Arrangements of Pseudocircles:
On Digons and Triangles[*]

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Abstract. In this article, we study the cell-structure of simple arrangements of pairwise intersecting pseudocircles. The focus will be on two problems from Grünbaum's monograph from the 1970's.

First, we discuss the maximum number of digons or touching points. Grünbaum conjectured that there are at most $2n - 2$ digon cells or equivalently at most $2n - 2$ touchings. Agarwal et al. (2004) verified the conjecture for cylindrical arrangements. We show that the conjecture holds for any arrangement which contains three pseudocircles that pairwise form a touching. The proof makes use of the result for cylindrical arrangements. Moreover, we construct non-cylindrical arrangements which attain the maximum of $2n - 2$ touchings and have no triple of pairwise touching pseudocircles.

Second, we discuss the minimum number of triangular cells (triangles) in arrangements without digons and touchings. Felsner and Scheucher (2017) showed that there exist arrangements with only $\lceil \frac{16}{11}n \rceil$ triangles, which disproved a conjecture of Grünbaum. Here we provide a construction with only $\lceil \frac{4}{3}n \rceil$ triangles. A corresponding lower bound was obtained by Snoeyink and Hershberger (1991).

Keywords: arrangement of pseudocircles · touching · empty lense · cylindrical arrangement · arrangement of pseudoparabolas · Grünbaum's conjecture

1 Introduction

An *arrangement $\mathcal{A}$ of pairwise intersecting pseudocircles* is a collection of $n(\mathcal{A})$ simple closed curves on the sphere or plane such that any two of the curves either touch in a single point or intersect in exactly two points where they cross. Throughout this article, we consider all arrangements to be *simple*, that is, no three pseudocircles meet in a common point. An arrangement $\mathcal{A}$ partitions the
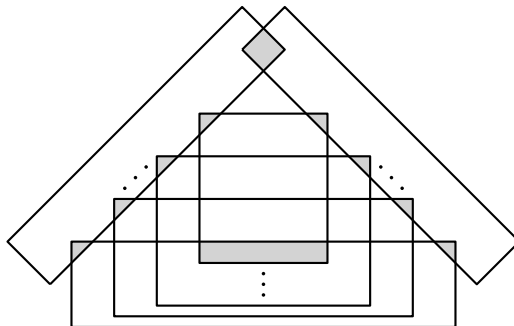
Fig. 1: An illustration of the construction by Grünbaum [6, Figure 3.28]: an arrangement of $n \geq 4$ pairwise intersecting pseudocircles with exactly $2n - 2$ digons. Digons are highlighted gray.

plane into cells. A cell with exactly k crossings on its boundary is a *k-cell*, 2-cells are also called *digons* and 3-cells are *triangles*. The number of k-cells of an arrangement $\mathcal{A}$ is denoted as $p_k(\mathcal{A})$.

The study of cells in arrangements started about 100 years ago when Levi [7] showed that, in an arrangement of at least three pseudolines in the projective plane, every pseudoline is incident to at least three triangles. In the 1970's, Grünbaum [6] intensively investigated arrangements of pseudolines and initiated the study of arrangements of pseudocircles.

1.1   Digons and touchings

Concerning digons in arrangements of pairwise intersecting pseudocircles, Grünbaum [6] presented a construction with $2n - 2$ digons (depicted in Figure 1) and conjectured that these arrangements have the maximum number of digons[1].

Conjecture 1 (Grünbaum's digon conjecture [6, Conjecture 3.6]). Every simple arrangement $\mathcal{A}$ of n pairwise intersecting pseudocircles has at most $2n-2$ digons, i.e., $p_2 \leq 2n - 2$.

It was shown by Agarwal et al. [1, Corollary 2.12] that Conjecture 1 holds for simple cylindrical arrangements. An intersecting arrangement of pseudocircles is *cylindrical* if there is a pair of cells which are separated by each pseudocircle of the arrangement. More specifically, they showed that the number of touchings in an intersecting arrangement of n pseudo-parabolas is at most $2n - 4$ [1, Theorem 2.4]. An *intersecting arrangement of pseudoparabolas* is a collection of infinite x-monotone curves, called *pseudoparabolas*, where each pair of them either have a single touching or intersect in exactly two points where they cross. Every cylindrical arrangement of pseudocircles can be represented as an arrangement

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[1] Originally the conjecture was stated as to include non-simple arrangements which are *non-trivial*, i.e., non-simple arrangements with at least 3 crossing points.
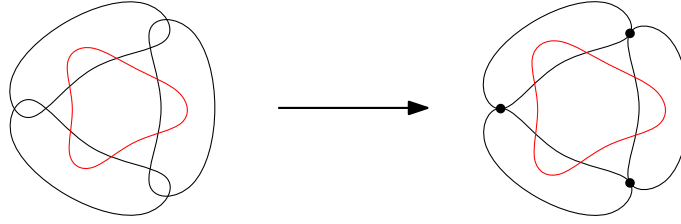
Fig. 2: Contracting some of the digons to touchings.

of pseudoparabolas and vice versa. From an arrangement of pseudoparabolas one can directly obtain a drawing of an arrangement of pseudocircles on the lateral surface of a cylinder so that the pseudocircles wrap around the cylinder. The two separating cells correspond to the top and the bottom of the cylinder.

Agarwal et al. [1, Theorem 2.13] showed for intersecting arrangements of pseudocircles that the number of digons is at most linear in n. The proof is based on the fact that every arrangement of intersecting pseudocircles can be stabbed by constantly many points. That is, there exists an absolute constant k, called the *stabbing number*[2], such that for every arrangement of n pseudocircles in the plane there exists a set of k points with the property that each pseudocircle contains at least one of the points in its interior [1, Corollary 2.8]. Therefore, the arrangement can be decomposed into constantly many cylindrical subarrangements. The linear upper bound then follows from the fact that each pair of subarrangements contributes at most linearly many digons. In [5] we verified Grünbaum's digon conjecture for up to 7 pseudocircles.

Here we show that Grünbaum's digon conjecture (Conjecture 1) holds for arrangements which contain three pseudocircles that pairwise form a digon. Before we state the result as a theorem, let us introduce some notation. For an arrangement $\mathcal{A}$ of pseudocircles and any selection of its digons, we can perform a perturbation so that the selected digons become touching points. Figure 2 gives an illustration. It is therefore sufficient to find an upper bound on the number of touchings to prove Grünbaum's digon conjecture. We define the *touching graph* $T(\mathcal{A})$ to have the pseudocircles of $\mathcal{A}$ as vertices, and two vertices form an edge if the two corresponding pseudocircles touch.

Theorem 1. *Let $\mathcal{A}$ be a simple arrangement of n pairwise intersecting pseudocircles. If the touching graph $T(\mathcal{A})$ contains a triangle, then there are at most $2n - 2$ touchings, i.e., $p_2 \leq 2n - 2$.*

Theorem 1 in particular shows that Grünbaum's construction with $2n - 2$ touchings is maximal for arrangements with triangles in the touching graph. However, the maximum number of touchings in general arrangements remains unknown. In Section 3 we construct a family of arrangements of n pseudocircles

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[2] In the literature, the stabbing number is also referred to as *piercing number* or *transversal number*.

which have exactly $2n - 2$ touchings and a triangle free touching graph. This family witnesses that the conjectured upper bound (Conjecture 1) can also be achieved in the cases not covered by Theorem 1.

Proposition 1. *For $n \in \{11, 14, 15\}$ and $n \geq 17$ there exists a simple arrangement $\mathcal{A}_n$ of n pairwise intersecting pseudocircles with no triangle in the touching graph $T(\mathcal{A}_n)$ and with exactly $p_2(\mathcal{A}_n) = 2n - 2$ touchings.*

1.2   Triangles in digon-free arrangements

In this context we assume that all arrangements are digon- and touching-free. It was shown by Levi [7] that every arrangement of n pseudolines in the projective plane contains at least n triangles. Since arrangements of pseudolines are in correspondence with arrangements of *great-pseudocircles* (see e.g. [4, Section 4]), it follows directly that an arrangement of n great-pseudocircles contains at least $2n$ triangles, i.e., $p_3 \geq 2n$.

Grünbaum conjectured that every digon-free intersecting arrangement on n pseudocircles contains at least $2n - 4$ triangles [6, Conjecture 3.7]. Snoeyink and Hershberger [10] proved a sweeping lemma for arrangements of pseudocircles. Using this powerful tool, they concluded that in every digon-free intersecting arrangement every pseudocircle has two triangles on each of its two sides (interior and exterior). This immediately implies the lower bound $p_3(\mathcal{A}) \geq 4n/3$; see Section 4.2 in [10].

In [5] we constructed an infinite family of digon-free arrangements with $p_3 < \frac{16}{11}n$ which shows that Grünbaum's conjecture is wrong and verified that the lower bound $p_3 \geq 4n/3$ by Snoeyink and Hershberger is tight for $6 \leq n \leq 14$. Here we show that their bound is tight for all $n \geq 6$:

Theorem 2. *For every $n \geq 6$, there exists a simple digon-free arrangement $\mathcal{A}_n$ of n pairwise intersecting pseudocircles with $p_3(\mathcal{A}_n) = \lceil \frac{4}{3}n \rceil$ triangles. Moreover, these arrangements are cylindrical.*

All arrangements constructed in Section 4 contain a specific arrangement $\mathcal{A}_6$ (depicted on the left of Figure 11) as a subarrangement. This remarkable arrangement has been studied as the arrangement $\mathcal{N}_6^\Delta$ in [4] where it was shown that $\mathcal{N}_6^\Delta$ is *non-circularizable*, i.e., $\mathcal{N}_6^\Delta$ cannot be represented by an arrangement of proper circles. As a consequence, all arrangements constructed in Section 4 are as well non-circularizable. In fact, all known counter-examples to Grünbaum's triangle conjecture contain $\mathcal{N}_6^\Delta$ and are therefore non-circularizable. Hence, Grünbaum's conjecture may still be true when restricted to arrangements of proper circles.

Conjecture 2 (Weak Grünbaum triangle conjecture, [5, Conjecture 2.2]). Every simple digon-free arrangement $\mathcal{A}$ of n pairwise intersecting circles has at least $2n - 4$ triangles.

1.3   Related Work and Discussion

In the proof of Theorem 1 we make use of a triangle (K_3) in the touching graph to bound the number of digons in the arrangement. It would be interesting whether other subgraphs like C_4 or $K_{3,3}$ can also be used to bound the number of digons.

The focus of this article is on arrangements of pairwise intersecting pseudocircles. For the setting of arrangements, where pseudocircles do not necessarily pairwise intersect, a classical construction of Erdős [3] gives arrangements of n unit circles with $\Omega(n^{1+c/\log\log n})$ touchings. An upper bound of $O(n^{3/2+\epsilon})$ on the number of digons in *circle* arrangements was shown by Aronov and Sharir [2]. The precise asymptotics, however, remain unknown. Moreover, we are not aware of an upper bound for *pseudocircles*.

Problem 1. Determine the maximum number of touchings among all simple arrangements of n circles and pseudocircles, respectively.

It is also worth noting that, for the very restrictive setting of arrangements of n pairwise intersecting unit-circles, Pinchasi showed an upper bound of $p_2 \leq n + 3$ [8, Lemma 3.4 and Corollary 3.10].

Concerning arrangements with digons, the number of triangles behaves different than in digon-free arrangements. While our best lower bound so far is $p_3 \geq 2n/3$, we managed to verify that $p_3 \geq n - 1$ is a tight lower bound for $3 \leq n \leq 7$ using a computer-assisted exhaustive enumeration [5]. It remains open, whether $p_3 \geq n - 1$ is a tight lower bound for every $n \geq 3$.

Conjecture 3 ([5, Conjecture 2.10]). Every simple arrangement of $n \geq 3$ pairwise intersecting pseudocircles has at least $n - 1$ triangles, i.e., $p_3 \geq n - 1$.

Concerning the maximum number of triangles in intersecting arrangements, in [5] we have shown an upper bound $p_3 \leq \frac{4}{3}\binom{n}{2} + O(n)$ which is optimal up to a linear error term. In fact, while $\frac{4}{3}\binom{n}{2}$ is an upper bound for arrangements of great-pseudocircles, we managed to find an intersecting arrangement with no digons, no touchings, and $\frac{4}{3}\binom{n}{2} + 1$ triangles. However, since we are not aware of an infinite family of such arrangements, it remains an interesting question to determine the exact maximum number of triangles.

Problem 2. Determine the maximum number of triangles among all simple arrangements of n pairwise intersecting pseudocircles.

2   Proof of Theorem 1

Since the touching graph $T(\mathcal{A})$ contains a triangle, there are three pseudocircles in $\mathcal{A}$ that pairwise touch. Let $\mathcal{K}$ be the subarrangement induced by these three pseudocircles and let $\triangle$ and $\triangle'$ denote the two open triangle cells in $\mathcal{K}$. We label the three touching points, which are also the corners of $\triangle$ and $\triangle'$, as a, b, c. Furthermore, we label the three boundary arcs of $\triangle$ (resp. $\triangle'$) as α, β, γ (resp. α', β', γ'), as shown in Figure 3(a).
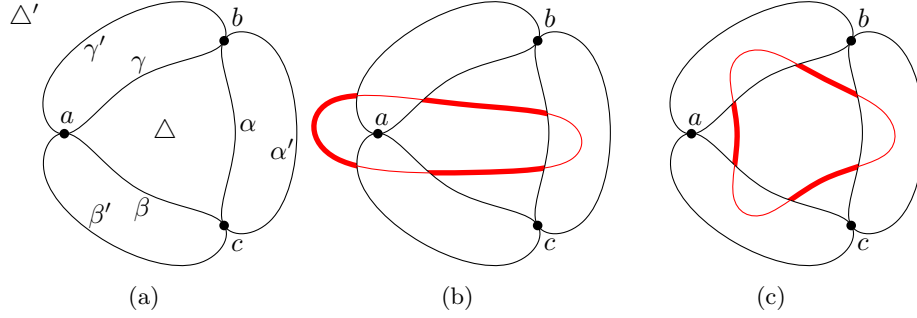
Fig. 3: (a) An illustration of the subarrangement $\mathcal{K}$. (b) and (c), respectively, illustrate an additional pseudocircle C (red). The pc-arcs inside $\triangle$ and $\triangle'$, respectively, are highlighted.

Assume that all digons in $\mathcal{A}$ are contracted to touchings. The intersection of a pseudocircle $C \in \mathcal{A} \setminus \mathcal{K}$ with $\triangle \cup \triangle'$ results in three connected segments, which we denote as the three *pc-arcs* of C, see Figures 3(b) and 3(c). Note that two of the pc-arcs induced by C may share an endpoint if C forms a touching with one of the pseudocircles from $\mathcal{K}$; Figure 5 shows such a touching.

Each pc-arc in $\triangle$ connects two of α, β or γ while a pc-arc in $\triangle'$ connects two of α', β' and γ'. Depending on the boundary arcs on which they start and end, they belong to one of the types $\alpha\beta$, $\beta\gamma$, $\alpha\gamma$, $\alpha'\beta'$, $\beta'\gamma'$ or $\alpha'\gamma'$.

Claim 1 *If two pc-arcs inside $\triangle$ (resp. $\triangle'$) have a touching or cross twice, then they are of the same type.*

Proof. We prove the claim for $\triangle$; the argument for $\triangle'$ is the same. Suppose towards a contradiction that two distinct pseudocircles C, C' from $\mathcal{A} \setminus \mathcal{K}$ contain pc-arcs $A \subset C \cap \triangle$ and $A' \subset C' \cap \triangle$ of different types that have a touching or cross twice. For simplicity, consider only the arrangement induced by the five pseudocircles $\mathcal{K} \cup \{C, C'\}$. By symmetry we may assume that A is of type $\alpha\gamma$ and A' is of type $\alpha\beta$. We may further assume that A and A' have a touching, since otherwise, if they cross twice, they form a digon and we can contract it. This allows us to distinguish four cases which are depicted in Figure 4 (up to further possible contractions of digons formed between C and the pseudocircles of $\mathcal{K}$).

    Case 1: C separates a from b and c.
    Case 2: C separates b from a and c.
    Case 3: C separates c from a and b.
    Case 4: C does not separate a, b, c.
In the next paragraph we show that in neither case, it is possible to extend the arc A' to a pseudocircle C' intersecting the three pseudocircles of $\mathcal{K}$. This is a contradiction.

Extend A' starting from its endpoint on α. The only way to reach γ or γ', avoiding an invalid, additional intersection with C, is via the pseudocircle $\beta \cup \beta'$.
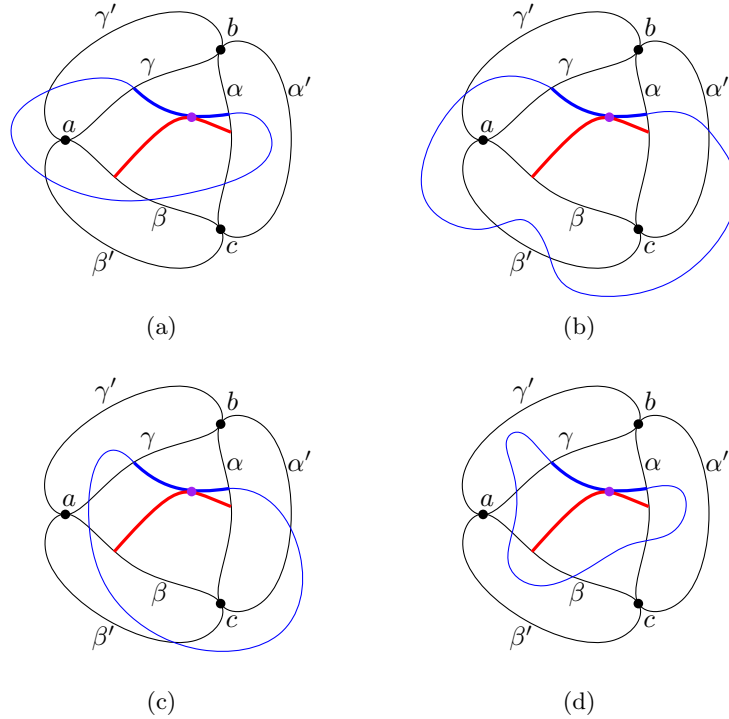
Fig. 4: (a)–(d) illustrate Cases 1–4 from the proof of Claim 1. The pseudocircles C and C' are highlighted blue and red, respectively. The pc-arcs A and A' are emphasized.

But the other endpoint of A' already lies on β, so either the pseudocircle extending A' has at least three intersections with $\beta \cup \beta'$ or it misses $\gamma \cup \gamma'$. Both are prohibited in an intersecting arrangement extending $\mathcal{K}$. This completes the proof of Claim 1. □

Next we transform $\mathcal{A}$ into another intersecting arrangement $\mathcal{A}'$ by redrawing the pc-arcs within $\triangle$ and $\triangle'$ such that the pairwise intersections and touchings are preserved and all crossings and touchings of each arc type are concentrated in a narrow region as depicted in Figure 5. First we apply an appropriate homeomorphism on the drawing so that $\triangle$ becomes a proper triangle ($\triangle'$ will be treated in an analogous manner). For the arc type $\alpha\beta$ we place a small rectangular region $R_{\alpha\beta}$ within $\triangle$ that lies close to the vertex c. We now redraw all pc-arcs of type $\alpha\beta$ so that

 – all crossings and touchings between pc-arcs of type $\alpha\beta$ lie inside $R_{\alpha\beta}$,
 – every pc-arc of type $\alpha\beta$ intersects $R_{\alpha\beta}$ on opposite sites, and
 – for every pc-arc of type $\alpha\beta$, the removal of $R_{\alpha\beta}$ leaves two straight line segments which connect $R_{\alpha\beta}$ to α and β (i.e., the boundary segments of $\triangle$).

We proceed analogously for the arc types $\alpha\gamma$ and $\beta\gamma$. By Claim 1 touchings and double crossings only occur between pc-arcs of the same type and therefore lie
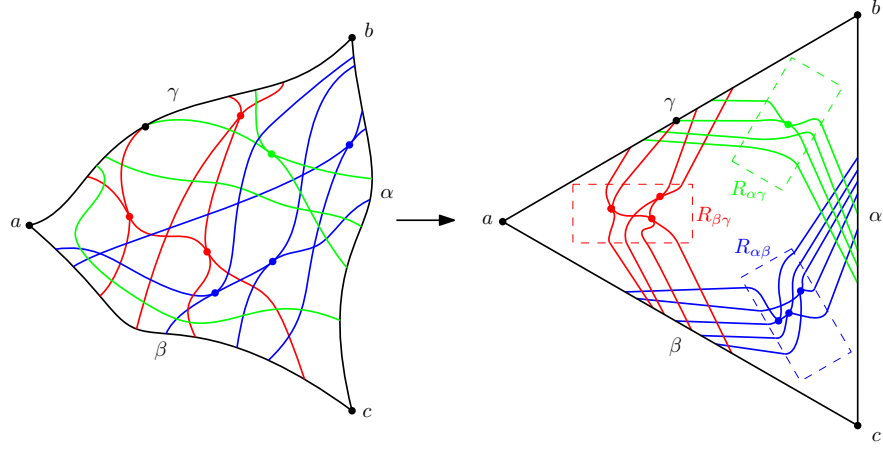
Fig. 5: Concentrate all crossings and touchings of one arc type in a narrow region. The narrow regions are indicated by dashed rectangles.

in the rectangular regions. Since the rectangular regions are placed close enough to the vertices a, b, c of the triangle $\triangle$, no additional intersections or touching points are introduced and we obtain an arrangement $\mathcal{A}'$ of pseudocircles with the same intersections and touchings as $\mathcal{A}$. The combinatorics of the resulting arrangement $\mathcal{A}'$ may however differ from $\mathcal{A}$ since the transformation typically changes the intersection orders of the pseudocircles. We conclude:

Observation *The transformation preserves the incidence relation between any pair of pc-arcs, that is, two pc-arcs in $\mathcal{A}$ are disjoint/cross in one point/cross in two points/touch if and only if the two corresponding pc-arcs in $\mathcal{A}'$ are disjoint/cross in one point/cross in two points/touch.*

This implies that $\mathcal{A}'$ is indeed again an arrangement of $n(\mathcal{A}') = n(\mathcal{A})$ pairwise intersecting pseudocircles with identical touching graph $T(\mathcal{A}') = T(\mathcal{A})$. In particular, the number of touchings is preserved.

Claim 2 *The arrangement induced by $\mathcal{A}' \setminus \mathcal{K}$ is cylindrical.*

Proof. For each pseudocircle $C \in \mathcal{A}' \setminus \mathcal{K}$, the intersection

$$C \cap (\triangle \cup \triangle') = (C \cap \triangle) \cup (C \cap \triangle')$$

consists of three pc-arcs, and each of these three pc-arcs is of a different type. The first arc is of type $\alpha\beta$ or $\alpha'\beta'$ (depending on whether it is inside $\triangle$ or $\triangle'$), the second is of type $\beta\gamma$ or $\beta'\gamma'$, and the third is of type $\alpha\gamma$ or $\alpha'\gamma'$.

Now we redraw $\mathcal{A}'$ on a cylinder as illustrated in Figure 6. Since all crossings and touchings of the arc type are within a small region, all pseudocircles from $\mathcal{A}' \setminus \mathcal{K}$ wrap around the cylinder, and hence the arrangement induced by $\mathcal{A}' \setminus \mathcal{K}$ is cylindrical. This completes the proof of Claim 2.   □
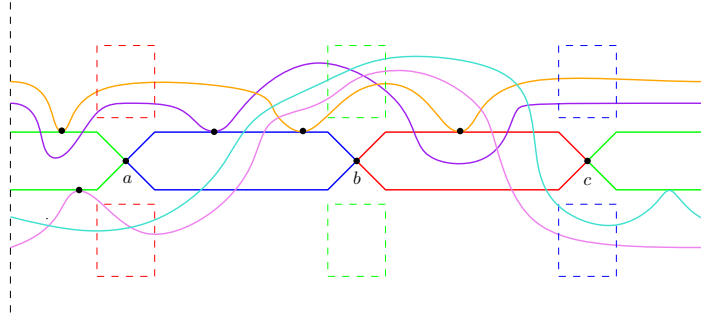
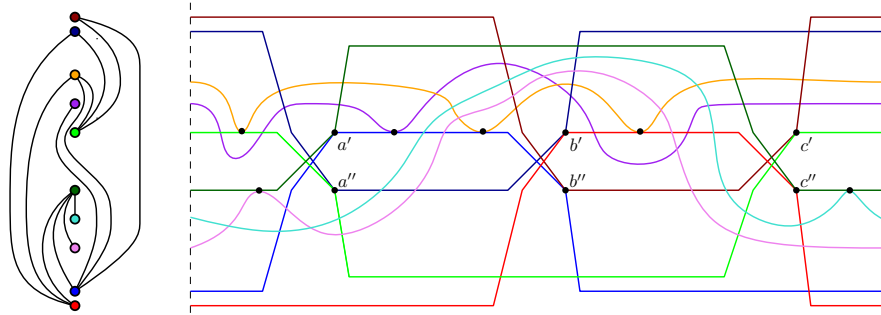**Fig. 6:** A cylindrical drawing of $\mathcal{A}' \setminus \mathcal{K}$.

Fig. 7: Replace each of the three pseudocircles of $\mathcal{K}$ by two new pseudocircles so that the entire arrangement is now cylindrical. The green (resp. red and blue) pseudocircle from Figure 6 is replaced by a new green and a new darkgreen (resp. red and darkred, and blue and darkblue) pseudocircle. On the left: the touching graph $T(\mathcal{A}'')$ of the arrangement.

Next we replace the three pseudocircles of $\mathcal{K}$ by six pseudocircles as illustrated in Figure 7, so that the resulting arrangement $\mathcal{A}''$ is cylindrical. Each of the three touching points a, b, c in $\mathcal{K}$ is replaced by two new touching points and altogether we obtain touchings $a', a'', b', b'', c', c''$. Hence, when transforming $\mathcal{A}$ into $\mathcal{A}''$, the number of pseudocircles is increased by 3 and the number of touchings is also increased by 3.

Agarwal et al. [1] proved the $p_2 \leq 2n - 2$ upper bound on the number of touchings in cylindrical arrangements of n pairwise intersecting pseudocircles by bounding the number of touchings in an arrangement of pairwise intersecting pseudoparabolas. They show that their touching graph is planar and bipartite [1, Theorem 2.4]. In fact, the drawing of $\mathcal{A}''$ in Figure 7 can be seen as an intersecting arrangement of pseudoparabolas. We review the ideas of their proof to verify the following claim.

Claim 3 $T(\mathcal{A}'')$ *is planar, bipartite, and has at most* $2n - 5$ *edges.*

Proof. Label the pseudoparabolas with starting segments sorted from top to bottom as $P_1, \ldots, P_n$. In the touching graph $T(\mathcal{A}'')$, we label the corresponding vertices as $1, \ldots, n$.

Bipartiteness: The bipartition comes from the fact that the digons incident to a fixed pseudoparabola P_j are either all from below or all from above. Suppose that a pseudoparabola P_j has a touching from above with P_i and from below with P_k. It follows that P_i is above P_j everywhere and P_k is below P_j everywhere. Hence, P_i and P_k are separated by P_j and cannot intersect – this contradicts the assumption that the pseudocircles are pairwise intersecting.

We now further observe that the uppermost pseudoparabola P_1 and the lowermost pseudoparabola P_n belong to distinct parts of the bipartition, because P_1 has all touchings below (i.e. with parabolas of greater index); P_n has all touchings above (i.e. with parabolas of smaller index). Hence, the touching graph remains bipartite after adding the edge $\{1, n\}$.

Planarity: For the planarity of $T(\mathcal{A}'')$, Agarwal et al. [1] create a particular drawing: The vertices are drawn on a vertical line and each edge $e = \{u, v\}$ is drawn as y-monotone curve according to the following *drawing rule*: For each w with $u < w < v$, we route e to the left of w if the pseudoparabola P_w intersects P_u before P_v, and to right otherwise. It is then shown that in the so-obtained drawing $\mathcal{D}$, each pair of independent edges has an even number of intersections. Hence, the Hanani–Tutte theorem (cf. Section 3 in [9]) implies that $T(\mathcal{A}'')$ is planar.

Notice that $\{1, n\}$ is not an edge in $T(\mathcal{A}'')$, since by construction, the lowermost and uppermost pseudocircles do not touch. We further observe that, since all edges in $\mathcal{D}$ are drawn as y-monotone curves, the entire drawing lies in a box which is bounded from above by vertex 1 and from below by vertex n. Hence, we can draw an additional edge from 1 to n which is routed entirely outside of the box and does not intersect any other edge. Again, by the Hanani–Tutte theorem, we have planarity. Since any planar bipartite graph on n vertices has at most $2n - 4$ edges, we conclude that $T(\mathcal{A}'')$ has at most $2n - 5$ edges. This completes the proof of Claim 3. □

We are now ready to finalize the proof of Theorem 1. From Claim 3 we obtain that $p_2(\mathcal{A}) + 3 = p_2(\mathcal{A}'') \leq 2(n + 3) - 5$, and therefore $p_2(\mathcal{A}) \leq 2n - 2$. This completes the argument.

3   Proof of Proposition 1

The proof of Proposition 1 is based on the *blossom operation*, which allows to dissolve certain triangles in the touching graph. We will apply the blossom operation to arrangements whose touching graphs are wheel graphs to obtain arrangements with the desired properties.

The blossom operation. Let $\mathcal{A}$ be an arrangement of pairwise intersecting pseudocircles, let v be a pseudocircle in $\mathcal{A}$, and let $w_1, \ldots, w_d$ be the pseudocircles
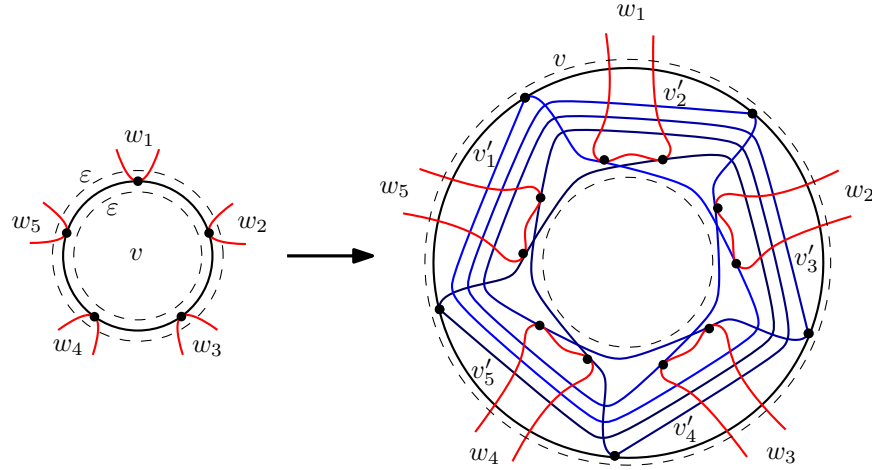
Fig. 8: An illustration of the blossom operation applied on the pseudocircle v of an arrangement.
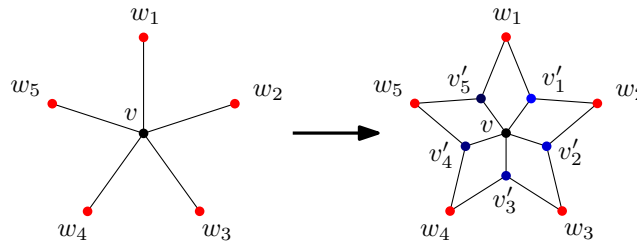

Fig. 9: Blossom operation applied on v: Modification of the touching graph.

in $\mathcal{A}$ which form touchings with v in this particular circular order along v. As illustrated in Figure 8, the blossom operation relaxes the touchings between v and $w_1, \ldots, w_d$ to digons and inserts d new pseudocircles $v'_1, \ldots, v'_d$ inside and very close to v so that

- $v'_1, \ldots, v'_d$ form a cylindrical arrangement,
- v touches $v'_1, \ldots, v'_d$, and
- w_i touches v'_{i-1} and v'_i (indices modulo d).

Since the new pseudocircles $v'_1, \ldots, v'_d$ are added in an ε-small area close to v, it is ensured that each v'_i intersects all other pseudocircles. Hence, the obtained arrangement is again an arrangement of pairwise intersecting pseudocircles.

Figure 9 shows the effect of the blossom operation on the touching graph. Note that in these graph drawings the circular orders of the edges incident to a vertex coincide with the orders in which the touchings appear on the corresponding pseudocircle.

The blossom operation increases the number of pseudocircles $n(\mathcal{A})$ by d while it increases the number of touchings $p_2(\mathcal{A})$ by $2d$. Hence, when applied to an
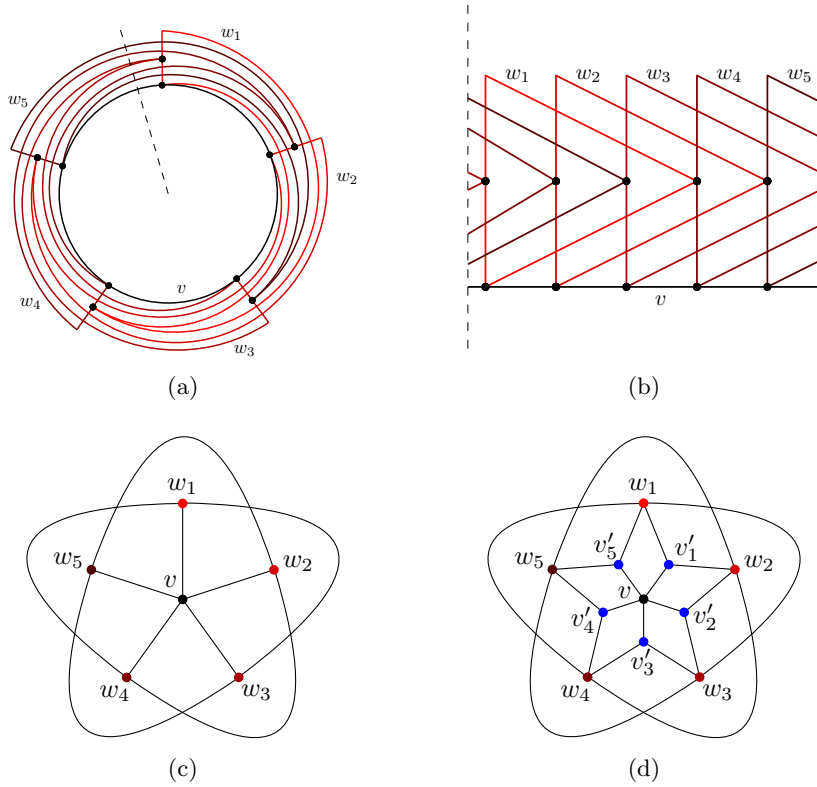
(a)

(b)

(c)

(d)

Fig. 10: (a) An arrangement $\mathcal{A}$ of 6 pseudocircles, (b) its cylindrical representation, (c) its touching graph $T(\mathcal{A})$, and (d) the touching graph $T(\mathcal{A}')$ after applying the blossom operation to v.

arrangement $\mathcal{A}$ with exactly $p_2(\mathcal{A}) = 2n(\mathcal{A}) - 2$ touchings, the blossom operation yields again an arrangement $\mathcal{A}'$ with $p_2(\mathcal{A}') = 2n(\mathcal{A}') - 2$ touchings.

Moreover, the blossom operation can be used to eliminate certain triangles in the touching graph. Assume w_i and w_j have a common touching, so v, w_i, w_j form a triangle in the touching graph. Then the blossom operation on v destroys this triangle without creating a new one if and only if, along the pseudocircle v, the two touchings with w_i and w_j are not consecutive. In Figure 9 a triangle $\{v, w_1, w_2\}$ would result in the new triangle $\{v_1', w_1, w_2\}$, while a triangle $\{v, w_1, w_3\}$ would not yield a new triangle.

Using the blossom operation, we are now able to prove Proposition 1.

Proof (of Proposition 1). Let $n' \geq 11$ be an integer with $n' \equiv 3 \pmod 4$. Then $n = \frac{n'+1}{2}$ is an even integer with $n \geq 6$. As illustrated in Figure 10(a) and Figure 10(b), we can construct an arrangement $\mathcal{A}$ of n pseudocircles with $p_2 = 2n - 2$ touchings such that the touching graph $T(\mathcal{A})$ is the wheel graph W_n.
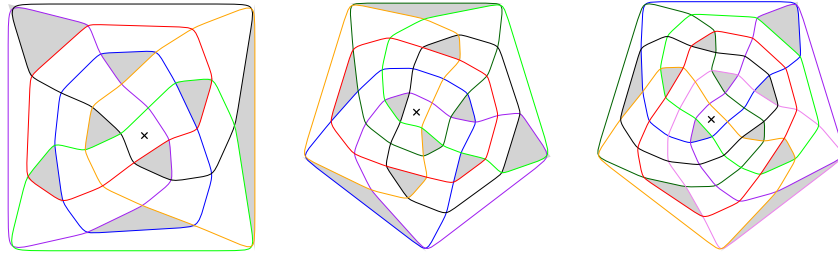
Fig. 11: Digon- and touching-free intersecting arrangements of $n = 6, 7, 8$ pseudocircles with 8, 10, 11 triangles, respectively. Each of the three arrangements is cylindrical, the common interior is marked with a cross. Triangular cells are highlighted gray. [5, Fig. 2]

In this construction the *central* pseudocircle v has a touching with each of the pseudocircles $w_1, \ldots, w_{n-1}$ and each w_i touches v, $w_{i+n/2}$, and $w_{i-n/2}$ (indices modulo $n - 1$); see Figure 10(c).

All triangles in $T(\mathcal{A})$ contain the central vertex v and for each such triangle $\{v, w_i, w_j\}$, the touchings of the pseudocircles w_i and w_j with the pseudocircle v are not consecutive on v. Therefore, applying the blossom operation to v eliminates all triangles and the resulting arrangement $\mathcal{A}'$ of $n' = 2n - 1$ pairwise intersecting pseudocircles has $p_2(\mathcal{A}') = 2n' - 2$ touchings and a triangle-free touching graph $T(\mathcal{A}')$; see Figure 10(d). This completes the argument for $n' \geq 11$ with $n \equiv 3 \pmod 4$.

To give a construction for $n'' = 14$ and for all integers $n'' \geq 17$, note that the blossom operation can be applied to pseudocircles with exactly three touchings. The constructed examples with $n \equiv 3 \pmod 4$ have pseudocircles with three touchings and the blossom operation applied to such a pseudocircle preserves the property.

Since $n'' = 14$ and every integer $n'' \geq 17$ can be written as $n' + 3k$ with $n' \in \{11, 15, 19\}$ and $k \in \mathbb{N} \cup \{0\}$ we obtain arrangements $\mathcal{A}''$ of n'' pseudocircles with $p_2(\mathcal{A}'') = 2n'' - 2$ touchings. This completes the proof of Proposition 1. $\square$

4   Proof of Theorem 2

We denote by $\mathcal{A}_6$, $\mathcal{A}_7$, and $\mathcal{A}_8$ the three arrangements shown in Figure 11. These three arrangements on 6, 7, and 8 pseudocircles, respectively, are digon- and touching-free and contain 8, 10, and 11 triangles, respectively. In each of the three arrangements, there is a pseudocircle C and four incident triangles which are alternatingly inside and outside of C in the cyclic order around C. In fact, this *alternation property* holds for each pseudocircle of these three arrangements.

To recursively construct $\mathcal{A}_n$ for $n \geq 9$, we replace a pseudocircle C with the alternation property from $\mathcal{A}_{n-3}$ by a particular arrangement of four pseudocircles as depicted in Figure 12.
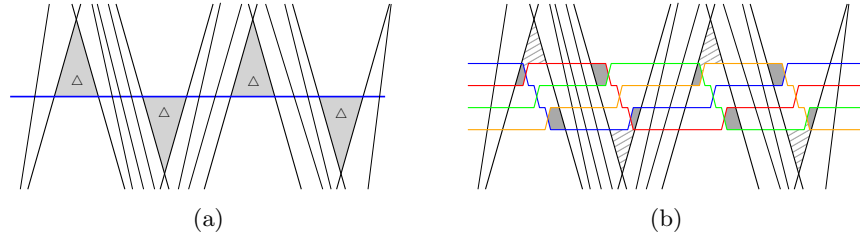
Fig. 12: Replacing one pseudocircle with the alternation property (i.e., four triangles on alternating sides) by a particular arrangement of four pseudocircles.
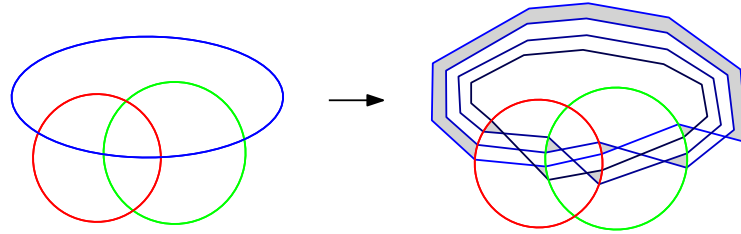

Fig. 13: Extending the Krupp arrangement (left) to the arrangement $\mathcal{A}_6$ (right).

With this replacement we destroy 4 triangles incident to C in the original arrangement, and in total the four new pseudocircles are incident to eight new triangles. Hence, we have $p_3(\mathcal{A}_n) = p_3(\mathcal{A}_{n-3}) + 4 = \lceil \frac{4}{3}(n-3) \rceil + 4 = \lceil \frac{4}{3}n \rceil$.

Moreover, the so-obtained arrangement is cylindrical as the cell marked with the cross lies inside each pseudocircle, and for each of the four new pseudocircles, there are four new triangles (among the eight new triangles) that lie on alternating sides. This allow us to recurse by using one of the four new pseudocircles in the role of C for the next iteration. This completes the proof.

It is worth noting that $\mathcal{A}_6$ can be created with the same construction as illustrated in Figure 13 by extending the Krupp arrangement of three pseudocircles, in which all cells are triangles.

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