x}, f_i^{a_y}, f_i^{b_x}, f_i^{b_y}) = (0, 1, 1, 1)$.

If $y \to x$ is zero-dominant, then let j be the least component in which the vectors $f^{a_x}, f^{a_y}, f^{b_x}$ are not all identical. We call j the *tie-breaking row* and write $j = tb(y \to x)$. Of course $d(y \to x) < tb(y \to x)$.

If $y \to x$ is one-dominant, then let j be the least component in which the vectors $f^{a_y}, f^{b_x}, f^{b_y}$ are not all identical. We call j the *tie-breaking row* and write $j = tb(y \to x)$. Again $d(y \to x) < tb(y \to x)$.

We now define a labeling ψ of arcs with $\psi(y \to x) \in \{0, 1, \dots, t\}$. The classes $F_0, F_1, \dots, F_t$ are then defined via this labeling: $y \to x \in F_{\psi(y \to x)}$.

- If $y \to x$ is balanced we let $\psi(y \to x) = 0$.
- If $y \to x$ is zero-dominant, $i = d(y \to x)$ and $j = tb(y \to x)$ we choose some $c \in M_i \setminus M_j$ and let $\psi(y \to x) = c$.
- If $y \to x$ is one-dominant, $i = d(y \to x)$ and $j = tb(y \to x)$ we choose some $c \in M_j \setminus M_i$ and let $\psi(y \to x) = c$.

Lemma 3.6 *Each of the classes F_c, $0 \le c \le t$, is semi-transitive.*

Proof. In fact, we will proof that F_0 is transitive, and no class F_c, $1 \le c \le t$, does contain consecutive arcs, i.e., they are proper arc color classes in $S(I_n)$.

We first deal with F_0. Let $z \to y$ and $y \to x$ both be balanced. As arcs in the step graph they are steps and we have $a_x < a_y \le b_x < b_y$ and $a_y < a_z \le b_y < b_z$. Note that, since they are balanced we have $f^{a_y}_{d(y \to x)} = f^{a_y}_{d(z \to y)} = 0$ and $f^{b_y}_{d(y \to x)} = f^{b_y}_{d(z \to y)} = 1$. Therefore, $d(y \to x) = d(y \to z)$, let $d = d(y \to x)$.

If $a_z \le b_x$ we have an arc $z \to x$ in $S(I_n)$. We claim that $z \to x$ is also balanced. Since $f_d^{a_x} \neq f_d^{b_x}$ we have $d(z \to x) \le d$. If $d(z \to x) = d$ then $z \to x$ is balanced.

Otherwise, assume $d' = d(z \to x) < d$. In the distinguishing row d' of $z \to x$ we have either $f_{d'}^{a_x} \neq f_{d'}^{b_x}$ or $f_{d'}^{a_z} \neq f_{d'}^{b_z}$. This contradicts either $d(y \to x) > d'$ or $d(z \to y) > d'$.

The remaining case is $b_x < a_z$, i.e., $x < z$. Recall that $d = d(z \to y) = d(y \to x)$ and note that d is the first component in which the vectors $f^{a_x}, f^{a_y}, f^{b_x}, f^{a_z}, f^{b_y}, f^{b_z}$ are not all identical. In the distinguishing row we then find the pattern $(f_d^{a_x}, f_d^{a_y}, f_d^{b_x}, f_d^{a_z}, f_d^{b_y}, f_d^{b_z}) = (0,0,1,0,1,1)$ which violates the lexicographic property. Therefore, if x, z is not a step we cannot have both of $z \to y$ and $y \to x$ in F_0.

Now let $y \to x$ be in F_c, $1 \le c \le t$. Our claim is that $z \to y$ is not in F_c. Let $d = d(y \to x)$ and $d' = d(z \to y)$.

Assume that $y \to x$ is zero-dominant, then $(f_d^{a_y}, f_d^{b_y}) = (0,1)$, therefore, $d' \le d$.

- If $z \to y$ is balanced then it is not in F_c:
- If $z \to y$ is one-dominant then $(f_{d'}^{a_y}, f_{d'}^{b_y}) = (0,1)$ and $d = d'$. As $x \to y$ is zero-dominant $c = \psi(y \to x) \in M_d$ but $\psi(z \to y) \notin M_d$ since $z \to y$ is one-dominant. Hence, $c \neq \psi(z \to y)$.
- If $z \to y$ is also zero-dominant then $(f_{d'}^{a_y}, f_{d'}^{b_y}) = (0,0)$ and necessarily $d' < d$. In the tie-breaking row $t = tb(z \to y)$, however, the lexicographic property ensures $(f_t^{a_y}, f_t^{b_y}) = (0,1)$, therefore, $t = d$. By definition $c = \psi(y \to x) \in M_d$ and $\psi(z \to y) \in M_{d'} \setminus M_d$. Hence, $c \neq \psi(z \to y)$.

Now assume that $y \to x$ is one-dominant.

- If $z \to y$ is balanced then it is not in F_c:
- If $z \to y$ is zero-dominant then $t = tb(y \to x) = tb(z \to y)$ is the first coordinate with $(f_t^{a_y}, f_t^{b_y}) = (0,1)$. By definition $c = \psi(y \to x) \in M_t \setminus M_d$ while $\psi(z \to y) \in M_{d'} \setminus M_t$. Hence, $c \neq \psi(z \to y)$.
- If $z \to y$ is also one dominant, then $(f_d^{a_y}, f_d^{b_y}) = (1,1)$ and $(f_{d'}^{a_y}, f_{d'}^{b_y}) = (0,1)$, hence $d' > d$. The tie-breaking coordinate $t = tb(y \to x)$ is the first

with $(f_t^{a_y}, f_t^{b_y}) = (0,1)$, therefore, $t = d'$. By definition $c = \psi(y \to x) \in M_t \setminus M_d$ while $\psi(z \to y) \notin M_{d'} = M_t$. Hence, $c \neq \psi(z \to y)$.

□

3.3.2 General Interval Orders

For an interval order I with $\mathsf{height}(I) = n$ we now define a mapping $\lambda : I \to I_{n+1}$. Let $A_1, A_2, \ldots, A_n$ be an ordered sequence of maximal antichains covering all the elements of I. For $x \in X$ let $\lambda(x) = [\alpha_x, \beta_x]$ if A_{α_x} is the first and $A_{\beta_x - 1}$ is the last antichain in this sequence containing x. Note that

- $\beta_x < \alpha_y$ implies $x < y$
- $x < y$ implies $\beta_x \leq \alpha_y$.

Let $F_1, F_2, \ldots, F_t$ be the family of semi-transitive classes constructed in Theorem 3.8 to cover the arcs of $S(I_{n+1})$. We have seen that $\mathsf{dim}(I_{n+1}) \leq t + 3$.

Now define $F_i^* = \{x \to y \in S(I) : \lambda(x) \to \lambda(y) \in F_i\}$.

Lemma 3.7 *With $F_1^*, F_2^*, \ldots, F_t^*$ we have a family of semi-transitive classes of arcs of $S(I)$.*

Proof. If $c \geq 1$ the class F_c does not contain consecutive arcs. This property is transferred to F_c^*, hence for $c \geq 1$ the class F_c^* is semi-transitive.

For the case of $c = 0$ note that in F_0 we never have an arc $y \to x$ with $\alpha_y = \beta_x$. This is true since $f^{\alpha_y} = f^{\beta_x}$ prevents $y \to x$ from being balanced. With this additional information F_0^* is easily seen to be semi-transitive as well. □

It remains to take care for those incomparabilities which are realized by the two linear extensions L_{up} and L_{down} of I_{n+1}, as well as for the steps x, y with $\alpha_y = \beta_x$. All this can be done with the two linear extensions:

L^*_{up} takes x before y if $\beta_x < \beta_y$ or $\beta_x = \beta_y$ and $\alpha_x > \alpha_y$.

L^*_{down} takes x before y if $\alpha_x < \alpha_y$ or $\alpha_x = \alpha_y$ and $\beta_x > \beta_y$

As a consequence we obtain:

Theorem 3.9 *If I is an interval order with $\mathsf{height}(I) \leq n$ then $\mathsf{dim}(I) \leq t+3$, if $t \geq N(\log(n+1))$.*

3.4 Colorings of Digraphs and Shift-Graphs

In the first part of this section we develop a theory of colorings for directed graphs. In Lemma 3.8–3.11 we exhibit relations between the chromatic number and the arc-chromatic number. As an application, we obtain estimates for the chromatic numbers of shift-graphs.

In the second part, we prove a combinatorial interpretation of the chromatic numbers of shift-graphs (Theorem 3.12).

I guess that all the material presented here belongs to the folklore of the field, parts of it can be found in [HE] and [FHRT]. However, when I first heard of the Erdös and Hajnal result on the chromatic numbers of shift-graphs (Theorem 3.11) it took me some time to reproduce their estimates. The techniques I used are given in the first part.

3.4.1 Colorings of Directed Graphs

An *arc-coloring* of a directed graph is an assignment of colors to arcs such that consecutive arcs obtain different colors. The *arc-chromatic number* $A(D)$ of a digraph D is the minimal number of colors in an arc-coloring of D. The *chromatic number* $\chi(D)$ of a digraph D is defined as the chromatic number of the underlying undirected graph G_D, i.e., $\chi(D) = \chi(G_D)$. With the next lemmas we exhibit some connections between $A(D)$ and $\chi(D)$.

Lemma 3.8 *For every digraph D chromatic and arc-chromatic numbers are related by: $A(D) \geq \lceil \log \chi(D) \rceil$.*

Proof. Given an arc-coloring of D with l colors we will construct a vertex-coloring with $\leq 2^l$ colors. As colors for the vertices we use boolean vectors of length l. Let c_x^i denote the i^{th} component of the color of x and define $c_x^i = 1$ iff there is an arc (y, x) of color i.

Assume that two vertices u, v of the same color are connected by an arc (v, u). Then the color of the arc (v, u) has to appear as the color of some arc (w, v) conflicting with the proper arc-coloring. □

With the next lemmas we investigate converses of Lemma 3.8. We begin with a special case which turns out to be paradigmatic.

Lemma 3.9 *Let T_n be a transitive orientation of the complete graph K_n, i.e., a transitive tournament, then $A(T_n) = \lceil \log n \rceil$.*

Proof. Since $\chi(T_n) = \chi(K_n) = n$ we obtain $A(T_n) \geq \lceil \log n \rceil$ from the previous lemma.

For the converse, let $n = 2^m$, and let the arcs of T_n be given by the pairs (i, j) with $i < j$ and $i, j \in \{0, 1, \ldots, 2^m - 1\}$. Assign color m to all the arcs (i, j) with $i < 2^{m-1}$ and $j \geq 2^{m-1}$. The uncolored arcs then induce two independent subtournaments on 2^{m-1} points each. The first consist of all arcs (i, j) with $i, j \in \{0, 1, \ldots, 2^{m-1} - 1\}$, the second of all arcs (i, j) with $i, j \in \{2^{m-1}, \ldots, 2^m - 1\}$. By induction we can arc-color each of them with colors from $\{1, \ldots, m - 1\}$. We obtain a proper arc-coloring of T_{2^m} using m colors. □

Lemma 3.10 *Every graph $G = (V, E)$ admits an acyclic orientation D_G so that $A(D_G) = \lceil \log \chi(G) \rceil$.*

Proof. Let $\chi(G) = k$, and let $c : V \to [k]$ be an optimal coloring of G. In D_G an edge $\{u, v\} \in E$ is oriented as (u, v) iff $c(u) < c(v)$. We use an arc-coloring of the transitive tournament T_k with $\lceil \log k \rceil$ colors (Lemma 3.9) to color (u, v) with the color of $(c(u), c(v))$ in T_k. Therefore $A(D_G) \leq \lceil \log \chi(G) \rceil$.

From Lemma 3.8 we obtain the converse inequality. □

Lemma 3.11 *For every digraph D chromatic and arc-chromatic numbers are related by: $A(D) \leq N(\chi(D))$.*

Proof. Let $\chi(G) = k$ and let an optimal coloring $c : V \to [k]$ be given. If $t = N(k)$, i.e., $\binom{t}{\lceil \frac{t}{2} \rceil} \geq k$, then let $M_1, M_2, \ldots, M_k$ be an antichain of subsets on level $\lceil \frac{t}{2} \rceil$ of $\mathcal{B}_t$. To an arc (u, v) assign as color some element from the set $M_{c(v)} \setminus M_{c(u)}$. Consecutive arcs (u, v) and (v, w) obtain different colors since the color of (u, v) is element of $M_{c(v)}$ while the color of (v, w) is not in $M_{c(v)}$. □

The *line graph* $\mathbf{L}(D)$ of a digraph $D = (V, A)$ is the directed graph with vertex set A and arcs (a, b), $a, b \in A$ if $\mathsf{head}(a) = \mathsf{tail}(b)$, i.e., $a = (u, v)$ and $b = (v, w)$ for vertices $u, v, w \in V$. From this definition we immediately obtain:

Lemma 3.12 *If $D = (V, A)$ is a digraph and $\mathbf{L}(D)$ is the line graph of D then $\chi(\mathbf{L}(D)) = A(D)$.*

From the definition of shift-graphs (page 39) the next lemma is immediate.

Lemma 3.13 *The shift-graph* $\mathbf{G}(n, 1)$ *is the transitive tournament* T_n. *Higher shift-graphs can be obtained as line graphs, i.e.,* $\mathbf{G}(n, k+1) = \mathbf{L}(\mathbf{G}(n, k))$.

We conclude this part with our main theorem on the chromatic number of shift-graphs

Theorem 3.10 $\log^k n \leq \chi(\mathbf{G}(n, k+1) \leq N^{k-1}(\log n)$

Proof. This is an easy combination of the preceding lemmas. □

3.4.2 Asymptotics

Here we give the results of some computations which are required to obtain the Erdös, Hajnal theorem as a consequence of Theorem 3.10.

Theorem 3.11 (Erdös,Hajnal)
a) *The chromatic number of the double shift-graph* $\mathbf{G}(n, 3)$ *satisfies*

$$\chi(\mathbf{G}(n, 3)) = \log\log n + (\frac{1}{2} + o(1)) \log\log\log n \ .$$

b) *The chromatic numbers of higher shift-graphs satisfy*

$$\chi(\mathbf{G}(n, k)) = (1 + o(1)) \log^{k-1} n \ .$$

Proof. From Theorem 3.12 and Remark 3.4.3 below we obtain that $\chi(\mathbf{G}(n, 3))$ is the least integer t with $2^{\binom{t}{\frac{t}{2}}} \geq n$, that is, $t \geq N(\log n)$.

Note that by Sterling's formula

$$\binom{n}{\frac{n}{2}} = \frac{2^n}{\sqrt{\frac{\pi}{2} n}}(1 + o(1)) \ .$$

This and the fact that with $n = N(m)$ we certainly have $\log n = o(1) \log m$ gives

$$N(m) = \log m + (\frac{1}{2} + o(1)) \log\log m \ .$$

That is the $\leq$ inequality of **a)**.

Relax the previous formula to obtain

$$N(m) = (1 + o(1)) \log m$$

and note that this implies

$$N^k(m) = (1 + o(1)) \log^k m ,$$

i.e., the conclusion of **b)**. □

3.4.3 A Connection with Lattices of Antichains

A *down set* in an order $P = (X, <)$ is a set $M \subseteq X$ such that $x \in M$ and $y < x$ implies $y \in M$. There is a one-to-one mapping between down sets and antichains given by $M \to \mathsf{Max}(M)$ and $A \to \widehat{A} = \{x \in P \ : \ x \leq y \text{ for some } y \in A\}$. Due to a well known theorem of Dilworth the down sets (antichains) of an order P form a lattice denoted by $A(P)$. The order relation of this lattice is given by set inclusion, i.e., if $M, N \in A(P)$ then $M < N$ iff $M \subset N$. Note, that we can view $A \ : \ P \to A(P)$ as an operator which maps posets to posets. Let $A^k(P)$ denote the poset resulting from k applications of A.

The next theorem gives a surprising connection between the chromatic numbers of shift-graphs and the size of certain down set lattices. (The case $k = 3$ of the theorem can be found in [FHRT]). Let $[t]$ denote the t element antichain, i.e., the poset consisting of c pairwise incomparable elements.

Theorem 3.12 *For all integers n and k the shift-graph $\mathbf{G}(n, k)$ is colorable with t colors if there are at least n elements in the poset $A^{k-1}([t])$.*

Remark. Note that in the case of $k = 2$ this theorem will give an alternative proof for Lemma 3.9.

Proof. Suppose that a proper coloring of $\mathbf{G}(n, k)$ is given by $c \ : \binom{n}{k} \to [t]$. Let

$$C_0(x_1, x_2, \dots, x_k) = c(\{x_1, x_2 \dots, x_k\}),$$
$$C_1(x_2, \dots, x_k) = \{C_0(x_1, x_2, \dots, x_k) \ : \ \text{for some } x_1 < x_2\}$$

and $\widehat{C_1}$ be the down set generated by C_1 in $A([t])$. For $j = 2 \dots k-1$ we iteratively define

$$C_j(x_{j+1}, \dots, x_k) = \{\widehat{C_{j-1}}(x_j, x_{j+1}, \dots, x_k) \ : \ \text{for some } x_j < x_{j+1}\}$$

where $\widehat{C_{j-1}}$ is the down set generated by C_{j-1} in $A^{j-1}([t])$.

Next, we show that $\widehat{C_{k-1}}(x) \not\supseteq \widehat{C_{k-1}}(y)$ for all $k \leq x < y \leq n$. Therefore, all these sets are distinct. Suppose, on the contrary, that $k \leq x_{k-1} < x_k \leq n$ and $\widehat{C_{k-1}}(x_{k-1}) \supseteq \widehat{C_{k-1}}(x_k)$. From this we conclude the existence of some $x_{k-2} < x_{k-1}$ with $\widehat{C_{k-2}}(x_{k-2}, x_{k-1}) \supseteq \widehat{C_{k-2}}(x_{k-1}, x_k)$ and iterating this argument we find $x_1 < x_2 < \ldots < x_{k-2}$ such that $\widehat{C_1}(x_1, \ldots, x_{k-1}) \supset \widehat{C_1}(x_2, \ldots, x_k)$, i.e., $C_1(x_1, \ldots, x_{k-1}) \supset C_1(x_2, \ldots, x_k)$. We now let $d = C_0(x_1, x_2, \ldots, x_k)$, from $d \in C_1(x_1, \ldots, x_{k-1}) \supset C_1(x_2, \ldots, x_k)$ we obtain the existence of some $x_0 < x_1$ such that $C_0(x_0, x_1, \ldots, x_{k-1}) = d$. This contradicts the assumption that c is a proper coloring.

We close this direction of the proof with the observation that at the bottom of $A^{k-1}([t])$ there is a chain of $k-1$ 'empty' elements which can not occur as $\widehat{C_{k-1}}(x)$ for $x \geq k$. These elements are $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots, \{\{\cdots\{\emptyset\}\}\cdots\}$, the last element in this list is an $\emptyset$ enclosed by $k-2$ pairs of braces.

For the converse direction assume that $A^{k-1}([t])$ contains $s \geq n$ elements and let $C(1), C(2), \ldots, C(s)$ be a linear extension of $A^{k-1}([t])$, i.e., if $x < y$ we never have $C(x) \supseteq C(y)$. Therefore, we can choose a down set $C(x, y)$ in $A^{k-2}([t])$ which is an element of $C(y) \setminus C(x)$. We claim that if $w < x$ then there is an element $C(w, x, y) \in C(x, y) \setminus C(w, x)$, otherwise the down set $C(x, y)$ would be contained in the down set $C(w, x)$ but $C(w, x) \in C(x)$ and $C(x, y) \notin C(x)$.

Repeating this, we can associate a set of colors $C(x_2, x_3, \ldots, x_k)$ with each $k-1$ subset of $[n]$, such that $C(x_2, x_3, \ldots, x_k) \in C(x_3, \ldots, x_k) \setminus C(x_2, \ldots, x_{k-1})$. Finally, we find a color $C(x_1, x_2, \ldots, x_k) \in C(x_2, \ldots, x_k) \setminus C(x_1, \ldots, x_{k-1})$. This coloring of the k element subsets of $[n]$ is a proper coloring of $\mathbf{G}(n, k)$. □

Remark. The problem of counting the antichains in $A([t]) = \mathcal{B}_t$ is a classical one. The estimates assert that the number of antichains in $\mathcal{B}_t$ is approximately the number of subsets of the largest antichain, i.e., $2^{\binom{t}{\frac{t}{2}}}$.

3.5 References for Chapter 3

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Chapter 4

Coloring Interval Order Diagrams

4.1 Introduction and Overview

For a nonnegative integer k, let I_k be the interval order defined by the open intervals with endpoints in $\{1, \ldots, 2^k\}$. It has height $2^k - 1$ and is isomorphic to the *canonical* interval order I_{2^k-1} (see Chapter 3 for canonical interval orders).

Two vertices v and w in I_k are a *cover*, denoted by $v \prec w$, exactly if the right endpoint of the interval of v equals the left endpoint of the interval of w. The diagram D_{I_k} of I_k is thus recognized as the *shift graph* $\mathbf{G}(2^k, 2)$ (see Chapter 3 for shift graphs). In general we denote by D_I the diagram of an interval order I, and we denote the chromatic number of the diagram by $\chi(D_I)$.

We again include a proof for the next lemma since we will need similar methods in later arguments (for an alternative formulation and proof of this lemma see Lemma 3.9).

Lemma 4.1

$$\chi(D_{I_k}) = \lceil \log_2 \mathsf{height}(I_k) \rceil = k$$

Proof. Suppose we have a proper coloring of D_{I_k} with colors $\{1, \ldots, c\}$. With each point i associate the set C_i of colors used for the intervals having their right endpoint at i. Note that $C_1 = \emptyset$. For $1 \leq i < j \leq 2^k$, we have $C_j \not\subseteq C_i$; otherwise the interval (i, j) would have the same color as some interval (l, i). This proves that all of the 2^k subsets C_i of $\{1, \ldots, c\}$ are distinct; therefore $2^c \geq 2^k$ and $c \geq k$.

A coloring of D_{I_k} using k colors can be obtained by the following construction. Take a linear extension of the Boolean lattice $\mathcal{B}_k$ and let C_i be the i^{th} set in this list. Assign to the interval (i, j) any color from $C_j \setminus C_i$. A coloring obtained in this way is easily seen to be proper. □

We derive a lemma for later use and a theorem from this construction.

Lemma 4.2 *In a coloring of D_{I_k} which uses exactly k colors, every point $i \in \{1, \ldots, 2^k\}$ is incident with an interval of each color.*

Proof. The crucial fact here is that every subset of $\{1, \ldots, k\}$ is the C_i for some i. Now choose any $i \in \{1, \ldots, 2^k\}$ and a color $c \in \{1, \ldots, k\}$, we have to show that an interval of color c is incident with i.

If $c \in C_i$, then this is immediate from the definition of C_i. Otherwise, i.e., if $c \notin C_i$, then there is a $j_c > i$ such that $C_{j_c} = C_i \cup \{c\}$ and the interval (i, j_c) is colored c. □

With the next lemma we improve the lower bound: There are interval orders I with $\chi(D_I) \geq 1 + \log_2(\mathsf{height}(I))$. Compared with Lemma 4.1, this is a minor improvement, but we feel it worth stating, since later we will prove an upper bound of $2 + \log_2(\mathsf{height}(I))$ on the chromatic number of the diagram of I.

Lemma 4.3 *For each k there is an interval order I_k^* such that*

$$\chi(D_{I_k^*}) \geq 1 + \lceil \log_2 \mathsf{height}(I_k^*) \rceil = k$$

Proof. Take I_k^* as the order obtained from I_k (see Lemma 4.1) by removing the intervals of odd length, i.e., the interval order defined by the open intervals (i, j) with $i, j \in \left\{1, \ldots, 2^k\right\}$ and $j - i \equiv 0 \pmod 2$. The height of I_k^* is $2^{k-1} - 1$ which is the height of I_{k-1}; however, as we are now going to prove, a proper coloring of I_k^* requires at least k colors. Note that two intervals (i_1, j_1) and (i_2, j_2) with $j_1 \leq i_2$ induce an edge in the diagram of I_k^* if either $j_1 = i_2$ or $j_1 = i_2 - 1$.

In I_k^* we find an isomorphic copy of I_{k-1} consisting of the intervals (i, j) with both i and j odd. Call this the odd I_{k-1}. The even I_{k-1} is defined by the interval (i, j) with i and j even. Let C_i be the set of colors used for intervals with right end-point $2i - 1$, and let D_i be the set of colors used for intervals with right end-point $2i$. From Lemma 4.1, we know that if both the odd and the even copy only need $k - 1$ colors, then the C_i and the D_i have to form linear extensions of the Boolean lattice $\mathcal{B}_{k-1}$. Now define $\overline{C_i}$ as the set of colors used for intervals with left-endpoint $2i-1$. From Lemma 4.2, we know that $\overline{C_i}$ is exactly the complement of C_i. With the corresponding definition, $\overline{D_i}$ and D_i are seen to be complementary sets as well. Note that a proper coloring requires $C_i \cap \overline{D_i} = \emptyset$. We therefore have $C_i \subseteq D_i$. A similar argument gives $D_i \subseteq C_{i+1}$. Altogether we find that the C_i have to be a linear extension of $\mathcal{B}_{k-1}$ with $C_i \subseteq C_{i+1}$ for all i. This is impossible. The contradiction shows that at least k colors are required. □

Now we turn to the upper bound which we view as the more interesting aspect of the problem.

Theorem 4.1 *If I is an interval order, then*

$$\chi(D_I) \le 2 + \log_2 \mathsf{height}(I)$$

Proof. In this first part of the proof, we convert the problem into a purely combinatorial one. The next section will then deal with the derived problem.

Let $I = (V, <)$ be an interval order of height h, given together with an interval representation. For $v \in V$, let $(l_v, r_v]$ (left open, right closed) be the corresponding interval. With respect to this representation, we distinguish the 'leftmost' h-chain in I. This chain consists of the elements $x_1, \ldots, x_h$ where x_i has the leftmost right-endpoint r_v among all elements of height i. It is easily checked that $x_1, \ldots, x_h$ is indeed a chain. Now let $r_i = r_{x_i}$ be the right endpoint of x_i's interval and define a partition of the real axis into blocks. The i^{th} block is

$$B(i) = [r_i, r_{i+1}).$$

This definition is made for $i = 0, \ldots, h$ with the convention that $B(0)$ extends to minus infinity and $B(h)$ to plus infinity.

In some sense these blocks capture a relevant part of the structure of I. This is exemplified by two properties.

- The elements v with $r_v \in B(i)$ are an antichain for each i. This gives a minimal antichain partition of I.
- If $r_v \in B(j)$, then $l_v \in B(i)$ for some i less than j.

Suppose we are given a sequence $C_1, \ldots, C_h$ of sets (of colors) with the following property

(α) $C_j \not\subseteq C_{i-1} \cup C_i$ for all $1 < i < j \leq h$.

A sequence with this property will, henceforth, be called an α-sequence. The α-sequence $C_1, \ldots, C_h$ may be used to color the diagram D_I with the colors occurring in the C_i. The rule is: to an element $v \in V$ with $l_v \in B(i)$ and $r_v \in B(j)$ assign any color from $C_j \setminus (C_{i-1} \cup C_i)$. This set of colors is nonempty by the α property of the sequence C_i, since $i < j$. We claim that a coloring obtained this way is proper. Assume, to the contrary, that there is a covering pair $w \prec v$ such that w and v obtain the same color. Let $r_w \in B(k)$ and $l_v \in B(i)$. Since $w \prec v$, we know that $k \leq i$. Due to our coloring rule, we know that the color of w is an element of C_k and the color of v is not contained in $C_{i-1} \cup C_i$; hence $k < i - 1$. This, however, contradicts our assumption that $w \prec v$, since $l_{x_i} \in B(i-1)$ and $l_v \geq r_{x_i} = r_i$ gives $w < x_i < v$.

We have thus reduced the original problem to the determination of the minimal number of colors which admits a α-sequence of length h. We will demonstrate in the next section, Lemma 4.4 and Lemma 4.5, how to construct a α-sequence of length $2^{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor$ using n colors. This will complete the proof of the theorem. □

In the third section we give an upper bound of $2^{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor$ for the maximal length of a α-sequence. From the proof, we derive some further properties α-sequences of this length necessarily satisfy. Finally, we apply the construction of long α-sequences to the problem of finding long cycles between two consecutive levels of the Boolean lattice. A famous instance of this problem is the question whether there is a Hamiltonian cycle between the middle two levels of the Boolean lattice (see e.g. [KT], [Sa]). The best constructions known until now could guarantee cycles of length $\Omega(N^c)$ where N is the number of vertices and $c \approx 0.85$. We exhibit cycles of length $\geq \frac{1}{4}N$.

4.2 A Construction of Long α-Sequences

Let $t(n,k)$ denote the maximal length of a sequence C_i of sets satisfying:

(1) $C_i \subseteq \{1, \ldots, n\}$,

(2) $|C_i| = k$ and

(α) if $i < j$ then $C_j \not\subseteq C_{i-1} \cup C_i$.

Lemma 4.4

$$t(n,k) \geq \binom{n-1}{k} + 1$$

Proof. The sequences actually constructed will have the additional property

(4) $|C_{i-1} \cup C_i| = k+1$ for all $i \geq 2$.

The proof is by induction. For all n and $k=1$ or $k=n$ the claim is obviously true.

Now suppose that two α-sequences as specified have been constructed on $\{1,\ldots,n-1\}$. First, a sequence of k-sets $\mathbf{A} = A_1,\ldots,A_s$ of length $s = \binom{n-2}{k}+1$, and second, a sequence of $(k-1)$-sets $\mathbf{B} = B_1,\ldots,B_t$ of length $t = \binom{n-2}{k-1}+1$.

Property (4) guarantees that there is a permutation π of the colors such that $A_s = B_1^\pi \cup B_2^\pi$. Now let

$$C_i = \begin{cases} A_i & \text{for } 1 \leq i \leq s \\ B_{i-s+1}^\pi \cup \{n\} & \text{for } s+1 \leq i \leq s+t-1 \end{cases}$$

The length of the new sequence is $s+t-1 = \binom{n-1}{k}+1$. Properties (1) and (2) are obviously true for the sequence C_i and property (4) is true for both the $\mathbf{A}$ and the $\mathbf{B}$ sequence. These observations and the choice of π give property (4) for the $\mathbf{C}$ sequence. It remains to verify property (α). If $i<j<s+1$, this property is inherited from the $\mathbf{A}$ sequence. If $s+1<i<j$, it is inherited from the $\mathbf{B}$ sequence. In case $i<s+1\leq j$, we have $n \in C_j$ and $n \notin C_{i-1}\cup C_i$. The remaining case is $s+1=i<j$. Here the choice of π and the sacrifice of B_1 show that $C_s \cup C_{s+1} = A_s \cup B_2^\pi \cup \{n\} = B_1^\pi \cup B_2^\pi \cup \{n\}$. Again, property ($\alpha$) can be concluded from this property for the $\mathbf{B}$ sequence. □

For $k=2$ and $k=n-1$, we can prove that the inequality of Lemma 4.4 is tight, but in general the value of $t(n,k)$ is open.

Problem 4.1 *Determine the true value of* $t(n,k)$.

Let $T(n)$ denote the maximal length of a sequence C_i of sets satisfying:

(1) $C_i \subseteq \{1,\ldots,n\}$ and

(α) if $i<j$ then $C_j \not\subseteq C_{i-1}\cup C_i$.

Lemma 4.5

$$T(n) \geq \sum_{\substack{k\leq n \\ k \text{ odd}}} \left(\binom{n-1}{k} + 1 \right) = 2^{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof. Let $\mathbf{L}(n, k)$ be the (n, k)–sequence constructed in the preceeding lemma. We claim that $\mathbf{L} = \mathbf{L}^{\pi_1}(n, 1) \oplus \mathbf{L}^{\pi_3}(n, 3) \oplus \mathbf{L}^{\pi_5}(n, 5) \oplus \ldots$ with appropriate permutations π_j is a α-sequence of subsets of $\{1, \ldots, n\}$. The π_k's can be found recursively. $\pi_1 = id$ and if π_{k-2} has been determined, then π_k is chosen as a permutation, such that, the last set of the sequence $\mathbf{L}^{\pi_{k-2}}(n, k\text{–}2)$ is a subset of the first set of $\mathbf{L}^{\pi_k}(n, k)$. Let C_i be the i^{th} set in the sequence $\mathbf{L}$. We now check property (α). If the three sets C_{i-1}, C_i and C_j are in the same subsequence $\mathbf{L}^{\pi_k}(n, k)$, then the property is inherited from this subsequence. If $C_i \in \mathbf{L}^{\pi_k}(n, k)$ and $C_j \in \mathbf{L}^{\pi_{k'}}(n, k')$ with $k \leq k' - 2$, then $|C_{i-1} \cup C_i| < \left|C_j\right|$ is a consequence of property (4) for the subsequence $\mathbf{L}^{\pi_k}(n, k)$, and gives the claim in this case. There remains the situation where C_{i-1} is the last set of its subsequence. The choice of the π_k gives $C_{i-1} \subset C_i$ and the property reduces to $C_j \not\subseteq C_i$, which is obvious.

The length of $\mathbf{L}$ is the sum over the length of the $\mathbf{L}^{\pi_k}(n, k)$ used in $\mathbf{L}$. This is the sum over $\binom{n-1}{k} + 1$ with k odd, which is $2^{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor$. □

4.3 The Structure of Very Long α-Sequences

Theorem 4.2 *Let $\mathbf{C} = C_1, \ldots, C_t$ be a α-sequence of subsets of $\{1, \ldots, n\}$. Then $t \leq 2^{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor$.*

Proof. We start with some definitions. For $1 \leq i \leq t - 1$, let

$$S_i = \{ \ S \ : \ C_{i+1} \subset S \subseteq C_i \cup C_{i+1} \ \} \tag{4.1}$$

and $s_i = |S_i|$. Observe that with $r_i = |C_i \setminus C_{i+1}|$ we have the equation

$$s_i = 2^{r_i} - 1. \tag{4.2}$$

We now prove two important properties of the sets S_i

- $S_i \cap S_j = \emptyset$ if $i \neq j$.

  Assume, to the contrary, that $S \in S_i \cap S_j$ and let $i < j$. From the definition of the S_i, we obtain $C_{j+1} \subset S \subseteq C_i \cup C_{i+1}$. This contradicts with property (α) for the sequence $\mathbf{C}$.

- $\mathbf{C} \cap S_i = \emptyset$ for all i.

  Assume, $C_j \in S_i$. If $j \leq i$, then $C_{i+1} \subset C_j$ gives a contradiction. If $j = i+1$, note that $C_{i+1} \notin S_i$ from the definition. If $j < i+1$, the contradiction comes from $C_j \subseteq C_i \cup C_{i+1}$.

Therefore $\mathbf{C}$ and the S_i are pairwise disjoint subsets of $\mathcal{B}_n$, this gives the inequality

$$2^n \geq t + \sum_{i=1}^{t-1} s_i \tag{4.3}$$

We now partition the indices $\{1, \ldots, t-1\}$ into three classes

- $I_1 = \{i : |C_i| = |C_{i+1}|\}$; note, that $i \in I_1$ implies $s_i \geq 1$.
- $I_2 = \{i : |C_i| < |C_{i+1}|\}$; trivially, $s_i \geq 0$ for $i \in I_2$.
- $I_3 = \{i : |C_i| > |C_{i+1}|\}$; note, that if $i \in I_3$, then the corresponding s_i is relatively large, i.e., $s_i \geq 2^{|C_{i+1}|-|C_i|+1} - 1$. This estimate is a consequence of Equation 4.2 and the fact that C_{i+1} has to contain an element not contained in C_i.

We first investigate the case $I_3 = \emptyset$. This condition guarantees that the sizes of the sets in $\mathbf{C}$ is a nondecreasing sequence. Since $\mathcal{B}_n$ has $n+1$ levels, the size of the sets in $\mathbf{C}$ can increase at most n times, i.e., $|I_2| \leq n$ and $|I_1| \geq t-1-n$. It follows that:

$$\begin{aligned} 2^n &\geq t + \sum_{i \in I_1} s_i + \sum_{i \in I_2} s_i \\ &\geq t + |I_1| \\ &\geq t + (t-1-n) \end{aligned}$$

This gives $2t \leq 2^n + (n+1)$. Hence, $t \leq 2^{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor$ in this case.

The case $I_3 \neq \emptyset$ is somewhat more complicated. Let the number of descending steps be d and $I_3 = \{i_1, \ldots, i_d\}$. Let m_j denote the number of levels the sequence is decreasing when going from C_{i_j} to C_{i_j+1}, i.e., $m_j = |C_{i_j+1}| - |C_{i_j}|$ and $s_{i_j} \geq 2^{m_{i_j}+1} - 1$. Again, we can estimate the size of I_2, namely, $|I_2| \leq n + \sum_{j=1}^{d} m_{i_j}$. It follows that:

$$2^n \geq t + \sum_{i \in I_1} s_i + \sum_{i \in I_2} s_i + \sum_{i \in I_3} s_i$$

$$
\begin{aligned}
&\geq \; t + |I_1| + \sum_{j=1}^{d}(2^{m_{i_j}+1} - 1) \\
&\geq \; t + ((t-1) - |I_2| - |I_3|) + \sum_{j=1}^{d} 2^{m_{i_j}+1} - d \\
&\geq \; t + (t - 1 - n - \sum_{j=1}^{d} m_{i_j} - d) + \sum_{j=1}^{d} 2^{m_{i_j}+1} - d
\end{aligned}
$$

Comparing this with the calculations made for the case $I_3 = \emptyset$, we find that $t \geq 2^{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor$ would require $-\sum_{j=1}^{d} m_{i_j} - 2d + \sum_{j=1}^{d} 2^{m_{i_j}+1} \leq 0$. For each j, we have $2^{m_{i_j}} > m_{i_j} - 2$. Hence, the above inequality can never hold. □

Remark Let $T^*(n) = 2^{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor$ be the upper bound from the theorem. We have seen that a α-sequence $\mathbf{C}$ of length $T^*(n)$ can only exist if $I_3 = \emptyset$. Moreover, the following conditions follow from the argument given for Theorem 4.2.

1. There are exactly n increasing steps, i.e., $|I_2| = n$.

2. If $i \in I_1$, then $s_i = 1$, i.e., two consecutive sets of equal size have to be a *shift*: $C_{i+1} = (C_i \setminus \{x\}) \cup \{y\}$ with $x \in C_i$ and $y \notin C_i$.

3. If $i \in I_2$ then $s_i = 0$, i.e., if $|C_i| < |C_{i+1}|$, then there is an element $x \in C_i$, such that, $C_{i+1} = C_i \cup \{x\}$.

4. Every element of $\mathcal{B}_n$ is either an element of $\mathbf{C}$ or appears as the unique element of some S_i, i.e., as $C_i \cup C_{i+1}$.

From this observations, we obtain an alternate interpretation for a sequence $\mathbf{C}$ of length $T^*(n)$ in $\mathcal{B}_n$. In the diagram of $\mathcal{B}_n$, i.e., the n-hypercube, consider the edges (C_i, C_{i+1}) for $i \in I_2$ and for $i \in I_1$ the edges (C_i, T_i) and (T_i, C_{i+1}) where T_i is the unique member of S_i, i.e., $T_i = C_i \cup C_{i+1}$. This set of edges is a Hamiltonian path in the hypercube and respects a strong condition of being level accurate. *After having reached the k^{th} level for the first time the path will never come back to level $k-2$* (see Figure 4.1 for an example, the bullets are the elements of a very long α-sequence).

Problem 4.2 *Do sequences of length $T^*(n)$ exist for all n ?*

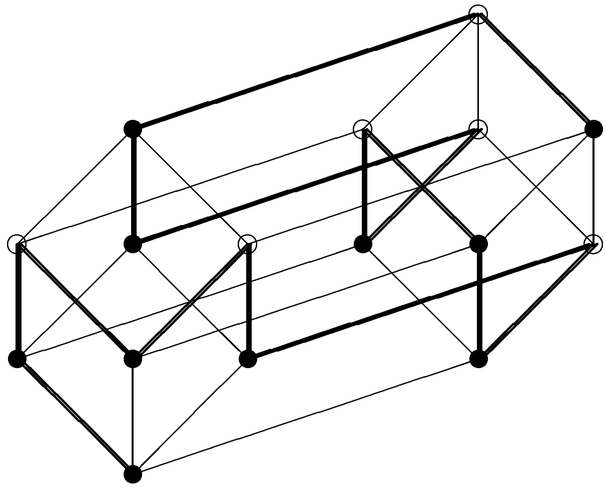

Figure 4.1: A level accurate path in $\mathcal{B}_4$

We are hopeful that such sequences exist. Our optimism stems in part from computational results. The number of sequences starting with $\emptyset, \{1\}, \{2\}, \ldots \{n\}$ is 1 for $n \leq 4$, 10 for $n = 5$, 123 for $n = 6$ and there are thousands of solutions for $n = 7$. The next case $n = 8$ could not be handled by our program, but Markus Fulmek wrote a program which also resolved this case affirmatively.

4.3.1 Long Cycles between Consecutive Levels in $\mathcal{B}_n$

Let $B(n, k)$ denote the bipartite graph consisting of all elements from levels k and $k+1$ of the Boolean lattice $\mathcal{B}_n$. A well known problem on this class of graphs is the following: Is $B(2k+1, k)$ *Hamiltonian for all* k ? Until now, it was known that this is the case for $k \leq 9$. Since the problem seems to be very hard, some authors have attempted to construct long cycles. The best results (see [Sa]) lead to cycles of length $\Omega(N^c)$ where $N = 2\binom{2k+1}{k}$ is the number of vertices of $B(2k+1, k)$ and $c \approx 0.85$.

Theorem 4.3 *In* $B(n, k)$, *there is a cycle of length*

$$4 \max\left\{\binom{n-3}{k-1} + 1, \binom{n-3}{n-k-2} + 1\right\}$$

Proof. Note that the graphs $B(n, k)$ and $B(n, n-k-1)$ are isomorphic, it thus suffices to exhibit a cycle of length $4\binom{n-3}{k-1} + 4$ in $B(n, k)$. To this end, take a

α-sequence $C_1, \ldots, C_t$ of $(k-1)$-sets on $\{1, \ldots, n-2\}$. From Lemma 4.4, we know that $t \geq \binom{n-3}{k-1} + 1$ can be achieved. Now consider the following set of edges in $B(n,k)$

- $\big(C_i \cup \{n\} \, , \, C_i \cup C_{i+1} \cup \{n\} \big)$ for $1 \leq i < t$,
- $\big(C_i \cup C_{i+1} \cup \{n\} \, , \, C_{i+1} \cup \{n\} \big)$ for $1 \leq i < t$,
- $\big(C_t \cup \{n\} \, , \, C_t \cup \{n-1, n\} \big)$ and $\big(C_t \cup \{n-1, n\} \, , \, C_t \cup \{n-1\} \big)$,
- $\big(C_i \cup \{n-1\} \, , \, C_i \cup C_{i+1} \cup \{n-1\} \big)$ for $1 \leq i < t$,
- $\big(C_i \cup C_{i+1} \cup \{n-1\} \, , \, C_{i+1} \cup \{n-1\} \big)$ for $1 \leq i < t$,
- $\big(C_1 \cup \{n-1\} \, , \, C_1 \cup \{n-1, n\} \big)$ and $\big(C_1 \cup \{n-1, n\} \, , \, C_1 \cup \{n\} \big)$.

The proof that this set of edges in fact determines a cycle of length $4t$ in $B(n,k)$ is straightforward. □

With a simple calculation on binomial coefficients, we obtain a final theorem

Theorem 4.4 *There are cycles in* $B(2k+1, k)$ *of length at least* $\frac{1}{4}N$.

4.4 References for Chapter 4

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Chapter 5

Interval Dimension and Dimension

5.1 Introduction and Overview

A family $\{Q_1, \ldots, Q_k\}$ of extensions of a partial order P is said to *realize* P or to be a *realizer* of P iff $P = Q_1 \cap \ldots \cap Q_k$, i.e., $x < y$ in P iff $x < y$ in Q_i for each i, $1 \leq i \leq k$. If we restrict the Q_i to belong to a special class of orders and seek for a minimum size realizer we come up with a concept of dimension with respect to the special class.

Interval dimension, denoted $\mathsf{idim}(P)$, is defined by using interval extensions of P. Since linear orders are interval orders we obtain the trivial inequality

$$\mathsf{idim}(P) \quad \leq \quad \mathsf{dim}(P) \tag{5.1}$$

It is well known that interval orders of large dimension exist (see Chapter 3). Hence, the gap between $\mathsf{idim}(P)$ and $\mathsf{dim}(P)$ may be arbitrarily large.

Let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be an open interval realizer of P and let (a_x^j, b_x^j) be the interval corresponding to $x \in P$ in a fixed representation of I_j. We now find a *box embedding* of P to $\mathbb{R}^k$. With $x \in P$ we associate the box $\prod_j (a_x^j, b_x^j) \subseteq \mathbb{R}^k$. Each of these boxes is uniquely determined by its *upper extreme corner* $u_x = (b_x^1, \ldots, b_x^k)$ and its *lower extreme corner* $l_x = (a_x^1, \ldots, a_x^k)$. Obviously, $x < y$ in P iff $u_x \leq l_y$ componentwise. The projections of a box embedding onto each coordinate yield an interval realizer, so the concepts of box embeddings and interval realizers are equivalent. For interval dimension the box embeddings thus play the role of the point embeddings into $\mathbb{R}^k$ introduced by Ore for order dimension.

A box embedding does not only depend on the realizer $\mathcal{I}$ of P, but also on the representations of the I_j. Now we define the *poset* $B(\mathcal{I})$ *of extreme corners* associated with a box embedding or, equivalently, with an interval realizer $\mathcal{I}$ of P. The vertices of $B(\mathcal{I})$ are l_x, u_x for $x \in P$. The order relation of $B(\mathcal{I})$ is given by the componentwise order in $\mathbb{R}^k$.

By definition we have an embedding of $B(\mathcal{I})$ in $\mathbb{R}^k$, so $\mathsf{dim}\, B(\mathcal{I}) \leq k = \mathsf{idim}(P)$. The starting point of our investigations was the following question concerning the interplay between dimension and interval dimension:

$$\text{Is } \mathsf{dim}\, B(\mathcal{I}) = \mathsf{idim}(P) \text{ ?}$$

In the next section we define a transformation $P \to B(P)$ such that $\mathsf{idim}(P) = \mathsf{dim}\, B(P)$. We provide two interpretations of this transformation, a combinatorial one and a geometrical one. In the combinatorial interpretation the elements of $B(P)$ are subsets of P. For the geometrical one we present a normalizing procedure $\mathcal{I} \to \mathcal{I}^*$ for box embeddings and find $B(P) = B(\mathcal{I}^*)$. From the proofs we obtain an affirmative answer to the question above.

In the third section we then investigate several consequences of the main result. First, we study the transformation $P \to B(P)$ on special partial orders of height 1. In particular we show that the standard example S_n of an n-dimensional order is an (almost) fixed point of the transformation $P \to B(P)$ in particular $\mathsf{dim}(S_n) = \mathsf{idim}(S_n)$.

Second, we investigate the relationship with the split operation [Tr1]. This has a surprising consequence for the iterated transformation $P \to B(P) \to B^2(P) \to \ldots B^k(P) \to \ldots$. For every n there are partial orders P such that $0 \leq \mathsf{dim}(P) - \mathsf{dim}\, B^k(P) \leq 2$ for all $k \leq n$ but $\mathsf{dim}(P) - \mathsf{dim}\, B^{n+1}(P) \geq m$, where m is arbitrary.

Third, we relate the interval dimension of subdivisions of P to the dimension of P, thus providing a theoretical framework for the examples of Spinrad [Sp].

Finally, we show the comparability invariance of the transformation $P \to B(P)$, which, as a consequence, gives another proof that the interval dimension is a comparability invariant.

5.2 Reducing Interval Dimension to Dimension

In the last section we defined the poset $B(\mathcal{I})$ of extreme corners associated with a box embedding of P in $\mathbb{R}^k$. We now show that $B(\mathcal{I})$ inherits some structure which is independent of the realizer $\mathcal{I}$ leading to the box embedding.

Lemma 5.1 *Let $B(\mathcal{I})$ be the poset of extreme corners of a box representation of a partial order P*

a) *If the lower extreme corners of x and y are comparable in $B(\mathcal{I})$, e.g., $l_x \leq l_y$, then the predecessor sets of x and y in P are ordered by inclusion, i.e., $\mathsf{Pred}_P(x) \subseteq \mathsf{Pred}_P(y)$.*

b) *If the upper extreme corners of x and y are comparable in $B(\mathcal{I})$, e.g., $u_x \leq u_y$, then the successor sets of x and y in P are ordered by (reversed) inclusion, i.e., $\mathsf{Succ}_P(x) \supseteq \mathsf{Succ}_P(y)$.*

c) *If the lower extreme corner of x and the upper extreme corner of y are related by $l_x \leq u_y$ then $\mathsf{Pred}_P(z) \supseteq \mathsf{Pred}_P(x)$ for all $z \in \mathsf{Succ}_P(y)$ or, equivalently, $\mathsf{Pred}_P(x) \subseteq \bigcap_{z \in \mathsf{Succ}_P(y)} \mathsf{Pred}_P(z)$.*

d) *If $u_x \leq l_y$ then $\bigcap_{z \in \mathsf{Succ}_P(x)} \mathsf{Pred}_P(z) \subseteq \mathsf{Pred}_P(y)$.*

Proof. a) From $l_x \leq l_y$ we obtain $a_x^j \leq a_y^j$ for all j. Therefore in each I_j, x has less predecessors than y, i.e., $\mathsf{Pred}_j(x) \subseteq \mathsf{Pred}_j(y)$. The claim now follows from $\mathsf{Pred}_P(x) = \bigcap_j \mathsf{Pred}_j(x)$ since the I_j realize P.
b) The proof of this part is symmetric to part a).
c) From $l_x \leq u_y$ we have $a_x^j \leq b_y^j$. If $z \in \mathsf{Succ}(y)$, then necessarily $a_z^j \geq b_y^j \geq a_x^j$. Hence, $\mathsf{Pred}_j(z) \supseteq \mathsf{Pred}_j(x)$ for all j. The claim follows.
d) From $u_x \leq l_y$ we immediately obtain $x \leq y$, i.e., $y \in \mathsf{Succ}(x)$, therefore, $\mathsf{Pred}(y) \supseteq \bigcap_{z \in \mathsf{Succ}(x)} \mathsf{Pred}(z)$. □

All statments except the conclusion part of b use only the sets $\mathsf{Pred}_P(x)$ and $\bigcap_{z \in \mathsf{Succ}_P(x)} \mathsf{Pred}_P(z)$. This irregularity is resolved with the next lemma.

Lemma 5.2 $\bigcap_{z \in \mathsf{Succ}(x)} \mathsf{Pred}(z) \supseteq \bigcap_{z \in \mathsf{Succ}(y)} \mathsf{Pred}(z)$ *if and only if* $\mathsf{Succ}(x) \subseteq \mathsf{Succ}(y)$.

Proof. The 'if' direction is trivial. We now prove the 'only if' direction. Let $z \in \mathsf{Succ}(x)$ and note that $y \in \bigcap_{z \in \mathsf{Succ}(y)} \mathsf{Pred}(z)$. From the assumed inclusion we obtain $y \in \mathsf{Pred}(z)$, hence $z \in \mathsf{Succ}(y)$. □

Definition 5.1 *With each vertex x of a partial order $P = (X, <)$ we associate the lower set $L(x) = \mathsf{Pred}_P(x)$ and the upper set $U(x) = \bigcap_{z \in \mathsf{Succ}_P(x)} \mathsf{Pred}_P(z)$,*

the case $x \in \mathsf{Max}(P)$ *is settled by the convention* $U(x) = X$.
Define $B(P) = \{L(x), U(x) : x \in X\}$ *ordered by setinclusion.*

Note that this construction is in fact equivalent with Cogis' construction in the context of Ferrers dimension [Co1]. Cogis also uses $L(x)$, but replaces $U(x)$ by the equivalent set $\{z \in X : \mathsf{Succ}(x) \subseteq \mathsf{Succ}(z)\}$. He also proves Theorem 5.2, but in a different way and without the geometrical interpretation that our approach is based on.

The preceding lemmas prove that $l_x \to L(x)$ and $u_x \to U(x)$ together form an order preserving mapping from $B(\mathcal{I})$ to $B(P)$, hence,

$$\mathsf{idim}(P) \geq \dim B(\mathcal{I}) \geq \dim B(P). \tag{5.2}$$

To get more structure into interval realizers we now introduce a procedure that transforms an interval extension $I = \{ (a_x, b_x) : x \in P\}$ of P into its *normalization* $I^* = \{ (a_x^*, b_x^*) : x \in P\}$.

- In the first step of the normalization we update left endpoints.
  $a_x^* = \max\{b_z : z \in \mathsf{Pred}(x)\}$ if $x \in P \setminus \mathsf{Min}(P)$,
  $a_x^* = \min\{a_z : z \in \mathsf{Min}(P)\}$ otherwise.

- In the second step we update right endpoints.
  $b_x^* = \min\{a_z^* : z \in \mathsf{Succ}(x)\}$ if $x \in P \setminus \mathsf{Max}(P)$,
  $b_x^* = \max\{b_z : z \in \mathsf{Max}(P)\}$ otherwise.

Note that the interval order I^* need not be isomorphic to I. In general I^* is a suborder of I and a *minimal interval extension* (see [HMR] for minimal interval extensions) of P if all the a_x, b_x were different.

If P is realized by $\mathcal{I} = \{I_1, \dots, I_k\}$ then $\mathcal{I}^* = \{I_1^*, \dots, I_k^*\}$ realizes P as well. We call the box embedding corresponding to $\mathcal{I}^*$ the *normalized* box embedding of $\mathcal{I}$. For an example see Figure 5.1.

After normalizing we have a realizer $\mathcal{I}^* = \{I_1^*, \dots, I_k^*\}$ of P, interval representations (a_x^{j*}, b_x^{j*}) and an associated poset of extreme corners $B(\mathcal{I}^*) = \{l_x^*, u_x^* : x \in P\}$. The next theorem shows that the geometrically defined $B(\mathcal{I}^*)$ and the combinatorially defined $B(P)$ are isomorphic.

Theorem 5.1 *If* $\mathcal{I}^*$ *is a normalized realizer of* P *then* $B(\mathcal{I}^*) = B(P)$.

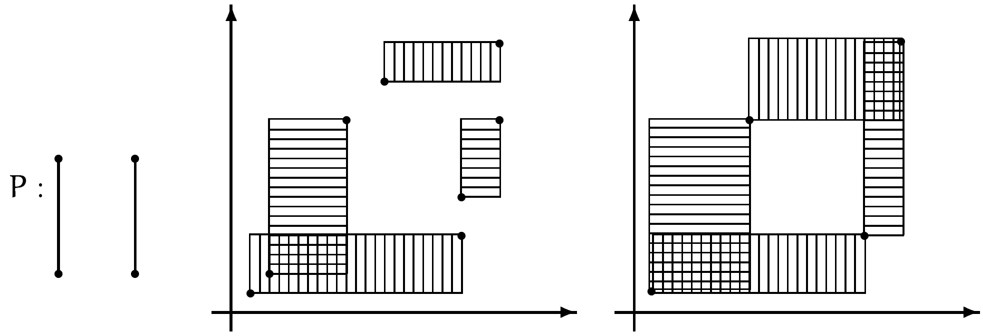

Figure 5.1: P, an interval realizer of P and its normalization

Proof. First observe that both partial orders have a least element generated by $x \in \mathsf{Min}(P)$ as l_x^* and $L(x)$, respectively, and a greatest element generated by $x \in \mathsf{Max}(P)$ as u_x^* and $U(x)$, respectively.

Moreover, $l_x^* \to L(x)$ and $u_x^* \to U(x)$ defines an order preserving mapping by the above remarks. To show the converse we distinguish four cases:

$U(x) \subseteq L(y)$. We know that $x \in U(x)$, so $x \in \mathsf{Pred}(y)$. Since I_j^* is an interval extension of P, we obtain $b_x^{j*} \leq a_y^{j*}$ for all j. Hence $u_x^* \leq l_y^*$.

$L(x) \subseteq L(y)$. Remember that $a_x^{j*} = \max\{b_z^j : z \in \mathsf{Pred}(x)\}$ and $a_y^{j*} = \max\{b_z^j : z \in \mathsf{Pred}(y)\}$. By assumption $\mathsf{Pred}(x) \subseteq \mathsf{Pred}(y)$, so $a_x^{j*} \leq a_y^{j*}$ and $l_x^* \leq l_y^*$.

$U(x) \subseteq U(y)$. By Lemma 5.2, this is equivalent to $\mathsf{Succ}(x) \supseteq \mathsf{Succ}(y)$. Now $u_x^* \leq u_y^*$ follows symmetrically to the second case.

$L(x) \subseteq U(y)$. Since I_j^* is normalized, there are $z_0 \in \mathsf{Pred}(x)$ and $z_1 \in \mathsf{Succ}(y)$ with $a_x^{j*} = b_{z_0}^j$ and $b_y^{j*} = a_{z_1}^{j*}$. The hypothesis provides $z_0 \leq z_1$, hence $a_x^{j*} = b_{z_0}^j \leq b_{z_0}^{j*} \leq a_{z_1}^{j*} = b_y^{j*}$. The validity of this inequality for all j again gives $l_x^* \leq u_y^*$. □

Now we are going to prove our main theorem about interval dimension and dimension.

Theorem 5.2 $\dim B(P) = \mathsf{idim}(P)$.

Proof. The inequality $\dim B(P) \leq \mathsf{idim}(P)$ was a simple consequence of the definition of $B(P)$. For the converse we need two arguments. We first show that a linear extension L of $B(P)$ induces an interval extension I_L of P. Secondly, we prove that if $L_1, \ldots, L_k$ is a realizer of $B(P)$, then the induced interval extensions $I_j = I_{L_j}$ form an interval realizer of P.

Let $L = M_1, M_2, \ldots, M_r$ be a linear extension of $B(P)$. For each $x \in P$ there are $i, j \in \{1, \ldots, r\}$ such that $M_i = L(x)$ and $M_j = U(x)$. From $L(x) \subseteq U(x)$ and $x \notin L(x)$, $x \in U(x)$ we obtain that $i < j$. So we can associate with x the unique interval $(a_x, b_x) = (i, j)$. We now show, that the interval order I_L induced by the interval representation $\{ (a_x, b_x) : x \in P\}$ is an extension of P. If $x < y$ in P, then $U(x) \subseteq L(y)$, and thus, with $M_i = U(x)$ and $M_j = L(y)$, $b_x = i \leq j = a_y$, which implies $x < y$ in I_L.

Let $\{L_1, \ldots, L_k\}$ be a realizer for $B(P)$. The induced family of interval extensions $\{I_1, \ldots, I_k\}$ of P is an interval realizer iff all incomparabilities $x||y$ of P are realized. If $x||y$ in P, then $U(x) \not\subseteq L(y)$ since $x \in U(x)$ but $x \notin L(y)$. Therefore, $L(y)$ precedes $U(x)$ in some L_j, which gives $a_y^j < b_x^j$. The symmetric argument yields an i with $a_x^i < b_y^i$. Both inequalities together give $x||y$ in $\bigcap_j I_j$. $\square$

This theorem together with Inequality 5.2 shows that $\dim B(\mathcal{I})$ is independent of the interval realizer $\mathcal{I}$.

5.3 Consequences

In the previous section we introduced the operation $P \to B(P)$ mapping partial orders to partial orders. We will now investigate several connections to other order-theoretical topics and results. Since $B(P)$ always has a greatest and a least element, we adopt the convention to call orders with this property closed and denote by $\widehat{Q}$ the closure of any partial order Q, i.e., the order resulting from Q by adjoining a new greatest and a new least element.

We first look at the effect of the operator B applied to special classes of orders.

5.3.1 Crowns and the Standard Examples

Let S_n denote the standard poset of dimension n, i.e., the set of all subsets of an n element set with either 1 or $n-1$ elements ordered by setinclusion. Then

$$B(S_n) = \widehat{S_n} \tag{5.3}$$

Let C_r denote the r-cycle. The r-cycle is the 3–dimensional poset on $2r$ elements $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r\}$ with comparabilities.

$$x_1 < y_1, y_1 > x_2, x_2 < y_2, \ldots x_r < y_r, y_r > x_1$$

$$B(C_r) = \widehat{C_r} \tag{5.4}$$

Let the diagram of T be a tree. The *truncation* of T, denoted by $tr(T)$, is the induced tree on the non-leaf vertices of T. Then

$$B(T) = \widehat{tr(T)} \tag{5.5}$$

In particular 5.3 shows that the standard example S_n of a n-dimensional order is (up to closures) a fixed point of the operation $P \to B(P)$, thus showing again that, for every $n \geq 3$, there is a poset P with $\mathsf{dim}(P) = \mathsf{idim}(P) = n$.

5.3.2 *B* and the Split Operation

We now turn to the natural question, whether, for every closed order Q there is some P with $Q = B(P)$. The next theorem answers this question affirmatively. Moreover it turns out that the operation $P \to B(P)$ is an almost left inverse of the *split* operation S which has applications in different branches of poset theory, see, e.g., [Tr1, Fr, Fe]. The split $S[P]$ of an order P is the poset of height one with minimal elements $\{x' : x \in P\}$ maximal elements $\{x'' : x \in P\}$ and ordered pairs $x' < y''$ iff $x \leq y$ in P.

Theorem 5.3 $B(S[P]) = \widehat{P}$.

Proof. For $x \in P$ let $\mathsf{Pred}[x] = \mathsf{Pred}(x) \cup \{x\}$. Obviously, P is isomorphic to the setsystem $\{\mathsf{Pred}[x] : x \in P\}$ ordered by inclusion. We will show that the sets of $B(S[P])$ are just the 'primed' sets $\mathsf{Pred}[x]$, i.e, $(\mathsf{Pred}[x])' = \{x' : x \in \mathsf{Pred}[x]\}$, together with the greatest element $\{x', x'' : x \in P\}$ and the least element $\emptyset$. We have

$L(x') = \emptyset$, and

$U(x') = \bigcap_{z'' \in \mathsf{Succ}(x')} \mathsf{Pred}(z'') = \bigcap_{z \in \mathsf{Succ}[x]} (\mathsf{Pred}[z])'$. Since $x \in \mathsf{Succ}[x]$, $U(x') \subseteq (\mathsf{Pred}[x])'$.

On the other hand, $\mathsf{Pred}[x] \subset \mathsf{Pred}[z]$ for $z \in \mathsf{Succ}(x)$. Together, this gives $U(x') = (\mathsf{Pred}[x])'$. Similarly, we obtain

$L(x'') = \mathsf{Pred}(x'') = (\mathsf{Pred}[x])'$. Finally,

$U(x'') = \{x', x'' : x \in P\}$ by definition since $\mathsf{Succ}(x'') = \emptyset$. □

It is easy to verify that

$$B(\widehat{P}) \quad = \quad \widehat{B(P)}. \tag{5.6}$$

So we may generalize the theorem to

$$B^n(S^n[P]) \quad = \quad \overset{\left.\hat{\vdots}\right\} n \text{ closures}}{\widehat{P}} \tag{5.7}$$

Investigations on the effect of iterated splitting to the dimension [TM1] lead to the inequality

$$\dim P \leq \dim S^n[P] \quad \leq \quad 2 + \dim P \quad \text{for all } n \tag{5.8}$$

As a consequence of (5.7) and (5.8) we obtain that, for every n there is an order P such that

$$\dim P - \dim B^k(P) \quad \leq \quad 2 \quad \text{for all } k < n.$$

Just take $P = S^n[Q]$ for some order Q. If we choose Q, however, to be an m dimensional interval order we obtain a large difference in dimension with the next iteration, i.e.,

$$\dim P - \dim B^n(P) \quad \geq \quad m - 1. \tag{5.9}$$

5.3.3 The Interval Dimension of Subdivisions

With the next theorem we relate the interval dimension of subdivisions of P to the dimension of P. Spinrad [Sp] showed that the dimension of a subdivision of a partial order can be an arbitrary multiple of its dimension, thus answering Trotter's Problem 4 in [Tr2]. With our result, we establish a theoretical framework for his examples.

In this context, partial orders and their diagrams are regarded as directed graphs whose edges (x, y) correspond to ordered pairs and cover pairs $x \prec y$ of P, respectively. An edge (x, y) is *subdivided* by placing a new vertex z in the 'middle' of the edge, i.e., (x, y) is replaced by (x, z) and (z, y). In the case of partial orders we then have to ensure transitivity, i.e., all edges (a, z) with $a \in \mathsf{Pred}[x]$ and (z, b) with $b \in \mathsf{Succ}[y]$ are also added.

The *complete diagram subdivision* $\mathsf{DS}(P)$ is the subdivision of all edges of the diagram of P. The *complete subdivision* $\mathsf{CS}(P)$ is the subdivision of all the edges of P, i.e., of the transitively closed relation. Since P is an induced suborder of each of its subdivisions $\mathsf{Sub}(P)$, and since $\mathsf{Sub}(P)$ is an induced suborder of $\mathsf{CS}(P)$, we obtain

$$\dim(P) \leq \dim \mathsf{Sub}(P) \leq \dim \mathsf{CS}(P). \tag{5.10}$$

With the next theorem we give an upper bound for $\mathsf{idim}\,\mathsf{Sub}(P)$.

Theorem 5.4 $\mathsf{idim}\,\mathsf{Sub}(P) \leq \mathsf{idim}\,\mathsf{DS}(P) = \mathsf{idim}\,\mathsf{CS}(P) = \dim(P)$.

Proof. Take any embedding of P into $\mathbb{R}^k$ with $k = \dim(P)$ and grow the points to obtain an embedding by 'miniboxes'. An interval embedding of a subdivision $\mathsf{Sub}(P)$ is then obtained by adding the box with lower extreme corner u_x and upper extreme corner l_y for the point z subdividing the edge (x, y) – see Figure 5.2. This gives $\mathsf{idim}\,\mathsf{Sub}(P) \leq \dim(P)$ for every subdivision $\mathsf{Sub}(P)$.

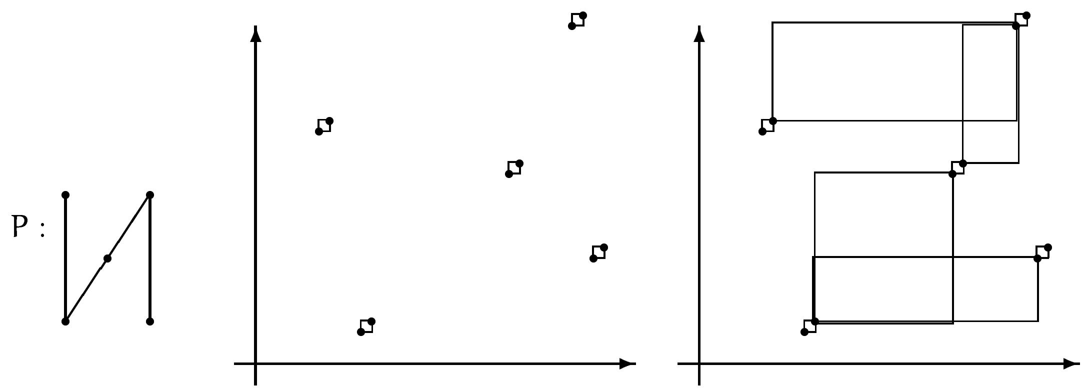

Figure 5.2: P, a minibox embedding of P and the box embedding of $\mathsf{DS}(P)$.

To prove that $\mathsf{idim}\,\mathsf{DS}(P) = \mathsf{idim}\,\mathsf{CS}(P) = \dim(P)$ we will show that P can be embedded in $B(\mathsf{DS}(P))$. This will give $\dim(P) \leq \dim B(\mathsf{DS}(P))$. From Theorem 5.2 we know $\mathsf{idim}\,\mathsf{DS}(P) = \dim B(\mathsf{DS}(P))$ so we obtain $\dim(P) \leq \mathsf{idim}\,\mathsf{DS}(P)$.

To show that P can be embedded in $B(\mathsf{DS}(P))$ we apply the normalizing procedure to the box embedding $\mathcal{I}$ of $\mathsf{DS}(P)$ constructed in the first paragraph of the proof. During the normalization we can only shift the left endpoints of intervals corresponding to elements in $\mathsf{Min}(P)$ and the right endpoints of elements in $\mathsf{Max}(P)$. We then embed P into the lower extreme corners of the miniboxes of the normalized representation $\mathcal{I}^*$. This gives $\dim(P) = k = \mathsf{idim}(\mathcal{I}^*) = \mathsf{idim}(\mathsf{DS}(P))$. □

Note that we obtained, in fact, a slightly stronger result: If $\mathsf{CS}(P) \supseteq \mathsf{Sub}(P) \supseteq \mathsf{DS}(P)$, then $B(\mathsf{Sub}(P)) = \mathsf{VS}(P)$, where $\mathsf{VS}(P)$ denotes the *vertical split* of P, i.e. the order obtained from P by substituting each vertex by a 2–chain. In [TM2] a distinct proof for $\dim(P) = \mathrm{idim}\,\mathsf{VS}(P)$ has been given.

5.3.4 Comparability Invariance of Interval Dimension

For the definition and basic facts on comparability invariance see [Ha]. Let $\mathsf{Comp}(P)$ be the comparability graph of P. We will show that $\mathsf{Comp}(B(P))$ is a comparability invariant of P in the sense that if $\mathsf{Comp}(P) = \mathsf{Comp}(Q)$ then $\mathsf{Comp}(B(P)) = \mathsf{Comp}(B(Q))$. Together with Theorem 5.2 and the known fact that dimension is a comparability invariant, this gives an alternative proof of the comparability invariance of interval dimension in the finite case. The comparability invariance of interval dimension was first shown in [HKM].

Theorem 5.5 *$\mathsf{Comp}(B(P))$ is a comparability invariant of P.*

Proof. Let A be an autonomous subset of P. It is enough (see e.g. [DPW]) to show that $\mathsf{Comp}(B(P)) = \mathsf{Comp}(B(P^A_{A^d}))$, where $P^A_{A^d}$ denotes the order resulting from substituting A by its dual A^d in P.

Note first that $B(A) = \{L(a), U(a) : a \in A\}$ is a closed suborder of $B(P)$. Let $\widetilde{B(A)}$ be $B(A)$ without its greatest element $1_{B(A)}$ and its least element $0_{B(A)}$. Our claim is that $\widetilde{B(A)}$ is autonomous in $B(P)$. To see this, observe first that, for each $a \in A$, we can decompose $\mathsf{Pred}(a)$ into $\mathsf{Pred}(a) = \mathsf{Pred}(A) \cup \mathsf{Pred}_A(a)$, hence, the same is valid for all elements of $\widetilde{B(A)}$. On the other hand, the elements of $B(P) \setminus B(A)$ either contain all of A or their intersection with A is empty. Now if $M \in B(P) \setminus B(A)$ contains all of A then it also contains $\mathsf{Pred}(A)$ and M is above all sets in $\widetilde{B(A)}$. If $M \subseteq \mathsf{Pred}(A)$ then M is below all sets in $\widetilde{B(A)}$. In all the other cases M is unrelated to all of $\widetilde{B(A)}$. This gives the claim.

To settle the theorem we again need an analogue of (5.6), namely $B(A^d) = B(A)^d$. Consider a normalized box embedding of A in $\mathbb{R}^k$. Its extreme corners are an embedding of $B(A)$ into $\mathbb{R}^k$. Flip the embedding, i.e., reverse the relations, this gives an embedding of A^d and the extreme corners form an embedding of $B(A)^d$. □

As we have seen, autonomous sets in P induce autonomous sets in $B(P)$. The converse, however, is far from being true. Take as P a prime interval order, then

B(P) is a chain. Hence P has none but B(P) has $\binom{|B(P)|}{2} - 1$ nontrivial autonomous sets.

5.3.5 Remarks on Computational Complexity

The transformation $P \to B(P)$ obviously can be computed in polynomial time, thus this paper also explains and reproves the earlier noticed coincidence in complexity for the recognition of partial orders of fixed (interval) dimension [Ya], and the related complexity status of recognizing trapezoid graphs [Ch, MS, HM]. In particular, it answers the questions of Dagan, Golumbic and Pinter [DGP] about the comparability invariance of interval dimension and the recognition of interval dimension at most 2 in a very direct and simple way.

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Chapter 6

Tolerance Graphs

6.1 Introduction and Overview

An undirected graph $G = (V, E)$ is called a *tolerance graph* if there exists a collection $\mathcal{I} = \{I_x \mid x \in V\}$ of closed intervals on the line and a (tolerance) function $t : V \to \mathbb{R}^+$ satisfying

$$\{x, y\} \in E \iff |I_x \cap I_y| \geq \min(t_x, t_y)$$

where $|I|$ denotes the length of the interval I. A tolerance graph is a *bounded tolerance graph* if it admits a tolerance representation $\{\mathcal{I}, t\}$ with $|I_x| \geq t_x$ for all $x \in V$.

Tolerance graphs were introduced by Golumbic and Monma [GM]. We summarize two of the results proved there. If all tolerances t_x equal the same value c, say, then we obtain exactly the class of all interval graphs. If the tolerances are $t_x = |I_x|$ for all vertices x, then we obtain exactly the class of all permutation graphs. Furthermore, the following theorem was proved.

Theorem 6.1 *Every bounded tolerance graph is the complement of a comparability graph, i.e., a cocomparability graph.*

The most important article on tolerance graphs is due to Golumbic, Monma and Trotter [GMT]. We summarize some of the results shown there in the next theorem.

Theorem 6.2

(1) *A tolerance graph must not contain a chordless cycle of length greater than or equal to 5.*

(2) *A tolerance graph must not contain the complement of a chordless cycle of length greater than or equal to 5.*

(3) *A tolerance graph admits an orientation such that every chordless 4 cycle is oriented as shown in Figure 6.1.*

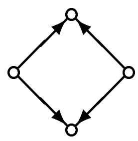

Figure 6.1: Alternating orientation of C_4

Remark. A graph G is called *alternatingly orientable* if there is an orientation of G such that around every chordless cycle of length greater than 3 the directions of arcs alternate. As a consequence of the preceding theorem, we obtain that tolerance graphs are alternatingly orientable, (see [Br] for more information on this class of graphs).

In this chapter we report on some results which may be useful for a complete answer to the following open problems.

Problem 6.1 *Characterize tolerance and bounded tolerance graphs.*

Problem 6.2 *Is the intersection of tolerance graphs and cocomparability graphs exactly the class of bounded tolerance graphs?*

6.2 Tolerance Graphs and Orders of Interval Dimension 2

The starting point of this work is a representation theorem for bounded tolerance graphs. Let $G = (V, E)$ be a bounded tolerance graph with representation $\{\mathcal{I}, t\}$ and $I_x = [a_x, b_x]$. We now define interval subgraphs G^1, G^2 of G. Let

G^1 be represented by the intervals $I_x^1 = [a_x + t_x, b_x]$ and G^2 by the intervals $I_x^2 = [a_x, b_x - t_x]$. It is easy to verify that $G = G^1 \cup G^2$. Since G^1 and G^2 are cocomparability graphs of interval orders P^1 and P^2, respectively, the comparability graph of $P = P^1 \cap P^2$ is the complement of G. Therefore G is the cocomparability graph of an order with interval dimension at most 2. A special feature of the interval realizer $\{\mathcal{I}^1, \mathcal{I}^2\}$ of P is that $|I_x^1| = |I_x^2|$ for all $x \in V$. In the spirit of the term box embedding introduced in Chapter 5, we call such a representation a *square embedding*. The construction is illustrated in Figure 6.2.

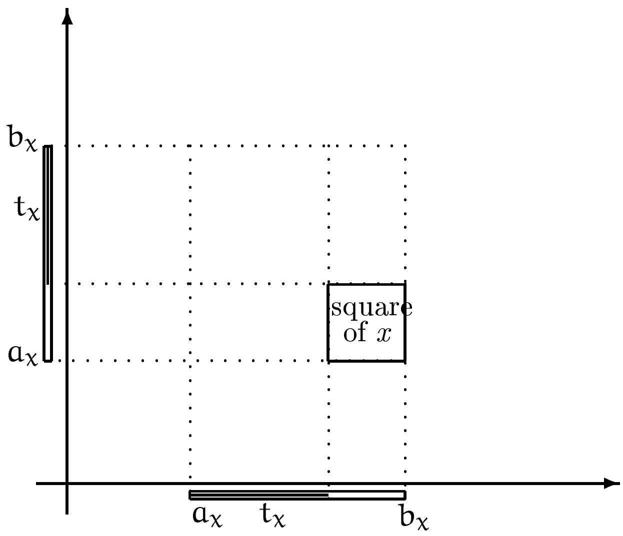

Figure 6.2: The square corresponding to $[a_x, b_x]$ and tolerance t_x

Conversely, let $P = (X, <)$ be an order of interval dimension ≤ 2 that admits a square embedding. We claim that the cocomparability graph of P ($\mathsf{CoComp}(P)$) is a bounded tolerance graph. Let $\{\mathcal{I}^1, \mathcal{I}^2\}$ constitute a square embedding of P, and let the corresponding intervals of x be given by $I_x^1 = [a_x^1, a_x^1 + l_x]$ and $I_x^2 = [a_x^2, a_x^2 + l_x]$. We now fix some $s \in \mathbb{R}$ such that $s \geq \max_{x \in X}(\ a_x^2 - a_x^1\ )$. The cocomparability graph of P is the bounded tolerance graph given by the intervals $I_x = [a_x^2, s + a_x^1 + l_x]$ and the tolerances $t_x = s + a_x^1 - a_x^2$.

We have thus shown the following strengthening of Theorem 6.1.

Theorem 6.3 *A graph G is a bounded tolerance graph iff G is the cocomparability graph of an order P with interval dimension at most 2 which has a square representation.*

Cocomparability graphs of orders with interval dimension at most 2 are known as *trapezoid graphs*. An observation similar to ours is used by Bogart et al. [BFIL] to show that the class of bounded tolerance graphs coincides with the class of parallelogram graphs, i.e., trapezoid graphs where every trapezoid is a parallelogram.

There exist orders of interval dimension 2 which do not admit a square representation. This is shown with the following example.

Example. It is easy to see that the graph G given in Figure 6.3 is not alternatingly orientable. An orientation which is alternating on the cycles $(3,4,7,8)$, $(7,8,5,6)$, $(5,6,1,2)$ and $(1,2,4,3))$ and contains $3 \to 4$ would require $7 \to 8, 5 \to 6, 1 \to 2, 4 \to 3$ a contradiction. Therefore, G is not a tolerance graph.

On the other hand, $\mathsf{G} = \mathsf{CoComp}(\mathsf{P})$, and the order P has a box embedding which proves that $\mathsf{idim}(\mathsf{P}) = 2$.

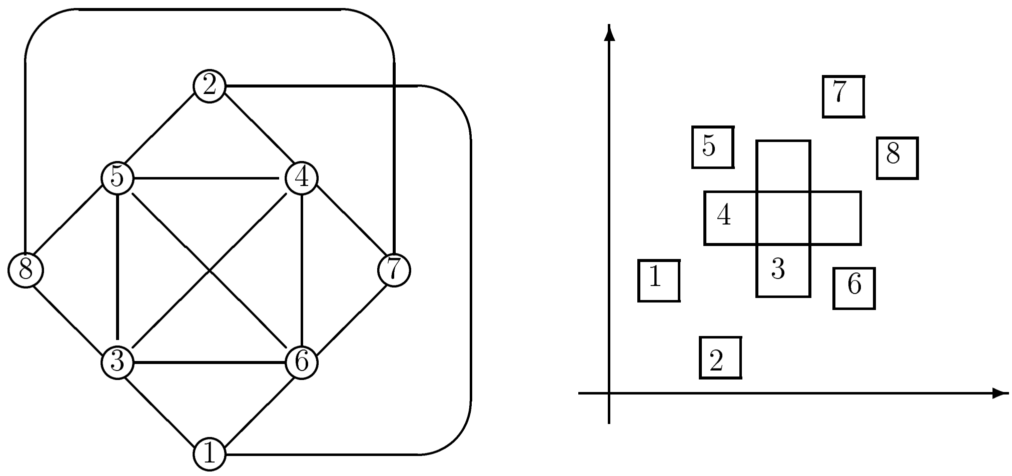

Figure 6.3: The graph $\mathsf{G} = \mathsf{CoComp}(\mathsf{P})$ and the order P.

Until now, we only dealt with bounded tolerance graphs. As a consequence of the next theorem, we will obtain: *Every tolerance graph which is a cocomparability graph is a trapezoid graph.* This will be useful later when we characterize all tolerance graphs that are complements of trees and could be of use as well for a solution to Problem 6.2.

Theorem 6.4 *The intersection of cocomparability graphs and alternatingly orientable graphs is contained in the class of trapezoid graphs.*

The main ingredient into the proof of this theorem will be a characterization of orders of interval dimension 2 (Lemma 6.1), which is due to Cogis [Co] (also [1]). Let $\mathsf{P} = (\mathsf{X}, <)$ and $x < y$, $z < t$ be a **2+2** in P. We call the pairs (x, t) and (z, y) the *diagonals* of the **2+2**. With P we now associate the *incompatibility graph* $\mathsf{F_P}$.

- As vertices of $\mathsf{F_P}$ we take the pairs (x, y) and (y, x) whenever $x||y$.
- Two vertices of $\mathsf{F_P}$ are connected by an edge iff they are the diagonals of a common **2+2** in P.

Lemma 6.1 (Cogis) $\mathsf{idim}(\mathsf{P}) \leq 2$ *iff* $\mathsf{F_P}$ *is bipartite.*

Cogis obtained this result in the more general context of the Ferrers-dimension of directed graphs. His definition of the incompatibility graph associated with a directed graph is somewhat more involved (see [HM], pages 5-7, for details). In the case of an antireflexive and transitive digraph (i.e., a partial order), however, an easy case analysis shows that we have chosen the right definition.

Proof. (Theorem) We have to show that $\mathsf{idim}(\mathsf{P}) > 2$ implies that $\mathsf{CoComp}(\mathsf{P})$ is not alternatingly orientable.

Let P with $\mathsf{idim}(\mathsf{P}) \geq 3$ be given. From Lemma 6.1 we know that $\mathsf{F_P}$ contains odd cycles. Fix an odd cycle $\mathsf{C} = [(x_1, y_1), (x_2, y_2), \dots, (x_{2k+1}, y_{2k+1}), (x_1, y_1)]$ in $\mathsf{F_P}$. If (x_i, y_i) and (x_{i+1}, y_{i+1}) are consecutive elements of C then, by the definition of $\mathsf{F_P}$, $[x_i, y_i, y_{i+1}, x_{i+1}, x_i]$ is a 4-cycle in $\mathsf{CoComp}(\mathsf{P})$. Therefore:

(∗) An alternating orientation of $\mathsf{CoComp}(\mathsf{P})$ will either contain the two arcs $x_i \to y_i$ and $y_{i+1} \to x_{i+1}$ or the two arcs $y_i \to x_i$ and $x_{i+1} \to y_{i+1}$.

Assume that an alternating orientation A of $\mathsf{CoComp}(\mathsf{P})$ is given, w.l.o.g. we may require $x_1 \to y_1$ to be in A. Using (∗) we obtain that $y_2 \to x_2$ is in A. Using (∗) again, we obtain that $x_3 \to y_3$ is in A. Repeating this argument we finally find $x_{2k+1} \to y_{2k+1}$ and hence $y_1 \to x_1$ in A. This contradicts the existence of an alternating orientation. □

6.2.1 Some Examples

After having obtained the previous theorem I had the idea that the intersection of cocomparability graphs with alternatingly orientable graphs and the intersection of cocomparability graphs with tolerance graphs could be the same. In this part we will separate several classes of graphs by examples.

We first need a definition. Let $P = (X, <)$ be an order of interval dimension 2 with a realizer, i.e., a box embedding, $I_1 = \{\ [a^1_x, b^1_x]\ :\ x \in X\}$, $I_2 = \{\ [a^2_x, b^2_x]\ :\ x \in X\}$. We say $x, y \in X$ have *crossing diagonals* if the line segments $(a^1_x, a^2_x) \to (b^1_x, b^2_x)$ and $(a^1_y, a^2_y) \to (b^1_y, b^2_y)$ intersect in $\mathbb{R}^2$.

Lemma 6.2 *If an order* P *of interval dimension 2 has a box embedding without crossing diagonals then* $G = \mathsf{CoComp}(P)$ *has an alternating orientation.*

Proof. We first indicate how to define an alternating orientation on G. Using the coordinates of the box of x we define two regions in the plane (see Figure 6.4).

$$R^1(x) = \{\ (u, v)\ :\ u \geq a^1_x \text{ and } v \leq b^2_x \text{ and } \mathsf{dist}[(u, v), (b^1_x, a^2_x)] \leq \mathsf{dist}[(u, v), (a^1_x, b^2_x)]\ \}$$
$$R^2(x) = \{\ (u, v)\ :\ u \leq b^1_x \text{ and } v \geq a^2_x \text{ and } \mathsf{dist}[(u, v), (b^1_x, a^2_x)] \geq \mathsf{dist}[(u, v), (a^1_x, b^2_x)]\ \}.$$

Note that if $x||y$ then the diagonal of the box of y has to intersect either $R^1(x)$

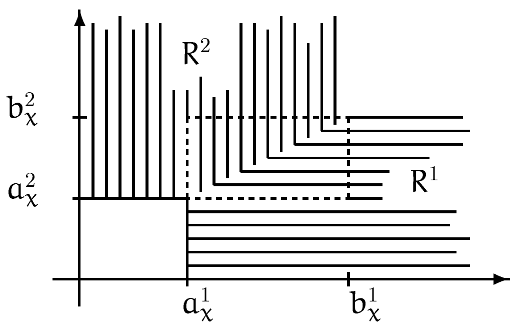

Figure 6.4: The regions defined by a box.

or $R^2(x)$. It can not intersect both, since, there are no crossing diagonals. If the diagonal of y intersects $R^1(x)$ we orient the edge $\{x, y\}$ from x to y and if the diagonal of y intersects $R^2(x)$ we orient it from y to x.

We now have to show that the orientation of G is alternating. Since G is a cocomparability graph we only have to deal with cycles of length 4. A C_4 in G

corresponds to a **2+2** in P. With this remark, it is not hard to verify that our orientation is alternating on the cycle. □

Consider the cocomparability graph G of the order P given in Figure 6.5. From Lemma 6.2 we know that G is alternatingly orientable, but due to a cyclic dependence among the boxes of a, b, c, and d there is no square representation of P. Therefore, G is not a bounded tolerance graph. But note that, whenever any box is deleted from the picture, the representation can be transformed into a square representation. Hence G is a minimal obstruction for the class of bounded tolerance graphs. The example is quite stable with respect to this property,

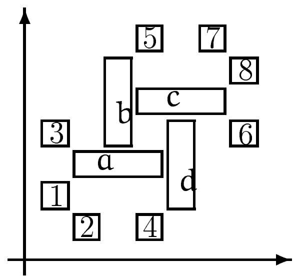

Figure 6.5: The complement of this order is alternatingly orientable but not a bounded tolerance graph.

we may add an arbitrary subset of the following comparabilities to P, without affecting it.

$$1 < 3, 2 < 4, 5 < 7, 6 < 8$$

In Theorem 6.6 and Theorem 6.7 we will exhibit more examples containing several infinite families of obstructions for the class of bounded tolerance graphs.

It would be quite hard to give a rigorous proof that the graph G from the previous example is no tolerance graph at all. Our aim, however, is to exhibit cocomparability graphs which possess an alternating orientation but are not tolerance graphs. This can be done using some more notation.

A vertex x of G is called *assertive* if for every tolerance representation $\{\mathcal{I}, t\}$ of G replacing t_x by $\min(t_x, |I_x|)$ leaves the tolerance graph unchanged. An assertive vertex never requires unbounded tolerance. Therefore, if every vertex of a tolerance graph G is assertive then G is a bounded tolerance graph. In [GMT] the following sufficient condition for a vertex to be assertive is shown.

Lemma 6.3 *Let x be a vertex in a tolerance graph $G = (X, E)$.*
If $\mathsf{Adj}(x) \setminus \mathsf{Adj}(y) \neq \emptyset$ for all y with $\{x, y\} \notin E$, then x is assertive.

Lemma 6.4 *If $G = (X, E)$ is not a bounded tolerance graph then $2G$ is not a tolerance graph. Here $2G$ is the graph on two copies X_1, X_2 of X with edges $\{x_1, x_2\}$ for all $x \in X$ and $\{x_i, y_j\}$ for all $\{x, y\} \in E$. That is, we replace each vertex of G by a 2 clique to obtain $2G$.*

Proof. It is easy to check that every vertex in $2G$ meets the conditions of Lemma 6.3, i.e., is assertive. Therefore, if $2G$ is a tolerance graph it will also be a bounded tolerance graph. This, however, is impossible since even G is not a bounded tolerance graph. □

Now consider the order given in Figure 6.6. The cocomparability graph of this order is just $2G$ if G is the graph corresponding to Figure 6.5. We have seen that G is not a bounded tolerance graph, so $2G$ is not a tolerance graph, but $2G$ has an alternating orientation since again there are no crossing diagonals.

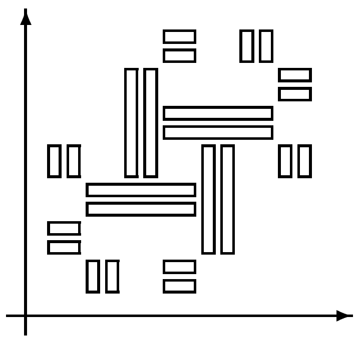

Figure 6.6: The complement of this order is alternatingly orientable but not a tolerance graph.

6.2.2 Cotrees and More Examples

Complements of trees are cocomparability graphs. In [GMT] it is suggested to take them as an initial step towards a solution of Problem 6.2. Using Theorems 6.3 and 6.4 we obtain

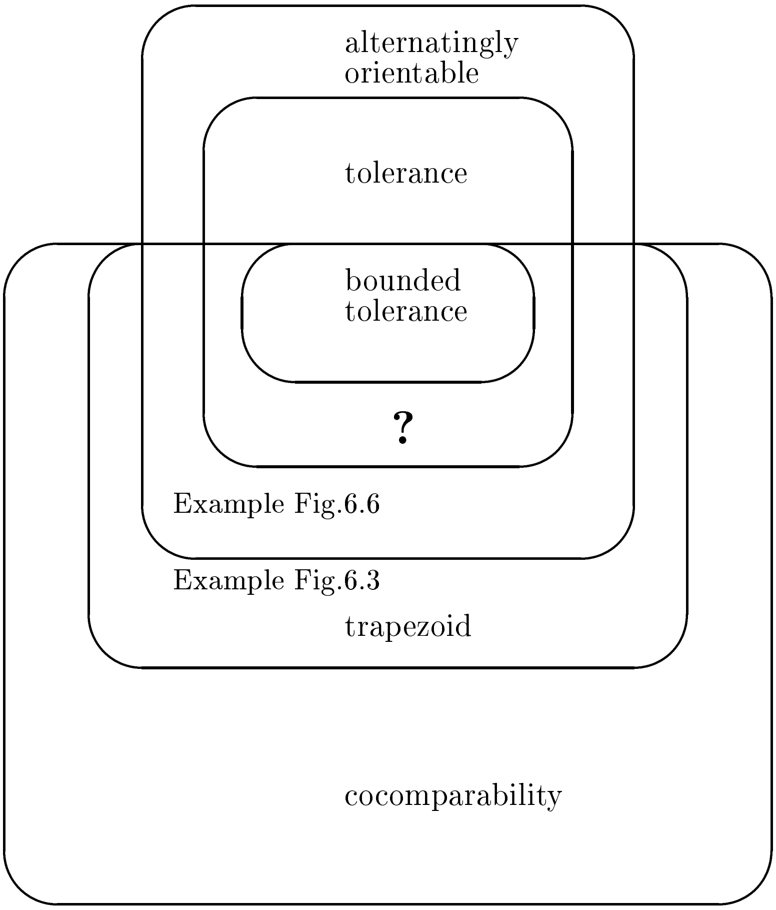

Figure 6.7: Relations of tolerance graphs to other classes of graphs.

Theorem 6.5 *If $\overline{T}$ is the complement of a tree T, then the following conditions are equivalent:*

(1) *$\overline{T}$ is a tolerance graph.*

(2) *$\overline{T}$ is a bounded tolerance graph.*

(3) *$\overline{T}$ is a trapezoid graph.*

(4) *T is a tolerance graph.*

(5) *T contains no subtree isomorphic to the tree T_3 of Figure 6.8.*

Proof. If $\overline{T}$ is a tolerance graph then, by Theorem 6.4, T is the comparability graph of an order of interval dimension 2, i.e., $\overline{T}$ is a trapezoid graph. Orders which have trees as comparability graphs are of height one. In a box realizer, of a height one order with interval dimension 2, we can grow the boxes of maxi-

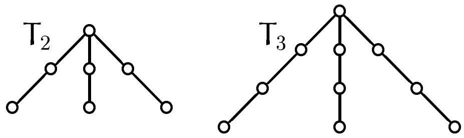

Figure 6.8: The trees T_2 and T_3.

mal elements upwards and boxes of minimal elements downwards to make them squares. Therefore, if $\overline{T}$ is a tolerance graph, then $\overline{T}$ is a bounded tolerance graph.

If $\overline{T}$ is a bounded tolerance graph then, trivially, it is a tolerance graph, and by Theorem 6.3 it is a trapezoid graph. This gives the equivalence of (1),(2) and (3).

For the equivalence of (3) and (5) we need a characterization of those trees which are the comparability graphs of orders of interval dimension 2. In chapter 5 we showed that for every poset $\mathsf{idim}(P) = \mathsf{dim}B(P)$. If T is a tree then $B(T)$ is the truncation of T, i.e., the tree obtained by cutting down the leaves of T (see page 75). Among the irreducible orders of dimension 3 there is only one tree, namely T_2. From this we obtain that T_3 is the unique tree among the obstructions against interval dimension 3.

For the remaining equivalence, i.e., to show the equivalence of (4) with (5) we refer to [GMT]. □

We now come back to minimal obstructions for the class of bounded tolerance graphs. A quite simple observation will provide us with many examples.

It is well known that a graph G is both a comparability graph and a cocomparability graph iff G and $\overline{G}$ are comparability graphs of orders of dimension 2. An order P is called *3–irreducible* if $\mathsf{dim}\ P = 3$, but whatever vertex x we remove from P we obtain an order of dimension 2, i.e., $\mathsf{dim}(P_x) = 2$ for all x.

Now let P be 3–irreducible and $G = \mathsf{Comp}(P)$. G is not a bounded tolerance graph since it is not a cocomparability graph. But if we remove any vertex x from G then G_x will be the cocomparability graph of an order of dimension 2. This order has an embedding by points, hence, a square embedding by minisquares. Therefore, G_x is a bounded tolerance graph.

Theorem 6.6 *If P is a 3–irreducible order then the comparability graph of P is a minimal obstruction for the class of bounded tolerance graphs.*

Remark. A complete list of 3–irreducible orders has independently been compiled by Kelly and by Trotter and Moore (see [KT]). They found 10 isolated examples and 7 infinite families.

We now turn to a second large class of obstructions. Recall that in the proof of Theorem 6.5 we gave evidence that a height 1 order of interval dimension 2 admits a square embedding, i.e., its cocomparability graph is a bounded tolerance graph. An order P is called *3–interval irreducible* if $\mathsf{idim}(P) = 3$ but whatever vertex x we remove from P we obtain an order of interval dimension 2, i.e., $\mathsf{idim}(P_x) = 2$.

Now let P be a 3–interval irreducible order of height 1 and $G = \mathsf{CoComp}(P)$. From $\mathsf{idim}(P) = 3$ we conclude that G is not a tolerance graph. But if we remove any vertex x from G then G_x is the cocomparability graph of an order posessing a square embedding. Hence G is a bounded tolerance graph.

Theorem 6.7 *If P is a 3–interval irreducible order of height 1 then the cocomparability graph of P is a minimal obstruction for both the class of tolerance graphs and the class of bounded tolerance graphs.*

Remark. A complete list of the 3–interval irreducible orders of height 1 has been compiled by Trotter [Tr]. There are 3 isolated examples and 6 infinite families.

We close this chapter with a last example. Let $N(x) = \mathsf{Adj}(x) \cup \{x\}$ denote the *neighbourhood* of a vertex x in G. A set of vertices $\{x_1, x_3, x_3\}$ is called an *asteroidal triple* if any two of them are connected by a path which avoids the neighbourhood of the remaining vertex. In [GMT] it is shown, that cocomparability graphs do not contain asteroidal triples, hence, bounded tolerance graphs are asteroidal triple-free as well. More information on asteroidal triple-free graphs is given in [COS].

All examples of tolerance graphs which are not bounded tolerance graphs given in [GMT] are not asteroidal triple-free. Therefore, it seems plausible to conjecture that every tolerance graph which is not bounded contains an asteroidal triple. Using Theorem 6.6 we now show that this is not true in general.

Example. Let G be the comparability graph of the order H_0 from the list of 3–irreducible orders, see Figure 6.9. This graph is a tolerance graph and asteroidal triple-free, but, by Theorem 6.6 it can not be a bounded tolerance graph.

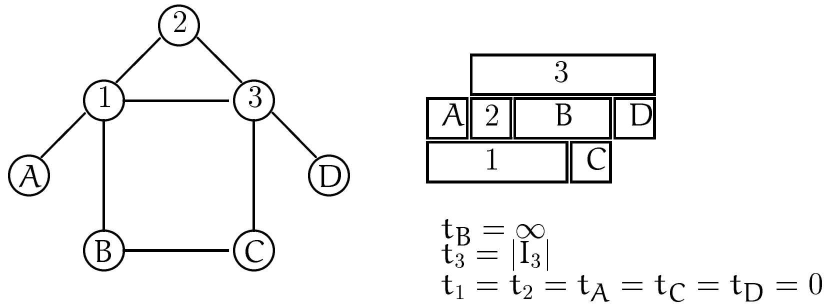

Figure 6.9: $G = \mathsf{Comp}(H_0)$ and a tolerance representation of G.

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