More Bounds for the Dimension of Interval Orders

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Abstract

Recently, by an ingenious construction Füredi, Rödel and Trotter could bound the dimension of interval orders by $4 \log\log h(P) + 5$. The contribution of this note are height independent bounds for the dimension of interval orders.

1 Introduction

Let $P = (X, <_P)$ and $Q = (X, <_Q)$ be two partial orders on a finite set X, we say Q is an *extension* of P if $x <_P y$ implies $x <_Q y$. Similarly a total order $L = x_1, \ldots, x_n$ is a *linear extension* of P if $x_i <_P x_j$ implies $i < j$.

A set of linear extensions of P whose intersection is P is called a *realizer* of P. In other words, a set $L_1, \ldots, L_r$ is a realizer of P if all incomparable pairs $x \| y$ are realized, i.e. there are $i, j \in \{1, \ldots, r\}$ with $x <_{L_i} y$ and $y <_{L_j} x$. The *dimension*, $\dim(P)$, of P is defined as the minimum size of a realizer (for a survey see [KT]). By a theorem of Yannakakis it is $\mathcal{NP}$-hard to compute $\dim(P)$.

A poset $P = (X, <_P)$ is an *interval order* if there is a set of intervals $(I_x)_{x \in X}$ on the real line such that $x <_P y$ iff I_x is on the left of I_y (for a survey see [F]). For interval orders, bounds on the dimension have been given in [BRT], [R] and [FRT]. The bound of Füredi, Rödel and Trotter [FRT] that dominates the others can be stated as

$$\dim(P) \leq 4 \log\log h(P) + 5$$

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here $h(P)$ denotes the height of the interval order P. The contribution of this note are height independent bounds for the dimension of interval orders. The dimension interval orders of is bounded by logarithmic functions of the 'staircase length' and the width of P.

In all subsequent considerations we assume that interval orders do not contain duplicate intervals and are given in their *canonical representation*; to obtain the canonical representation consider the lattice of maximal antichains of P. For interval orders this lattice is a chain $A_0 < A_1 < \ldots < A_m$. Associate with $x \in P$ the interval $[i, j]$ if i and j are the minimal respectively the maximal k with $x \in A_k$.

Most of our constructions of linear extensions rely on an easy observation. If in every interval there is a marked point, then shrinking the intervals to this points gives, if ties are broken, a linear extension. In fact every linear extension of an interval order can be obtained by appropriate marks.

2 Staircases and colorings

Let $P = (X, <)$ be an interval order and $\{(x_*, x^*) : x \in X\}$ be its canonical representation. Construct the linear extensions L_*, L^* by taking the lower, respectively the upper end of the intervals as marks. More formaly: L_* takes x before y if $x_* < y_*$ or $x_* = y_*$ and $x^* > y^*$; similarly L^* takes x before y if $x^* < y^*$ or $x^* = y^*$ and $x_* > y_*$. Let $x \| y$ be a pair not realized in $L_* \cup L^*$ if L_* takes x before y then we have $x_* < y_* < x^* < y^*$. Two elements x, y with this ordering of endpoints are called a *step*.

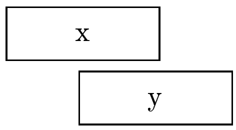

Figure 1: An example for a step

If we want to extend L_*, L^* to a realizer of P we just have to add the steps in reversed order. Let $S(P) = (X, U)$ be the directed graph with arcs $y \to x$ if x, y is a step, a path in $S(P)$ corresponds to a sequence of steps, i.e. a staircase. Since all paths in $S(P)$ are going down a staircase we obtain: $S(P)$ is acyclic. We now want to partition the arcs of $S(P)$ into classes $F_1, \ldots, F_r$, such that $P \cup F_i$ is acyclic for every i. We then can take linear extensions L_i of $P \cup F_i$ and obtain a realizer $L_*, L^*, L_1, \ldots, L_r$ of P. The next proposition gives an easy sufficient condition for the classes F_i.

Proposition 1 *If F does not contain consecutive arcs, then $P \cup F$ is acyclic.*

Proof Let F be a family of nonconsecutive steps and let us assume that $[x_0, \ldots, x_s, x_0]$ is a cycle in $P \cup F$. We may require $x_0 <_P x_1$. For all steps $(x_i, x_{i+1}) \in F$ we conclude $(x_{i-1}, x_i) \in P$. In the interval representation we then have $x^*_{i-1} < x^*_{i+1}$. Now let $[y_0, \ldots, y_t, y_0]$ be the sequence obtained by skipping the intermediate points x_i of all triplets x_{i-1}, x_i, x_{i+1} with $(x_i, x_{i+1}) \in F$. The interval endpoints of this sequence are strictly increasing, i.e. $y^*_0 < y^*_1 < \ldots < y^*_t < y^*_0$. This contradicts the assumption.$\Box$

Let an arc-coloring of a directed graph be an assignment of colors to arcs such that consecutive arcs obtain different colors. The arc-chromatic number of a digraph is the minimal number of colors in an arc-coloring. An inmediate consequence of proposition 1 then is:

Proposition 2 *An arc-coloring of $S(P)$ with k colors leads to a realizer of P with $k+2$ linear extensions.*

A first attempt could rely on the fact that an edge-coloring of $S(P)$ induces an arc-coloring. With Vizing's theorem we therefore get:

Proposition 3 $\dim(P) \leq \max_{x \in P}(\text{number of steps containing } x) + 3.$

We now discuss another strategy leading to an arc-coloring of $S(P)$. Take an antichain-partition $\mathcal{A} = \{A_1, \ldots, A_h\}$ of the transitive closure of $S(P)$ and melt every antichain of $\mathcal{A}$ into a single point, this gives $S_{\mathcal{A}}(P)$. Since consecutive arcs of $S(P)$ remain consecutive in $S_{\mathcal{A}}(P)$ we see that every arc-coloring of $S_{\mathcal{A}}(P)$ induces an arc-coloring of $S(P)$. The distinct arcs of $S_{\mathcal{A}}(P)$ can be seen as a subset of $U_h = \{(i,j) : i,j \in \{1, \ldots, h\}, i < j\}$, the arc set of the transitive tournament T_h on h points. Therefore the arc-chromatic number of T_h which is known to be $\lceil \log h \rceil$ (see [HE]) is an upper bound for the arc-chromatic number of $S_{\mathcal{A}}(P)$ and thus of $S(P)$.

Note that the number h of antichains in $\mathcal{A}$ is just the maximal staircase length of P. The bound obtained thus is:

Proposition 4 $\dim(P) \leq \log(\text{maximal staircase length of } P) + 2$

Remark In the transformation leading to the realizer of the size given in proposition 4 we did skip a lot of information which could be usefull to obtain better bounds. First of all note that $P \cup F$ remains acyclic if we only put the following relaxed condition on F: to each sequence $x_1 \to x_2 \to \ldots \to x_l$ of arcs in F there is an arc $x_1 \to x_l$ in $S(P)$. Furthermore the logarithm of the height may be arbitrary far from the arc-chromatic number of $S(P)$.

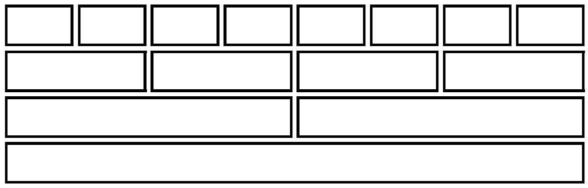

Figure 2: An order with large height but without steps

3 The width bound

The construction of this section is based on the following observation. If C_i and C_j are two chains in an interval order P, then all the incomparabilities among elements of C_i and C_j can be realized by two linear extensions $L_{1,0}$ and $L_{0,1}$. The order of $L_{1,0}$ is obtained by placing marks at the upper interval end of elements of C_i and at the lower ends of elements of C_j; for $L_{0,1}$ interchange the roles of C_i and C_j.

Let a minimal chain partition $C_1, \ldots, C_w$ of P be given. To a binary vektor $v = (v^1, \ldots, v^w)$ we associate the linear extension L_v obtained from marks placed at the upper interval end of elements contained in a chain C_i with $v^i = 1$ and at the lower ends of elements in chains C_i with $v^i = 0$. Call a set V of binary vectors of length w, such that for each pair $i, j \in \{1, \ldots, w\}$ there are $u, v \in V$ with $u^i = 1, u^j = 0$ and $v^i = 0, v^j = 1$ an A_w-set. From the previous considerations the next proposition should be obvious.

Proposition 5 *If V is an A_w-set then $\{L_v : v \in V\}$ is a realizer of P.*

Let $V = \{v_1, \ldots, v_k\}$ be an A_w-set and $v^i = (v^i_1, \ldots, v^i_k)$ then from the definition of A_w-sets we see that $W = \{v^1, \ldots, v^w\}$ is an antichain in the boolean lattice B_k. Therefore with Sperner's theorem we get: the minimal size of an A_w-set is

$$\beta(w) = \min_k [\binom{k}{\lceil \frac{k}{2} \rceil} \geq w]$$

and hence the proposition.

Proposition 6 $\dim(P) \leq \beta(w)$

Remark The magnitude of $\beta(w)$ can be estimated by

$$\log w < \beta(w) \leq \log w + \log(\log w + 1).$$

Conclusion

For each of the bounds given in proposition 3, 4 and 6 there are examples on which the bound dominates all the other known bounds. On the other side, however, we have not been able to decide wether the bounds are best possible for certain families of interval orders.

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