Contact representations of planar graphs with cubes

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Abstract. We prove that every planar graph has a representation using axis-parallel cubes in three dimensions in such a way that there is a cube corresponding to each vertex of the planar graph and two cubes have a non-empty intersection if and only if their corresponding vertices are adjacent. Moreover, when two cubes have a non-empty intersection, they just touch each other. This result is a strengthening of a result by Thomassen which states that every planar graph has such a representation using axis-parallel boxes.

1 Introduction

An axis-aligned d-dimensional box, or d-box in short, is defined to be the Cartesian product of d closed intervals. The boxicity of a graph G is defined as the smallest d such that G can be represented as the intersection graph of d-boxes in $\mathbb{R}^d$. A graph has boxicity at most one if and only if it is an interval graph.

Roberts [12] showed that removing a perfect matching from a complete graph on $2n$ vertices yields a graph of boxicity n. The octahedron graph is the instance with $n = 3$ from this series of examples. This shows that the boxicity of planar graphs can be as large as 3. Thomassen [16] proved that the boxicity of planar graphs is at most 3. In fact, he proved a stronger result: every planar graph is the intersection graph of a family of closed axis-aligned boxes in 3-space such that the intersection of any two boxes is contained in the boundary of both. We propose to call such a representation a *cuboidal layout*.

The main contribution of this paper is to show the following strengthening of Thomassen's theorem:

Theorem 1 *Every planar graph has a cube layout, i.e., a cuboidal layout where all cuboids are cubes.*

The proof of the theorem is based on the representation of 4-connected triangulations as contact graphs of homothetic triangles. As of today, the existence of such a contact representation only has a nonconstructive proof. We review the result and related matters in the next section.

In Section 4, the proof of Theorem 1 is presented.

In [16], Thomassen first characterizes those planar graphs that can be obtained as intersection graphs of closed axis-aligned rectangles in the plane such that two rectangles that intersect have a segment on the boundary in common. Following [4], we call such a representation a *rectangular*

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layout. Thomassen's result is: A planar graph G has a rectangular layout if and only if G is a proper subgraph of a 4-connected triangulation.

If we allow the rectangles of a rectangular layout to intersect by overlapping each other in arbitrary fashion, then the class of graphs that have such a representation is exactly the class of graphs with boxicity at most 2. It is known that outerplanar graphs have boxicity at most 2 [13]. But the same cannot be said about series-parallel graphs as there are series-parallel graphs that need boxicity 3 [3]. The boxicity of a special class of minimally 3-connected planar graphs was shown to be 2 in [5]. Bellantoni et al. show that every bipartite graph that has boxicity at most 2 is a "grid intersection graph", or the intersection graph of vertical and horizontal line segments in the plane [2]. Note that grid intersection graphs are a subclass of the class of graphs with boxicity at most 2. It is shown in [10] that every planar bipartite graph is a grid intersection graph.

The representation of graph as the intersection of axis-aligned cubes of equal size have also been studied. A d-cube, or axis-aligned d-dimensional cube, is defined to be the Cartesian product of d unit length intervals. The cubicity of a graph is defined to be the minimum d for which the graph has a representation as the intersection of d-cubes. Obviously, the cubicity of a graph is at least its boxicity. Chandran et al. [6] show that the ratio of the cubicity of a graph on n vertices to its boxicity is at most $\lceil \log_2 n \rceil$. It has to be noted that even though we show a contact representation for planar graphs using cubes in 3-space, this does not imply that planar graphs have cubicity at most 3, since the representation we construct uses cubes of varying sizes. In fact, the cubicity of the star graph $K_{1,n}$ itself is $\lceil \log_2 n \rceil$ [12], hence, there is no constant bound on the cubicity of planar graphs.

It seems interesting to consider the minimum dimension in which a given graph G has a representation as the intersection of axis-aligned cubes, not necessarily of the same size. Note that this parameter, which we shall denote by $\mathrm{vcub}(G)$, will always be at least the boxicity and at most the cubicity of the graph. The result of this paper implies that $\mathrm{vcub}(G) \leq 3$ if G is planar so that vcub and boxicity are bounded by the same constant on planar graphs.

A class of graphs where vcub behaves more like cubicity is given by balanced complete bipartite graphs. Roberts [12] observed that $K_{n,n}$ has cubicity equal to $2\lceil \log_2 n \rceil$ and boxicity equal to 2. We claim that $\mathrm{vcub}(K_{n,n})$ is equal to $\lceil \log_2 n \rceil + 1$. The upper bound on $\mathrm{vcub}(K_{n,n})$ can be seen as follows. Let A and B be the two parts of the bipartite graph $K_{n,n}$. Number the vertices of part B from 0 to $n-1$. For $v \in B$, let $b(v)$ denote number of v written in binary using $k = \lceil \log_2 n \rceil$ bits and let $b_i(v)$ denote the ith bit in $b(v)$. Define the interval graph I_i, for $1 \leq i \leq k$, by assigning the vertices intervals as follows: For every vertex $v \in A$, assign the interval $[0, 1]$. For a vertex $v \in B$, assign the interval $[-2n, 0]$ if $b_i(v) = 0$ and $[1, 2n+1]$ if $b_i(v) = 1$. The interval graph I_{k+1} is defined by assigning the interval $[0, 2n]$ to each vertex from B and by assigning disjoint unit length intervals in $[0, 2n]$ to the vertices in A. The intervals of the interval graph I_i can be seen to be the projections of the $k+1$-dimensional cubes assigned to the vertices on the ith coordinate axis. We thus have a representation of $K_{n,n}$ using cubes in $\lceil \log_2 n \rceil + 1$ dimensions. This bound on $\mathrm{vcub}(K_{n,n})$ can be shown to be tight as follows. Suppose there is a representation of $K_{n,n}$ using cubes in $k = \lceil \log_2 n \rceil$ dimensions. Consider the smallest cube, say Q, in the representation. Let us assume without loss of generality that the vertex corresponding to Q belongs to A. Let $\mathcal{B}$ be the set of cubes corresponding to the vertices in B. Every cube in $\mathcal{B}$ intersects Q and since they are bigger, each of them must contain at least one corner of Q. The cube Q has 2^k corners. It can be seen that if there are more than 2^{k-1} cubes in $\mathcal{B}$, then any other cube that intersects all the cubes in $\mathcal{B}$ will intersect the cube Q also. This cannot be the case if this is a valid representation as the cubes corresponding to other vertices in A do not intersect Q but intersect all the cubes in $\mathcal{B}$. Therefore, $|\mathcal{B}| \leq 2^{k-1}$ which implies that $n \leq 2^{\lceil \log_2 n \rceil - 1}$, which is a contradiction.

2 Contact representations by rectangles and squares

Suppose a graph G has a rectangular layout as described by Thomassen. Raising the rectangles of a rectangular layout representing G into a third independent dimension yields a cuboidal layout of G. This already yields Thomassen's theorem for a large class of planar graphs. To extend this construction to the general case it has to be shown that separating triangles can be handled. The basic idea is to represent the inner and the outer part of a separating triangle on different layers in the third dimension and enlarge the boxes of the separating triangle such that they can contribute to the representations on both layers. Details are given in [16].

A similar approach for cube layouts would start with a square layout, i.e., with a rectangular layout where all the rectangles are squares. Schramm [15] proves a representation theorem for square layouts. In his model, it is possible that four squares touch in a single point, in this case the representation does not specify which of the diagonals correspond to G, see Figure 1. We call such a layout a *weak square layout*.

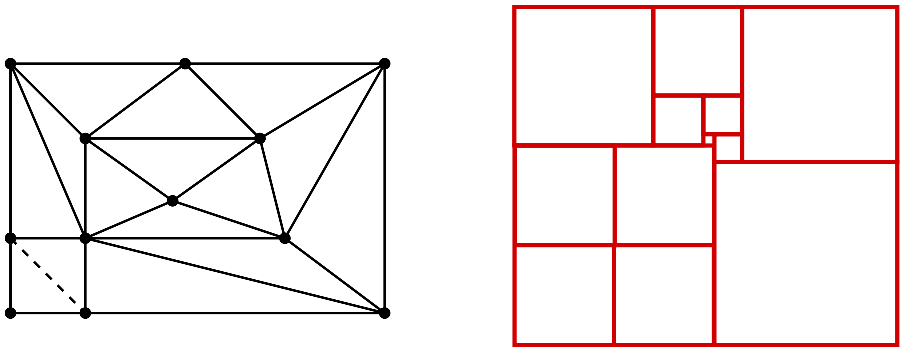

Figure 1: A planar graph with a weak square layout.

Theorem 2 *If G is an induced subgraph of T' and T' is obtained from a triangulation without separating cycle of length at most 4 by removing a vertex, then there is a weak square layout representing G.*

Schramm's beautiful proof is based on the notion of an extremal metric. Lovasz [11] explains the construction via a blocking pair of polyhedra. Planar graphs with separating cycle of length at most 4 also admit extremal metrics. However, in this case it may happen that the squares representing some of the vertices degenerate to single points. This would happen when in the graph shown in Figure 1, the dashed edge is removed and a more complex subgraph is engrafted into the resulting 4-face.

Schramm's theorem yields cube layouts for a large class of planar graphs. Unlike in the case of cuboidal layouts, we do not see a way of taking the 2-dimensional layout as a basis for the general representation theorem.

3 Contact representations by triangles

In a triangle contact representation of a graph, the vertices are represented by a set of interiorly disjoint triangles such that two triangles touch exactly if there is an edge between the corresponding vertices.

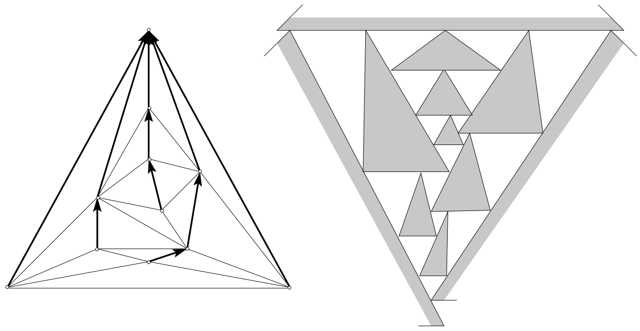

Figure 2: A planar triangulation with a triangle contact representation. The arrows indicate where the top corners of triangles touch.

De Fraysseix et al. [7] proved that every planar triangulation admits a triangle contact representation. Actually, they showed that the triangles can be chosen to be isosceles over a horizontal basis. During the Graph Drawing workshop in Bertinoro in 2007 [1], a stronger result was conjectured: Every 4-connected planar triangulation has a triangle contact representation with homothetic triangles.

As in the case of Schramm's square layouts, the connectivity condition comes from examples where some of the triangles degenerate to points. Such an example is obtained from the octahedron graph by placing two vertices of degree 3 in opposite faces.

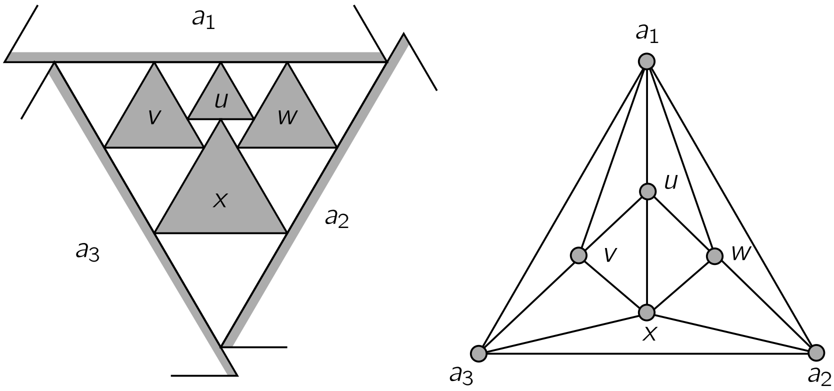

Figure 3: A homothetic triangle contact representation of a planar graph.

Gonçalves, Lévêque and Pinlou [9] observe that the conjecture follows from Schramm's Convex Packing Theorem. This extremly strong and surprising result is apparently not well known, therefore we include the statement:

Theorem 3 (Convex Packing Theorem) *Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v containing more than one point. Then there is a contact representation of a supergraph (on the same vertex set but possibly with more edges) of T where each interior vertex is represented with a homothetic copy of the convex set prescribed for it and each outer vertex by one of the three arcs of C.*

As in the cases discussed so far, the homothetic copies of some Q_v may degenerate to a point. Gonçalves et al. prove that if T is 4-connected, we can prevent this from happening by assigning homothetic triangles to each interior vertex of T and by choosing C to be a triangle in such a way

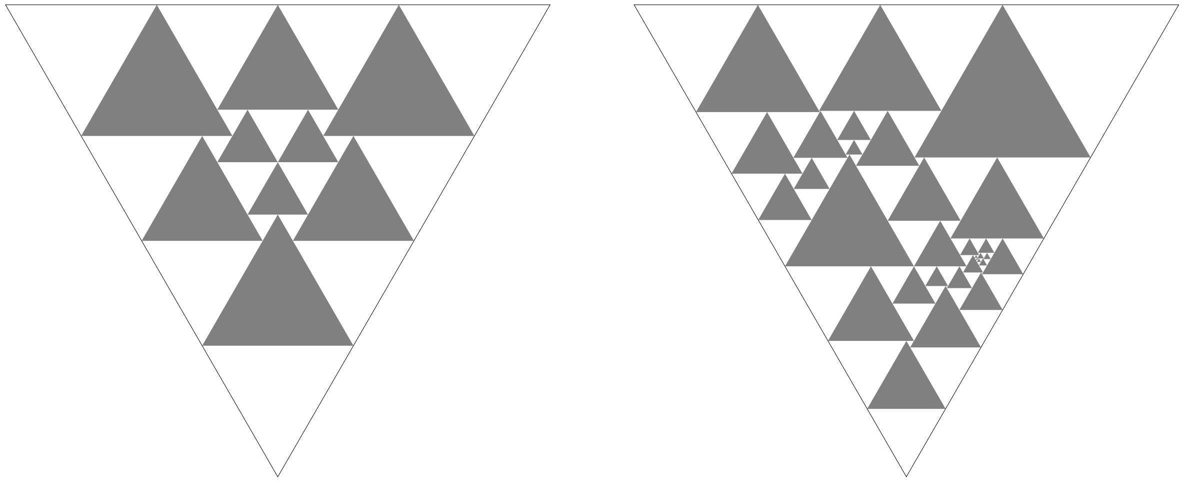

Figure 4: A homothetic triangle contact representation of the icosahedron graph and of a larger triangulation. The triangles corresponding to the three vertices on the outer face are not shown.

that each of its sides can be one side of the triangle assigned to one of the three outer vertices. The slopes of the three segments of C are chosen so that the triangles that are so assigned to the outer vertices are homothetic to those given to the interior vertices. They also show that in that case, we obtain a contact representation for T with homothetic triangles (i.e., with no additional edges).

With an appropriate affine transformation, the triangles of the contact representation with homothetic triangles can be made equilateral. Thus the result of Gonçalves et al. can be stated as:

Theorem 4 *Every 4-connected planar triangulation has a triangle contact representation with aligned equilateral triangles.*

In [14], Schramm deduces Theorem 3 as a corollary of his Monster Packing Theorem. It seems that Schramm's proof can be turned into an iterative procedure that allows to produce arbitrarily good approximations of the packings whose existence follows from the theorem. However, to apply such a procedure on concrete instances may be completely impractical.

Surprisingly, there is another approach to triangle contact representation with equilateral triangles that works well in practice. We include a brief sketch, some more details are given in [8]: The idea is to use a Schnyder wood of a triangulation T to prescribe how triangles in a triangle contact representation of T are supposed to touch. From this geometric information, a system of linear equations whose variables are the sidelengths of the triangles is extracted. If the system has a positive solution, this yields the intended triangle contact representation with equilateral triangles of T. If the solution of the system contains negative variables, these can be used as sign-posts indicating how to change the Schnyder wood for another try. The procedure has been implemented, it produces nice pictures as e.g. Figure 4, and it has never failed but so far there is no proof of Theorem 4 based on this approach.

4 Proof of Theorem 1

If $G = (V, E)$ has a contact representation with sets $(C_v)_{v \in V}$ and $H = (W, E[W])$ is an induced subgraph of G, then $(C_v)_{v \in W}$ is a contact representation of H. Since every planar graph is an induced subgraph of a planar triangulation, it suffices to prove Theorem 1 for triangulations.

From now on, let $G = (V, E)$ be a planar triangulation. We first assume that G is 4-connected and demonstrate the basic idea with this case. Later, we extend the argument to the general case.

Let $(T_v)_{v\in V}$ be a triangle contact representation of G with homothets of the triangle with corners $(0,0),(1,0)$ and $(0,1)$. We may assume that the three outer vertices a_1, a_2, a_3 are represented as shown in Figure 5 and all the other triangles $(T_v)_{v\in V\setminus\{a_1,a_2,a_3\}}$ are contained in the region R with corners $(0,1),(1,1),(1,0)$. Let x_v, y_v, t_v be such that triangle T_v has corners $(x_v, y_v), (x_v + t_v, y_v), (x_v, y_v + t_v)$. The cube Q_v representing the vertex v is the product of intervals I_v^x, I_v^y, I_v^z of length t_v on the three coordinate axes, where $I_v^x = [x_v, x_v + t_v]$, $I_v^y = [y_v, y_v + t_v]$, $I_v^z = [2 - x_v - y_v - t_v, 2 - x_v - y_v]$. Note that since for $v \in V \setminus \{a_1, a_2, a_3\}$, T_v is contained in R, the corresponding cube Q_v is contained in the unit cube in $\mathbb{R}^3$. Also, for $v \in V$, the intersection of Q_v with the plane $x + y + z = 2$ projects along the z-axis to T_v. From the following two claims, it follows that the family $(Q_v)_{v\in V}$ forms a cube layout of G.

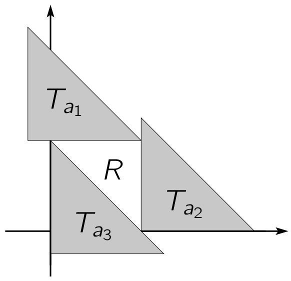

Figure 5: The triangles representing the outer vertices and the region R.

Claim A. If $v, w \in E$ then Q_v and Q_w are interiorly disjoint but intersect on the boundary.

Proof. From the assumptions about triangle contact representation, it follows that the triangles T_v and T_w are non-trivial, i.e., $t_v, t_w > 0$, and they have a contact point. We may assume that a corner c of T_v belongs to a boundary segment B of T_w. There are several cases.

Let B be the vertical segment of T_w. In this case, $x_v + t_v = x_w$ and therefore, all points in Q_v have x-coordinate at most x_w and all points in Q_w have x-coordinate at least x_w. It remains to be shown that the two cubes intersect in the plane with equation $x = x_w$. For the following considerations, we restrict attention to this plane and drop the x-coordinate. The intersection of Q_w with the plane consists of all points $p = (y_p, z_p)$ with $y_w \le y_p \le y_w + t_w$ and $2 - x_w - y_w - t_w \le z_p \le 2 - x_w - y_w$. The point $(y_v, 2 - x_v - y_v - t_v) = (y_v, 2 - x_w - y_v)$ is a corner of Q_v and satisfies these conditions, i.e., it belongs to Q_w.

The case where B is the horizontal segment of T_w is symmetric to the previous case. Just exchange the roles of x and y coordinates.

Now let B be the diagonal segment of T_w. The supporting line of B has the equation $x + y = x_w + y_w + t_w$. Since $c = (x_v, y_v)$ is on this line, we have $x_w + y_w + t_w = x_v + y_v$. All points of Q_v have z-coordinate at most $2 - x_v - y_v$ and all points of Q_w have z-coordinate at least $2 - x_w - y_w - t_w$. It remains to be shown that the two cubes intersect in the plane with equation $z = 2 - x_v - y_v$. For the following considerations, we restrict attention to this plane and drop the z-coordinate. The intersection of Q_w with the plane consists of all points $p = (x_p, y_p)$ with $x_w \le x_p \le x_w + t_w$ and $y_w \le y_p \le y_w + t_w$. The point (x_v, y_v) is a corner of Q_v and satisfies the conditions, i.e., it belongs to Q_w. $\triangle$

Remark. A slightly more detailed analysis shows that if the contact of T_v and T_w is a corner of both, then the intersection of Q_v and Q_w is part of an edge of both. Otherwise, if the contact of T_v and T_w is generic, then the intersection of Q_v and Q_w is a 2-dimensional rectangle, i.e., a 2-box. Thomassen's construction of cuboidal layouts of planar graphs has the property that any two cuboids that have a contact already intersect in a 2-box. In this sense, our cube layouts are not quite as strong as Thomassen's cuboidal layouts.

Claim B. If $v, w \notin E$ then Q_v and Q_w are disjoint.

Proof. Since $(T_v)_{v \in V}$ is a triangle contact representation of G, it follows that the triangles T_v and T_w are disjoint. A pair of disjoint homothetic triangles can be separated by a line which is parallel to an edge of the triangles. Let S be such a separating line. If S is vertical (respectively horizontal), the plane perpendicular to the x- (respectively y-) axis that contains S separates Q_v and Q_w. If S is diagonal with equation $x + y = r$, then the plane with equation $z = 2 - r$ is separating Q_v and Q_w. △

From the preceding discussion, it is clear that if $G = (V, E)$ is any graph that has a triangle contact representation with homothetic triangles, then G has a cube layout in which the cube corresponding to each interior vertex is contained in the unit cube in $\mathbb{R}^3$.

The general case. Let $G = (V, E)$ be an arbitrary planar triangulation. Using the Convex Packing Theorem, we obtain a packing with triangles homothetic to the triangle with corners $(0, 0), (1, 0)$ and $(0, 1)$ within the triangular region R with corners $(1, 1), (1, 0)$ and $(0, 1)$. The three boundary segments of R are assigned to the outer vertices, the horizontal segment belongs to a_1, the vertical to a_2 and the diagonal to a_3. Adding the outer triangles as in Figure 5, we obtain a triangle contact representation $(T_v)_{v \in V}$ with homothetic triangles. Some vertices of G, however, may be represented by degenerate triangles of size 0. Let V_0 be the set of all vertices whose triangle is of size 0 and let $V_1 = V \setminus V_0$. Note that $\mathcal{T} = (T_v)_{v \in V_1}$ is a proper triangle contact representation of the subgraph $G[V_1] = (V_1, E[V_1])$ of G induced by V_1. Consider a maximal subset W of V_0 such that the induced subgraph $G[W] = (W, E[W])$ of G is connected.

Edges are respected by contacts in the packing, therefore, all the degenerate triangles T_w corresponding to vertices $w \in W$ have to be the same point P_W.

Consider the set S consisting of all vertices s from V_1 whose triangle T_s contains P_W. One of the outer vertices a_i is not in S, therfore, S is a separating set. Since planar triangulations are 3-connected $|S| \geq 3$.

The triangles T_s of vertices in S are non-degenerate, interiorly disjoint, homothetic and share a common point. It follows that $|S| \leq 3$, hence, $|S| = 3$.

There is no vertex $v \notin W \cup S$ with $P_W \in T_v$, otherwise, S would pairwise separate the three vertices $w \in W$, v and $a_i \notin S$ and as G is 3-connected, there would be a $K_{3,3}$ minor in G. This is in contradiction to planarity.

Now, consider the cube layout $(Q_v)_{v \in V_1}$ of $G[V_1]$ that can be obtained from its triangle contact representation $\mathcal{T}$ as described previously. We focus our attention on the three cubes Q_s of the vertices in S. If P_W is the point (x_W, y_W), then all three cubes have $q = (x_W, y_W, 2 - x_W - y_W)$ as a corner and pairwise share a 1-box along edges of the cubes, Figure 6 illustrates the situation. The figure also shows another cube Q_W with a corner in q and sidelength small enough such that the intersection of Q_W with each of the cubes Q_s for $s \in S$ is a facet of Q_W.

We will use this cube Q_W together with the three cubes Q_s for $s \in S$ to accommodate a cube layout of the subgraph of G induced by $W \cup S$. To make the construction precise, we first label the elements of S as s_1, s_2, s_3 as in Figure 6. The sections of the cubes Q_{s_1}, Q_{s_2} and Q_{s_3} with the plane $x + y + z = 2$ are shown as black triangles. The figure also shows the section of Q_W

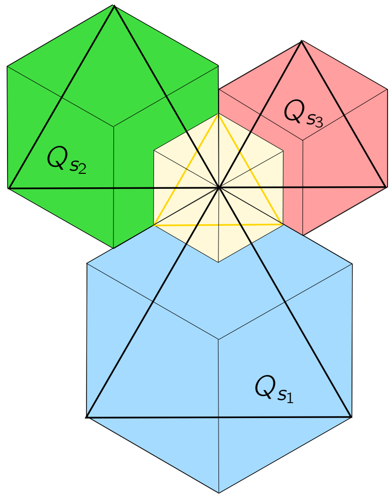

Figure 6: The three cubes Q_s for $s \in S$ and the cube Q_W between them.

with the plane $x + y + z = 2 + \ell$ where ℓ is the sidelength of Q_W as a light triangle. Recall that $q = (x_W, y_W, 2 - x_W - y_W)$ and note that the facet $y = y_W$ of Q_W is contained in Q_{s_1}, the facet $x = x_W$ of Q_W is contained in Q_{s_2}, and the facet $z = 2 - x_W - y_W$ of Q_W is contained in Q_{s_3}.

By induction on the number of vertices, we may assume that there is a cube layout of $G[W \cup S]$. In fact, we can assume the following additional properties: All the cubes Q_w for $w \in W$ are contained in the unit cube Q. The cubes representing the outer vertices s_i intersects the unit cube Q only on facets. The facet $y = 1$ of Q is contained in Q_{s_1}, the facet $x = 1$ of Q is contained in Q_{s_2}, and the facet $z = 1$ of Q is contained in Q_{s_3}. Recall that for 4-connected triangulations, we have already constructed such a cube layout and the cubes placed for the cube layout of $G[V_1]$ also obey these properties.

Consider the unit cube Q together with all the cubes from the cube layout of $G[W \cup S]$ that are contained in Q. We now describe the transformations that have to be applied to get these cubes into Q_W such that together with the cubes corresponding to V_1 they yield a cube layout of $G[V_1 \cup W]$: First, apply a point reflection at the origin. This yields the required change of role of the upper and lower facets of the bounding cube Q in each direction. Now it only remains to translate and scale Q in order to make it identical with Q_W.

Let $W_1, W_2, \ldots, W_k$ be the connected components of $V \setminus V_1$ and let P_{W_i} denote the point that is the degenerate triangle corresponding to each vertex in W_i. The set S_i is defined for W_i in exactly the same way as the set S was defined for W, i.e., S_i is the set of vertices of V_1 whose triangles contain the point P_{W_i}. As before, S_i is a separating set with $|S_i| = 3$. Like we did for W, we can find the cubes Q_{W_i} corresponding to each point P_{W_i} in the cube layout of $G[V_1]$. We can also find the cube layouts for each $G[W_i \cup S_i]$ and place them in to Q_{W_i} as described above so that we get a cube layout for G. □

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