On Primal-Dual Circle Representations

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The Koebe-Andreev-Thurston Circle Packing Theorem states that every triangulated planar graph has a circle-contact representation. The theorem has been generalized in various ways. The arguably most prominent generalization assures the existence of a primal-dual circle representation for every 3-connected planar graph. The aim of this note is to give a streamlined proof of this result.

1 Introduction

For a 3-connected plane graph $G = (V, E)$ with face set F, a *primal-dual circle representation* of G consists of two families of circles $(C_x \colon x \in V)$ and $(D_y \colon y \in F)$ such that:

(i) The vertex-circles C_x have pairwise disjoint interiors.

(ii) All face-circles D_y are contained in the circle D_o corresponding to the outer face o, and all other face-circles have pairwise disjoint interiors.

Moreover, for every edge $xx' \in E$ with dual edge yy' (i. e., y and y' are the two faces separated by xx'), the following holds:

(iii) Circles C_x and $C_{x'}$ are tangent at a point p with tangent line $t_{xx'}$.

(iv) Circles D_y and $D_{y'}$ are tangent at the same point p with tangent line $t_{yy'}$.

(v) The lines $t_{xx'}$ and $t_{yy'}$ are orthogonal.

Figure 1 shows an example.

Theorem 1. *Every 3-connected plane graph G admits a primal-dual circle representation. Moreover this representation is unique up to Möbius transformations.*

As a special case of this statement, G admits a circle packing representation: as a contact graph of nonoverlapping disks.

The proof presented here combines ideas from an unpublished manuscript of Pulleyblank and Rote, from Brightwell and Scheinerman [5] and from Mohar [23]. The motive for the write-up is that the amount of calculations needed for the proof has been reduced significantly. We decided to share this note with the community because Theorem 1 is an important result, and the proof seems to be suited for a presentation in a class on Graph Theory or Computational Geometry.

In the next section we give a rather comprehensive account of the history of the theorem and link to some of its applications. The proof of the theorem is given in Section 3.

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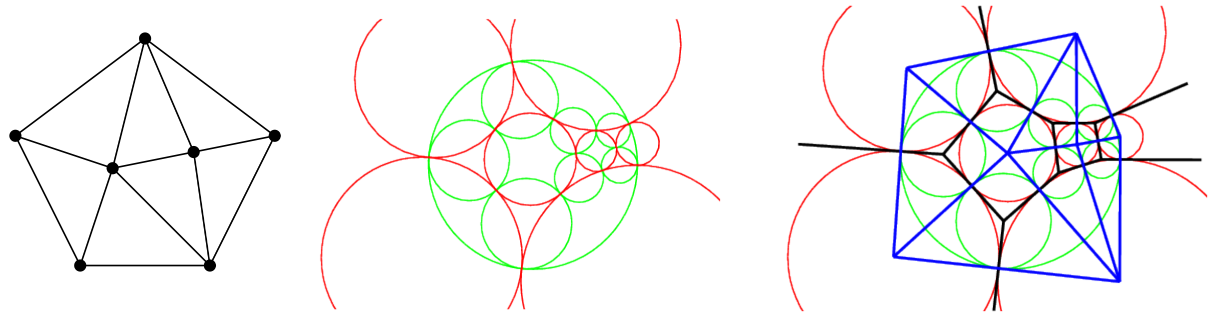

(a) (b) (c)

Figure 1: (a) a 3-connected graph G, (b) a primal-dual circle representation of G, (c) straight-line drawings of G and the dual graph G^*, yielding a tessellation by kites.

2 History and Applications of the Theorem

In graph theory the study of circle contact representations can be traced back to the 1970's and 1980's; the term "coin representation" was used there. Wegner [35] and Jackson and Ringel [17] conjectured that every plane graph has a circle representation. The problem was popularized by Ringel [25] who even included it in the textbook from 1990 [16]. In a note written in 1991 [27] Sachs mentions that he found a proof of the circle packing theorem which was based on conformal mappings. This eventually lead him to the discovery that the theorem had been proved by Koebe as early as 1936 [18].

In a the context of his study of 3-manifolds Thurston [33] proved that any triangulation of the sphere has an associated "circle packing" which is unique up to Möbius transformations. As it turned out, this result was also present in earlier work of Andreev [2]. Nowadays the result is commonly referred to as the *Koebe-Andreev-Thurston Circle Packing Theorem*. At a conference talk in 1985 Thurston emphasized on connections of circle packings and the Riemann Mapping Theorem. A precise version was obtained by Rudin and Sullivan [26]. This line lead to the study of discrete analytic functions and other aspects of discrete differential geometry, we refer to [31], [32], [4], and [20] for more on the topic.

In the early 1990's new proofs of the circle packing theorem where found. Colin de Verdière [6] gave an existential proof based on 'invariance of domain', this proof can also be found in [24] and in the primal-dual setting in [19]. Colin de Verdière [7] gave a proof based on the minimization of a convex function and extended circle packings to more general surfaces. Pulleyblank and Rote (unpublished) and Brightwell and Scheinerman [5] gave proofs of the primal-dual version (Theorem 1) based on an iterative algorithm, similar to the proof given in this note. Mohar [22] analyzed an iterative approach and showed that an ε-approximation for the radii and centers can be obtained in time polynomial in the size of the graph and $\log(1/\varepsilon)$.

Primal-dual circle representations yield *simultaneous orthogonal drawings* of G and its dual G^*, i. e., straight-line drawings of G and G^* such that the outer vertex of G^* is at infinity and each pair of dual edges is orthogonal. The existence of such drawings was conjectured by Tutte [34].

Another application of primal-dual circle representations is known as the *Cage Theorem*. It says that every 3-connected planar graph can be represented as the skeleton of a convex 3-polytope such that every edge of the polytope is tangent to a given sphere. This is a strengthening of Steinitz Theorem which comes with a very simple proof, see e. g. [20]. The Cage Theorem was generalized by Schramm [29] so that the polytope is caging any given smooth strictly convex body.

A stunning generalization of the Circle Packing Theorem is the *Monster Packing Theorem* of Schramm [28]. The statement is as follows: if each vertex v of a planar triangulation G has a prescribed convex prototype P_v, then there is a contact representation of G where each vertex is represented by a (possibly degenerate) homothet of its prototype. When the prototypes have a smooth boundary, then there are no degeneracies. Contact representations of planar graphs with other shapes than circles have received quite some attention over the years. here are some pointers to the literature: triangles [8, 14, 1]; rectangles and squares [11, 30]; k-gons [13].

The Circle Packing Theorem has been used to prove *Separator Theorems*. The approach was pioneered by Miller and Thurston and generalized to arbitrary dimensions by Miller et al. in [21]. The 2-dimensional case is reviewed in [24]. A slightly simpler proof was proposed by Har-Peled [15].

Not surprisingly the theorem also has applications in Graph Drawing. Eppstein [10] used circle representations to prove the existence of *Lombardi drawing* (a drawing in which the edges are drawn as circular arcs, meeting at equal angles at each vertex) for all subcubic planar graphs. Felsner et al. [12] used circle representations to show that 3-connected planar graphs have planar *strongly monotone drawings*, i. e., straight-line drawings such that for for any two vertices u, v there is a path which is monotone with respect to the connecting line of u and v.

Eppstein [3] and [9] relates circle packings to mesh generation techniques.

3 Primal-Dual Circle Representation: The Proof

Before diving into details we give a rough outline of the proof. A primal-dual circle representation of G induces a straight-line drawing of G and a straight-line drawing of the dual. Superimposing the two drawings yields a plane drawing whose faces are special quadrangles called kites, see Figures 1c and 3. After guessing radii for the circles, the shapes of the kites are determined. It is then checked whether the angles of kites meeting at a vertex sum up to 2π. If at some vertex the angle sum does not match 2π, the radii are changed to correct the situation. The process is designed to make the radii converge and to make the sum of angles meet the intended value at each vertex. The second part of the proof consists of showing that the kites corresponding to the final radii can be laid out to form a tessellation, thus giving the centers of a primal-dual circle representation of G.

Proof of Theorem 1. Given a primal-dual circle representation of G, we can use a stereographic projection to lift it to a primal-dual circle representation on the sphere. This spherical representation has the advantage that the circle D_o has no special role. On the sphere the face-circles can be viewed as a family with pairwise disjoint interiors. Rotating this representation and mapping it back to the plane, we can get a primal-dual circle representation of G or of the dual G^* where any prescribed element $z \in V \cup F$ has the role of the outer face. This process can be reverted. Therefore, we use the well-known fact that G or G^* has a triangular face, which follows easily from Euler's formula, and assume that the outer face o of the given plane graph is a triangle.

Given a primal-dual circle representation of G we can use the centers of the circles C_x for $x \in V$ to obtain a planar straight-line drawing of G. Similarly the centers of the circles D_z for $z \in F \setminus \{o\}$ yield a planar straight-line drawing of $G^* \setminus \{o\}$. Looking at the two drawings simultaneously and adding appropriate rays for the edges yo of G^* we see *kites*, i. e., quadrangular shapes with two opposite right angles, tessellating the polygon formed by the centers of the outer vertices of G, see Figure 1c.

The kites are in bijection with the incident pairs (x, y), where x is a primal vertex and y is a dual vertex. Since the involved circles intersect orthogonally, the kite of x and y (see Figure 3) is completely determined by the radii r_x of C_x and r_y of D_y. The angles at x and y are given by

$$\alpha_{xy} = 2 \arctan \frac{r_y}{r_x} \quad \text{and} \quad \alpha_{yx} = 2 \arctan \frac{r_x}{r_y}. \tag{1}$$

The *angle graph* of a plane graph $G = (V, E)$ is the graph $G^\diamond$ whose vertex set is $V \cup F$ and whose edges are the incident pairs xy with $x \in V$, $y \in F$, i. e., x is a vertex on the boundary of y. The graph $G^\diamond$ is plane, bipartite and every face is a 4-gon, i. e., $G^\diamond$ is a quadrangulation. Let

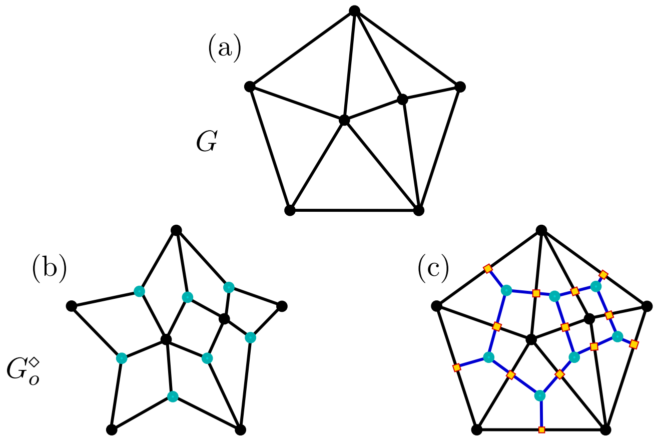

Figure 2: (a) A plane graph G. (b) Its reduced angle graph $G_o^\diamond$. (c) Its primal-dual completion (skeleton graph of kites).

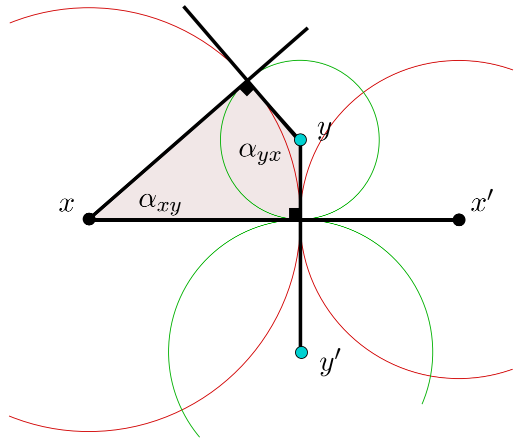

Figure 3: The kite corresponding to the incident vertex-face pair x,y.

$G_o^\diamond = (U, K)$ be the *reduced angle graph*, obtained by deleting the vertex corresponding to the outer face of G from $G^\diamond$, see Figure 2(b). (Note that the outer face of the graph G in this example is a pentagon, unlike in our setup, where we assume a triangular outer face.) The set K of edges of $G_o^\diamond$ is in bijection to the kites of a primal-dual circle representation of G. We will need the following property of $G_o^\diamond$.

Claim 1. *Every subset S of the vertices of $G_o^\diamond$ with $|S| \geq 5$ induces at most $2|S| - 5$ edges.*

Proof. Since $G_o^\diamond$ is bipartite, every subset S induces at most $2|S| - 4$ edges, with equality only if S induces a quadrangulation. Since the outer face of G is incident to 3 vertices we have $|K| = 2(|U| + 1) - 4 - 3 = 2|U| - 5$. Now let $S \subsetneq U$. Since G is 3-connected, there is no separating 4-cycle in $G^\diamond$. This implies that the outer face of the induced graph $G_o^\diamond[S]$ is not a 4-cycle, whence $G_o^\diamond[S]$ has at most $2|S| - 5$ edges. □

We specify that the triangle formed by the three outer vertices should be equilateral. This is no loss of generality, since it can be achieved for any primal-dual circle representation by applying a Möbius transformation. After this normalization, the following equations hold:

$$\sum_{w:\, uw \in K} \alpha_{uw} = \begin{cases} \pi/3 & \text{if } u \text{ is an outer vertex of } G \\ 2\pi & \text{else.} \end{cases}$$

Define the *target angles* $\beta(u)$ for $u \in U$ such that $\beta(u) = \pi/3$ if u is an outer vertex and $\beta(u) = 2\pi$ for all other vertices and all bounded faces of G.

Given an arbitrary assignment $r : U \to \mathbb{R}_+$ of radii, we can form the corresponding kites. The angle sum at $u \in U$ is then $\alpha(u) = \sum_{w:\, uw \in K} \alpha_{uw}$. We aim at finding radii such that $\alpha(u) = \beta(u)$ for all $u \in U$. Later we will show that such radii induce a primal-dual circle representation.

We first show that $\sum_u \alpha(u) = \sum_u \beta(u)$, i. e., any choice of radii attains the correct target angles *on average*. Indeed,

$$\sum_u \alpha(u) = \sum_{xy \in K} \pi = |K|\pi \qquad \text{and}$$

$$\sum_u \beta(u) = \big((|V| - 3) + (|F| - 1)\big)2\pi + 3\frac{\pi}{3} = \big(|V| + |F| - 1\big)2\pi - 5\pi = (2|U| - 5)\pi = |K|\pi.$$

As a consequence, whenever $\alpha(u) \neq \beta(u)$ for some u, the following two sets are both nonempty:

$$U_- = \{u \in U\colon \alpha(u) < \beta(u)\} \qquad \text{and} \qquad U_+ = \{u \in U\colon \alpha(u) > \beta(u)\}$$

If we increasing the radius r_u of a vertex $u \in U_+$, we observe from (1) that for every incident edge $uw \in K$, the angle α_{uw} decreases monotonically to 0 as $r_u \to \infty$. Hence, it is possible to increase r_u to the unique value where $\alpha(u) = \beta(u)$.

The core of the proof is the following infinite iteration.

$$\begin{array}{l} \textbf{repeat forever}\text{:} \\ \quad \textbf{for all } u \in U\text{:} \\ \qquad \textbf{if } u \in U_+ \textbf{ then } \text{ increase } r_u \text{ to make } \alpha(u) = \beta(u) \end{array} \tag{2}$$

We claim that the radii converge to an assignment with $\alpha(u) = \beta(u)$ for all u. The increase of r_u may cause another element $w \in U_-$ to move to U_+, but a transition from U_+ to U_- is impossible. It follows that some element u_0 must belong to U_- indefinitely unless the iteration comes to a halt with $U_- = U_+ = \emptyset$.

Since radii can only increase, $u \in D$ implies that $r_u \to \infty$. We want to show that the set $D \subseteq U_+ \subsetneq U$ of elements whose corresponding radii do not converge is empty. The subset of outer vertices of V in D is denoted by D_o. If $u \in D$ and $w \in U \setminus D$, then α_{uw} converges to 0 according to (1). Thus, for given $\varepsilon > 0$, the iteration will eventually lead to vectors of radii such that the inequality $\displaystyle\sum_{w \in U\setminus D\colon\, uw \in K} \alpha_{uw} \leq \frac{\varepsilon}{|U|}$ holds for each $u \in D$. We now consider the case $|D| \geq 5$ and use Claim 1:

$$\sum_{u\in D} \alpha(u) \leq \varepsilon + \sum_{\text{kite with } x,y\in D} (\alpha_{xy} + \alpha_{yx}) = \varepsilon + \sum_{xy \text{ edge of } G_o^\diamond[D]} \pi \leq \varepsilon + (2|D| - 5)\pi \tag{3}$$

$$\sum_{u\in D} \alpha(u) = \sum_{u \in D\cap U_+} \alpha(u) > \sum_{u\in D} \beta(u) = 2\pi|D| - \frac{5|D_o|}{3}\pi.$$

By comparing these bounds, we see that $|D_o| = 3$ and the subgraph $G_o^\diamond[D]$ of $G_o^\diamond$ induced by D has $2|D| - 5$ edges. This implies that $G_o^\diamond[D]$ is connected and that the outer face of $G_o^\diamond[D]$ includes the three outer vertices of G. Thus, by the edge count, $G_o^\diamond[D]$ is an internal quadrangulation, and the outer face of $G_o^\diamond[D]$ coincides with the hexagonal outer face of $G_o^\diamond$ because this face bounds the unique shortest cyclic walk through the 3 nonadjacent vertices of D_o. Since $G_o^\diamond$ has no separating 4-cycles, we conclude that $G_o^\diamond[D] = G_o^\diamond$. This contradicts $D \subsetneq U$ and shows that D must be empty.

If $3 \leq |D| \leq 4$ and $|D_o| = 3$, then we conclude from the above that D induces less than $2|D| - 5$ edges of $G_o^\diamond$. This yield the contradiction $\varepsilon + (2|D| - 6)\pi > 2\pi|D| - 5\pi$. if $|D_o| \leq 2$, then $\varepsilon + (2|D| - 4)\pi > 2\pi|D| - \frac{5|D_o|}{3}\pi$ is again a contradiction.

If $1 \leq |D| \leq 2$ and there is no edge, then $\varepsilon > 2|D|\pi - \frac{5|D_o|}{3}\pi$ is a contradiction because $D_o \subseteq D$. If there is an edge, then $|D_o| \leq 1$ and $\varepsilon + \pi > 4\pi - \frac{5|D_o|}{3}\pi$ is a contradiction.

We have shown that all radii are bounded, and hence they converge. It follows that the angle sums $\alpha(u)$ converge as well, and by the nature of the iteration (2), their limits $\hat\alpha(u)$ are bounded by $\hat\alpha(u) \leq \beta(u)$. Since $\sum_u \hat\alpha(u) = \sum_u \beta(u)$, we must have $\hat\alpha(u) = \beta(u)$ for all u.

Uniqueness up to scaling. Let r and r' be two vectors of radii such that $\alpha_r(u) = \alpha_{r'}(u) = \beta(u)$ for all u and $r_{u_0} = r'_{u_0}$ for some u_0. Suppose that $S = \{u : r_u > r'_u\}$ is nonempty and observe that $u_0 \in \bar S = U \setminus S$.

$$0 = \sum_{u\in S} \alpha_r(u) - \sum_{u\in S} \alpha_{r'}(u) = \sum_{u\in S} \sum_{w\in U\colon\, uw\in K} \alpha_{uw}(r) - \sum_{u\in S} \sum_{w\in U\colon\, uw\in K} \alpha_{uw}(r') \tag{4}$$

$$= \sum_{u\in S, w\in \bar S, uw\in K} \big(\alpha_{uw}(r) - \alpha_{uw}(r')\big) < 0 \tag{5}$$

The equality between (4) and (5) holds because the equation $\alpha_{uw} + \alpha_{wu} = \pi$ is independent of the radii, and hence the contributions of the edges uw with $u, w \in S$ cancel. For the last inequality, note that $\alpha_{uw}(r) < \alpha_{uw}(r')$ due to (1), because $r_u > r'_u$ and $r_w \leq r'_w$, and there is some pair $uw \in K$ with $u \in S$ and $w \in \bar{S}$. The contradiction proves uniqueness.

Laying out the kites. To finish the proof of Theorem 1, it remains to show that the kites defined by the limiting radii r can be laid out in the plane with the intended side-to-side contacts, and that the circles with radii given by r and centers as given by the laid-out kites have the properties (i)–(v), i. e., they form a primal-dual circle representation of G.

We first show that if the kites can be laid out without overlap, they yield a primal-dual circle representation. The kites induce a straight-line drawing of G and a straight-line drawing of the dual G^* with the outer vertex o at ∞ and edges yo being represented by rays. The point p where an edge xx' crosses its dual edge yy' is a right angle of kites. This implies (v).

For a vertex $u \in U$, consider the set of kites containing u. These kites can be put together in the cyclic order given by the rotation of u in $G_o^\diamond$ to form a polygon P_u. If u is not one of the three outer vertices V_o, P_u is a convex polygon surrounding u, because $\hat{\alpha}(u) = \beta(u) = 2\pi$. By the geometry of the kites, all edges incident to u have the same length r_u, and the circle C_u of radius r_u centered at u is inscribed in P_u and touches P_u at the common corners of neighboring kites. For $u \in V_o$, the polygon P_u has u as a corner, but the circle C_u still goes through the right-angle corners of the kites. From the incidences of the kites, and since the polygons P_u for $u \in V$ are pairwise disjoint, we obtain that the family $(C_x \colon x \in V)$ satisfies (i) and (iii).

The union of all kites forms a triangle T. T is an equilateral triangle, because its angles are 60°. This forces the radii of the three outer vertices to be equal, whence the touching points of the outer circles are the midpoints of the sides of T. Now, define D_o as the inscribed circle of T. Let the family of circles defined for dual vertices be $(D_y \colon y \in F)$. Properties (ii) and (iv) follow from the layout of kites.

The layout of kites is warranted by the following Lemma 2. When we apply this lemma, the graph H is the bipartite *primal-dual completion* of $G = (V, E)$. The vertices of H are $V \cup F \setminus \{o\} \cup E$, and the edges of H are the pairs $(z, e) \in (V \cup F \setminus \{o\}) \times E$ for which z is incident to the edge $e \in E$ in G. This graph is the skeleton of the laid-out kites, see Figure 2(c).

This concludes the proof of Theorem 1. □

Lemma 2. *Let H be a 3-connected plane graph. For every inner face f of H let P_f be a simple polygon whose corners are labeled with the vertices from the boundary of f in the same cyclic order. The corner of P_f labeled with v is denoted $p(f, v)$ and $\alpha_{i,v}$ denotes the angle of P_f at $p(f, v)$. If the following conditions are satisfied:*

(i) $\sum_{i=1}^{k} \alpha_{i,v} = 2\pi$ *for every inner vertex v of H with incident faces $f_1, \ldots, f_k$.*

(ii) $\sum_{i=1}^{k} \alpha_{i,v} \leq \pi$ *for every outer vertex v of H with incident faces $f_1, \ldots, f_k$.*

(iii) $\|p(f_1, v) - p(f_1, w)\| = \|p(f_2, v) - p(f_2, w)\|$ *for every inner edge vw of H with incident faces f_1, f_2.*

Then there is a crossing-free straight-line drawing of H in which the drawing of every inner face f can be obtained from P_f by a rigid motion, i. e., translation and rotation.

Proof. Let H^* be the dual graph of H without the vertex corresponding to the outer face of H. Further let S be a spanning tree of H^*. Then by (iii) we can glue the polygons P_f of all inner faces f of H together along the edges of S. This determines a unique position for every polygon, up to a global motion. We need to show that the resulting shape has no holes or overlaps. For the edges of S we already know that the polygons of the two incident faces are touching such that corners corresponding to the same vertex coincide. For the edges of the complement $\overline{S}$ of S we need to show this. Considering $\overline{S}$ as a subset of the edges of H, the set $\overline{S}$ is a forest in H. Let v be a leaf of this forest that is an inner vertex of H, and let e be the edge of $\overline{S}$ incident to v. Then for all incident edges $e' \neq e$ of v we already know that the polygons of the two incident faces of e touch in the right way. But then also the two polygons of the two incident faces of e touch in the right way because v fulfills property (i). Since the set of edges we still need to check remains always a forest, we can iterate this process until all inner edges of H are checked. After gluing all the polygons P_f, every vertex v has a unique position, and because of property (ii), all angles at the boundary of the union are convex. Let V_o be the set of outer vertices of H and let $d = |V_o|$. We claim that

$$\sum_{v \in V_o} \sum_i \alpha_{i,v} = (d-2)\pi.$$

If $\deg(f) = k$, then the sum of angles of P_f is $(k-2)\pi$. Summing this over all polygons P_f we obtain $\sum_f (\deg(f) - 2)\pi = (2|E| - 2|F|)\pi - (d-2)\pi$. Using properties (i) and (ii) We also have $\sum_f (\deg(f) - 2)\pi = (|V| - d)2\pi + \sum_{v \in V_o} \sum_i \alpha_{i,v}$. Using Euler's Formula this yields the claim.

The claim show that the sum of angles at the outer vertices is just the right value for a d-gon, whence the boundary of the union of the glued polygons P_f is a convex polygon and therefore nonintersecting. □

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