Parameters of Bar k-Visibility Graphs

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January 23, 2007

Abstract

Bar k-visibility graphs are graphs admitting a representation in which the vertices correspond to horizontal line segments, called bars, and the edges correspond to vertical lines of sight which can traverse up to k bars. These graphs were introduced by Dean et al. [3] who conjectured that bar 1-visibility graphs have thickness at most 2. We construct a bar 1-visibility graph having thickness 3, disproving their conjecture. Furthermore, we define semi bar k-visibility graphs, a subclass of bar k-visibility graphs, and show tight results for a number of graph parameters including chromatic number, maximum number of edges and connectivity. Then we present an algorithm partitioning the edges of a semi bar 1-visibility graph into two plane graphs, showing that for this subclass the (geometric) thickness is indeed bounded by 2.

1 Introduction

Visibility is a major topic in discrete geometry where a wide range of classes of visibility graphs has been studied, see e.g. [1], [2], [5], [10], [11]. Among the best studied variants are the traditional bar visibility graphs, they admit a complete characterization which has been obtained independently by Wismath [16] and Tamassia and Tollis [15]. On the Graph Drawing Symposium 2005, Dean, Evans, Gethner, Laison, Safari and Trotter [3], [4] introduced the class of bar k-visibility graphs (BkVs) which 'interpolates' between two classes of graphs with a representation by a family of intervals, namely between bar visibility graphs and interval graphs. Dean et al. are mainly interested in measurements of closeness to planarity of bar k-visibility graphs. They prove a bound of 4 for the thickness of bar 1-visibility graphs and conjecture that they actually have

*Research of M. Massow supported by the Research Training Group "Methods for Discrete Structures" (DFG-GRK 1408) and the Berlin Mathematical School.

thickness at most 2. In Section 2 we disprove this conjecture by showing that there are bar 1-visibility graphs with thickness 3.

The second main focus of this paper is a subclass of bar k-visibility graphs called semi bar k-visibility graphs (SBkVs). These graphs emerge when considering bars which extend in only one direction (called *semi bars*). In Section 3, we show a number of properties for this subclass, including tight results on their chromatic number, clique number, maximum number of edges and connectivity. We also show how to reconstruct a bar representation of a given SBkV with maximum number of edges.

In Section 4 we turn back to thickness and show that semi bar 1-visibility graphs have thickness at most 2, thus, this subclass of B1Vs fulfills the conjecture of Dean et al.. The proof is based on an algorithm that partitions the edges of a given SB1V G into two planar graphs. We also construct a straight-line embedding of G for which the algorithm can be used, thus showing that even the geometric thickness of SB1Vs is bounded by 2.

1.1 Bar k-Visibility Graphs

Let a collection of pairwise disjoint horizontal line segments (called *bars*) in the Euclidean plane be given. Construct a graph based on these bars as follows: Take a set of vertices representing the bars. Two vertices are joint by an edge iff there is a line of sight between the two corresponding bars (we then say that the bars *see each other*). A *line of sight* is a vertical line segment connecting two bars and intersecting at most k other bars. A graph is a *bar k-visibility graph* (BkV) if it admits such a bar representation.

We call the lines of sight that do not intersect any bar (other than the two it connects) *direct*, all others are *indirect* lines of sight. We also use these adjectives for the corresponding edges.

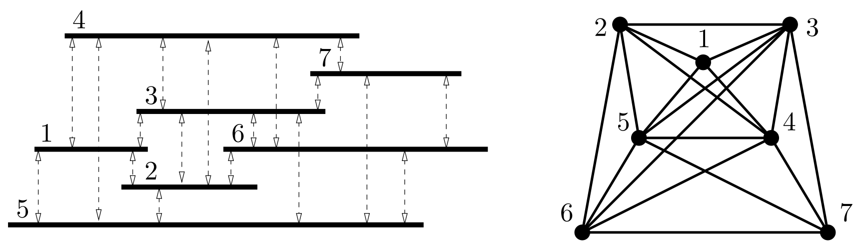

Figure 1: Example of a bar 1-visibility graph.

Note that ordinary bar visibility graphs can be regarded as bar 0-visibility graphs, and interval graphs as bar ∞-visibility graphs. Given a bar representation, we can consider the induced BkV for any k. We will most often deal with the case $k = 1$. Figure 1 shows an example of a B1V where lines of sight are indicated by dashed lines.

Throughout this paper, we assume that all bars are located at different heights. This can easily be obtained by slightly altering the y-coordinates of some bars. We also assume all endpoints of bars to have pairwise different x-coordinates by slightly permutating the x-coordinates of the endpoints in a given bar representation. (This might result in additional edges, but since we consider problems that only get harder when the number of edges increases, our results extend to general BkVs.) Note that with this assumption, lines of sight can be thought of as pillars of positive width.

1.2 Semi Bar k-Visibility Graphs

A *semi bar k-visibility graph* (SBkV) is a bar k-visibility graph admitting a representation in which all left endpoints of bars are at $x = 0$. Note that for $k = 0$, these graphs have been investigated in [2] where they are identified as the graphs graphs of *representation index* $1 + 1/2$.

For the class of SBkVs the assumption that all endpoints of bars have different x-coordinates only refers to the right endpoints. For convenience, we always think of semi bar representations rotated counterclockwise as shown in Figure 2.

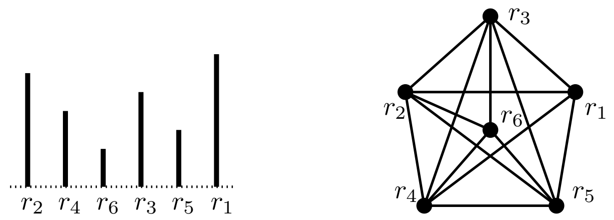

Figure 2: Example of a semi bar 1-visibility graph.

Label the bars of a semi bar representation $r_1, r_2, \ldots, r_n$ by decreasing y-coordinate of the upper endpoint, i.e., by decreasing height. Reading these labels from left to right we obtain a permutation of $[n]$ which completely determines the graph. The SB1V in Figure 2 is encoded by the permutation $(2, 4, 6, 3, 5, 1)$.

This simple way of encoding SBkVs is one reason why it is interesting to further explore this subclass: It yields a strong combinatorial structure which enables us to find tight results for many graph parameters, as we will see in Section 3.

1.3 Thickness

Thickness is a parameter that measures how far a graph is from being planar: The (graph-theoretic) *thickness* of a graph G, denoted by $\theta(G)$, is defined as the minimum number of planar subgraphs whose union is G.

Determining the thickness of a graph is NP-hard (see [12]). Exact values are only known for very few classes of graphs. For a survey on theoretical and practical aspects see [14].

Note that in the definition of thickness, the planar embeddings of the subgraphs do not have to coincide. The *geometric thickness* of G asks for the minimum number of subgraphs/colors in the following setting: Choose a straight-line embedding of G and a coloring of the edges such that crossing edges have different colors, the edges of each color then form a plane graph. Geometric thickness was introduced by Dillencourt, Eppstein and Hirschberg in [6]. In [7], Eppstein showed that graph-theoretic thickness and geometric thickness are not even asymptotically equivalent.

2 A Bar 1-Visibility Graph with Thickness 3

In [3], Dean et al. used the Four Color Theorem to show that the thickness of B1Vs is bounded by 4. They conjectured that the correct bound is 2. In this section we construct a B1V with thickness 3. We will often talk about a 2-*coloring* of a graph $G = (V, E)$, meaning a 2-coloring of the edges such that each color class is the edge set of a planar graph on V. Given a 2-coloring (with blue and red) we define G_{blue} and G_{red} as the graphs on V with all blue and all red edges, respectively.

Here is a brief outline of the construction: First we analyze a quite simple type of graph which has thickness 2 but with the property that every 2-coloring has uniform substructures, so called lampions. Assuming that the original graph is large enough we can assume arbitrarily large lampions. In a second step we introduce a series of slight perturbations into the original graph. It is shown that most of these perturbations have to be incorporated into lampions and the number of perturbations in one lampion is proportional to its size. However a lampion can only absorb a constant number of the perturbations. This yields a contradiction to the assumption that a 2-coloring exists.

We start with an Autobahn where we have heaped up the median strip: Consider the bar representation of the graph A_n shown in Figure 3. This graph has four outer vertices A, B, C, D and a set V_{inner} of n inner vertices. Each inner vertex v_i is adjacent to all outer vertices, and additionally to the inner vertices $v_{i-2}, v_{i-1}, v_{i+1}$ and v_{i+2}.

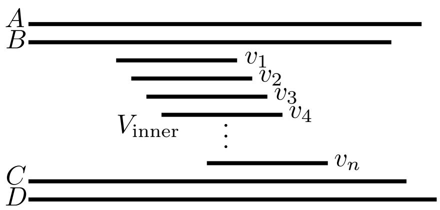

Figure 3: The Autobahn-graph A_n

Since A_n contains a $K_{4,n}$ as subgraph we know that A_n is non-planar (assuming $n \geq 3$), hence, $\theta(A_n) \geq 2$. To show that $\theta(A_n) = 2$ we let G_{blue} consist of all direct edges and G_{red} consist of all indirect edges. Figure 4 shows the partition.

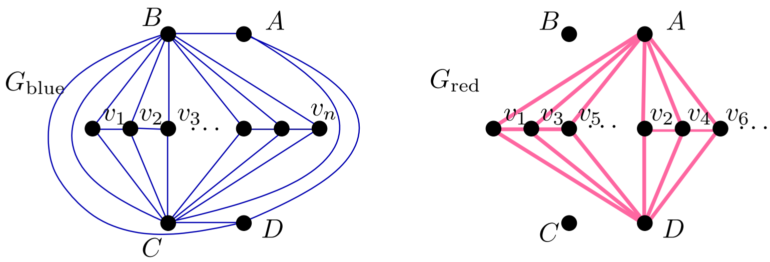

Figure 4: A partition of A_n into two planar graphs.

Let $V_{\text{inner}} = \{v_1, v_2, \ldots, v_n\}$, such that the indices represent the order of the right endpoints of the bars from left to right. The inner neighbors of v_i are $v_{i-2}, v_{i-1}, v_{i+1}$ and v_{i+2}. The graph $G[V_{\text{inner}}]$ induced by the inner vertices is maximal outerplanar with an interior zig-zag.

A *lampion* in a 2-coloring of A_n consists of a set $W = \{v_i, v_{i+1}, \ldots v_j\}$ of consecutive inner vertices and a partition $\{S_1, S_2\}$, $\{S_3, S_4\}$ of the four outer vertices such that G_{blue} consists of all zig-zag edges of $G[W]$ and all edges connecting vertices from W with S_1 and S_2, while G_{red} consist of the two outer paths of $G[W]$ and all edges connecting vertices from W with S_3 and S_4 (of course exchanging red and blue again yields a lampion). The set W is the *core of the lampion*. See Figure 5 for a lampion coloring of the core. Note that Figure 4 shows a lampion with core V_{inner} together with the additional edges between the four outer vertices.

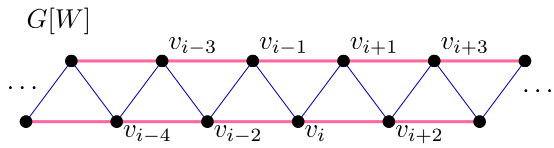

Figure 5: A lampion coloring of $G[W]$.

Lemma 1. *For every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that in every 2-coloring of A_n there is a $W \subset V_{\text{inner}}$ with $|W| \geq k$ such that W is the core of a lampion.*

Proof. Each inner vertex has four outer neighbors. Let us call an edge connecting an inner and an outer vertex a *transversal edge*. Consider the blue transversal edges; at each vertex there can be $0, 1, 2, 3$ or 4 of them. But G_{blue} is planar and therefore does not contain a $K_{3,3}$. Thus, at most two inner vertices can have the same three outer neighbors in G_{blue}. There are $\binom{4}{3} = 4$ different triples of outer neighbors in G_{blue}, so there can be at most eight inner vertices with more than two outer neighbors in G_{blue}. We might find another eight in

G_{red}. These irregular vertices break the sequence $v_1, v_2, v_3, \ldots, v_n$ of inner vertices of A_n into at most 17 pieces. By pigeon-holing, there must be at least one piece with size $n' \geq (n-16)/17$ such that in the induced 2-coloring of $A_{n'}$ all inner vertices are incident to exactly two blue and two red transversal edges.

Considering only the blue transversal edges of $A_{n'}$, the resulting subgraph G'_{blue} is a subgraph of a blown-up K_4 as illustrated in Figure 6. This subgraph is not arbitrary but has the property that every inner vertex has exactly two incident edges.

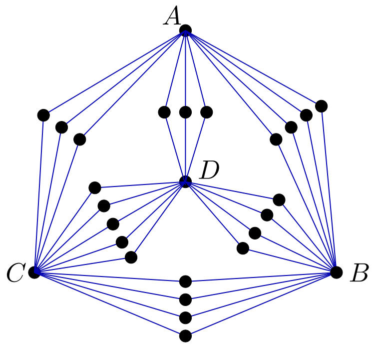

Figure 6: Blown-up K_4.

Now it remains to planarly embed the *inner edges* of $A_{n'}$, i.e. those of $G[V'_{\text{inner}}]$. To our disposition we have the ≤ 4 'large faces' of the blown-up K_4, which makes eight faces in total for the two planar graphs. In each of these faces we can embed at most three inner edges. There are other cases with fewer large faces which have in turn more inner vertices at the boundary. In all cases it is impossible to embed more than 12 edges between inner vertices with different outer neighbors in G'_{blue}, the red subgraph may contain another 12 irregular edges. These at most 24 irregularities break the sequence of inner vertices of $A_{n'}$ into at most 25 pieces, we remain with a 2-coloring of $A_{n''}$ with $n'' \geq (n'-24)/25$ such that all inner vertices are incident to the same two outer vertices in G'_{blue} and to the other two outer vertices in G'_{red}.

We now have $K_{2,n''}$ as a subgraph in both G''_{blue} and in G''_{red}. The good thing about this is that $K_{2,n''}$ has an (essentially) unique planar embedding. Consequences for the inner edges of $A_{n'}$ are exploited in the following facts.

Fact 1. Every inner vertex of $A_{n''}$ has at most two incident inner edges of each color.

It follows that the 2-coloring of $A_{n''}$ induces a 2-coloring of $G[V''_{\text{inner}}]$ such that each color consists of a set of paths and cycles.

Fact 2. The set of blue inner edges of $A_{n''}$ contains at most one cycle. The same holds for the red inner edges. If there is a monochromatic cycle, then it is a spanning cycle of V''_{inner}.

There is not much freedom for a 2-coloring of the inner edges of $A_{n''}$ with these properties: We almost have a lampion coloring on $G[V''_{\text{inner}}]$. The exception is that there can be a single *Z-structure* (see Figure 7) in one color.

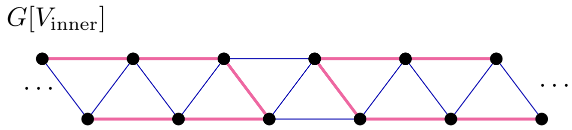

Figure 7: One Z-structure and no monochromatic cycle force all other colors.

Removing the Z-structure leaves two consecutive pieces of the sequence of inner vertices. These pieces of V''_{inner} have a lampion coloring. The size of the larger piece can be estimated as $n''' \geq (n'' - 4)/2$. This proves the lemma. ☐

Well-prepared we can now look at the variant B_n of A_n in which we have slightly perturbed some of the inner bars, see Figure 3.

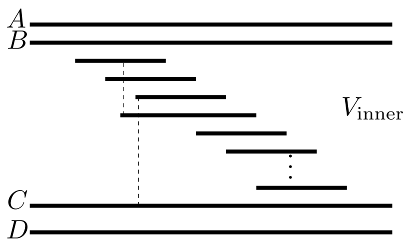

Figure 8: The modified Autobahn-graph B_n.

To obtain B_n, we have elongated every (say) tenth inner bar by pulling its left endpoint to the left, such that it is further left than the left endpoint of the bar directly above. With this modification we introduce an additional edge between the elongated bar v_i and v_{i-3}, but in turn we lose the edge between the bar v_{i-1} and the lowest outer bar D. Let us call the vertices corresponding to the elongated bars *modified vertices*.

Theorem 2. *The graph B_n is a bar 1-visibility graph with thickness 3, for n large enough.*

Proof. We will show that B_n has no 2-coloring. It follows that its thickness is at least 3, and since we can easily use a 2-coloring of A_n and embed the independent additional edges in a third graph, B_3 has thickness exactly 3.

Assume that B_n admits a 2-coloring. We first show that any 2-coloring of B_n would have to be very much alike a 2-coloring of A_n.

The vertices corresponding to a bar directly above a modified one – let us call them *reduced* – have only three outer neighbors. To avoid a $K_{3,3}$ in one color there can be at most 16 inner vertices incident to more than two transversal edges. In particular, most reduced vertices have to divide their three incident transversal edges into two of one color and one of the other. As in the proof of the lemma we consider a continuous piece in the sequence of inner vertices such that all inner vertices have at most two transversal edges of each color. The graph induced by the largest of these pieces and the outer vertices is $B_{n'}$.

The blue subgraph G'_{blue} of $B_{n'}$ is a subgraph of a blown-up K_4 where at least 9/10 of the inner vertices have degree 2 and the remaining reduced vertices have degree 1. As in the proof of the lemma it can be argued that there is only a constant number c of edges in G'_{blue} which join two inner vertices such that there are at least three different outer neighbours of these two vertices, i.e., which join two inner vertices which not belonging to the same blown-up edge of K_4. The constant can be bounded as $c \leq 24$. The red graph may contribute another set of c irregular edges. Removing the irregularities will break the sequence of inner vertices into at most $2c+1$ pieces. The graph induced by the largest of these pieces and the special vertices is $B_{n''}$. Assuming that the edges between inner vertices and D are blue in the 2-coloring of $B_{n''}$ the transversal edges of G''_{blue} and G''_{red} are shown in Figure 9.

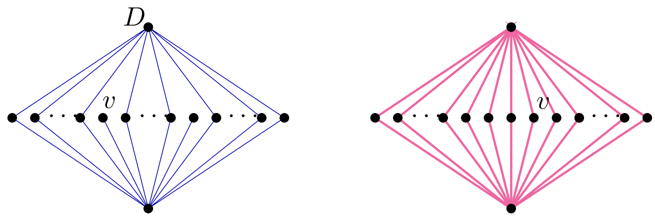

Figure 9: Embedding of a reduced vertex v in G''_{blue} and G''_{red}.

Let $v = v_{i-1}$ be a reduced vertex, the neighbors v_i and v_{i-3} of v both have inner degree 5. In G''_{red} they have degree (at most) 2, hence, they must have degree 3 in G''_{blue}. This is only possible if v_i, v_{i-1} and v_{i-3} form a blue triangle. Since v_{i-1} can have no further blue inner neighbors it follows that the edges $v_{i-1}v_{i-2}$ and $v_{i-1}v_{i+1}$ must be red.

Now consider the edge $v_{i-2}v_{i-3}$. Let us first suppose that this edge is colored blue. To avoid closing a blue cycle, the edge $v_{i-2}v_i$ must be red. Then to avoid a red cycle the edge v_iv_{i+1} must be blue. Continuing that way the colors of all edges to the right of the blue triangle in Figure 10 are uniquely determined. To the other side consider the parity of blue and red edges at v_{i-2}, this forces $v_{i-2}v_{i-4}$ to be blue, while the parity at v_{i-3} forces two red edges. To avoid a red cycle $v_{i-4}v_{i-5}$ must be blue, whence, parity forces $v_{i-4}v_{i-6}$ to be red. That way the color of edges left of the blue triangle is determined. The complete picture is shown in Figure 10: We have found a blue Z-structure.

Now suppose that the color of the edge $v_{i-3}v_{i-2}$ is red. Recalling that $v_{i-1}v_{i-2}$ and $v_{i-1}v_{i+1}$ are red, consider the parity at v_{i-2} which forces $v_{i-2}v_i$ and $v_{i-2}v_{i-4}$ to be blue. Now, parity at v_i forces the red edges v_iv_{i+1} and

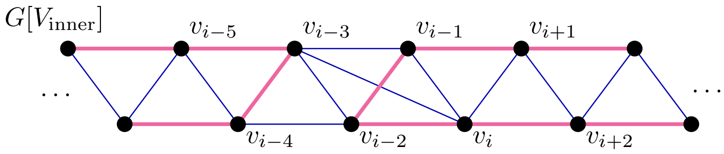

Figure 10: The blue edge $v_{i-3}v_{i-2}$ implies a blue Z.

v_iv_{i+2}. Hence, we have a red Z-structure in this case, see Figure 11.

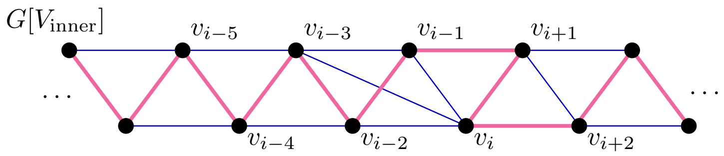

Figure 11: The red edge $v_{i-3}v_{i-2}$ implies a red Z.

We have seen that, given a modified vertex in $B_{n''}$, the parity condition and the cycle-freeness of the colored graphs induced by the inner vertices of $B_{n''}$ enforce a Z-structure. The zig-zag emanating from such a Z-structure in one direction has to run into the Z-structure of a second modified vertex. The outer paths of the other color make a turn at a modified vertex – and close a cycle at a second one. This is a contradiction since the blue triangles of the modified vertices are the only monochromatic cycles in $G[V''_{\text{inner}}]$. The contradiction shows that for large[1] n there is no 2-coloring of B_n, hence, the thickness of the graph is 3. $\square$

Our construction of a B1V with thickness 3 and the upper bound of 4 by Dean et al. [3] leave a gap waiting to be closed. Considering that the graph of our proof only requires few independent edges in the third planar layer, the following conjecture seems plausible:

Conjecture 1. *The thickness of bar* 1*-visibility graphs is at most* 3*.*

3 Parameters of Semi Bar k-Visibility Graphs

In this section we present some structural properties of semi bar k-visibility graphs, before turning to their thickness in Section 4. We will see why SBkVs form an interesting subclass of BkVs: Their strong combinatorial structure gives rise to a number of proof techniques yielding tight results for different graph parameters.

[1] A rough computation yields that $n \geq 25000$ should do.

3.1 Basic Properties

Semi bar 0-visibility graphs are outerplanar, as observed in [2] (for a proof see [13]). For $k > 0$, SBkVs in general are non-planar. Here we show that they are $(2k + 2)$-degenerate, which is a useful property for limiting their minimum degree and always provides a point of attack for induction proofs. We will see that the upper bounds on the chromatic number and the clique number of SBkVs follow easily.

Definition 3. *A graph G is called ℓ-degenerate if every subgraph of G has a vertex of degree at most ℓ.*

Lemma 4. *Semi bar k-visibility graphs are $(2k + 2)$-degenerate for all $k \geq 0$.*

Proof. The vertex corresponding to the shortest bar in a bar representation always has degree at most $2(k + 1)$. □

Corollary 5. *The chromatic number of a semi bar k-visibility graph is at most $2k + 3$, for all $k \geq 0$.*

Proof. This can be seen with a standard inductive argument. For the induction step, take out a vertex v of degree $2k + 2$, color the remaining graph, and use the free color when re-inserting v. □

Example 6. *To see that there are SBk Vs attaining the chromatic number $2k+3$ for all $k \geq 0$, consider Figure 12 which shows a schematic bar representation of K_{2k+3}. Note that for any complete graph on $n \leq 2k + 2$ vertices, a bar representation can be found by leaving out some of the bars in Figure 12.*

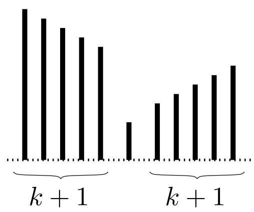

Figure 12: A complete SBkV on $2k + 3$ bars.

Corollary 7. *The clique number of SBk Vs is $2k + 3$.*

Proof. Since SBkVs are $(2k + 2)$-degenerate, no such graph can contain a complete subgraph on more than $2k + 3$ vertices. □

We have seen above that the largest possible chromatic number of SBkVs coincides with the size of the largest complete SBkV. However, this does not transfer to every particular SBkV and its induced subgraphs.

Example 8. *Figure 13 shows an SBk V containing C_5 as induced subgraph.*

Corollary 9. *In general, SBk Vs are not perfect.*

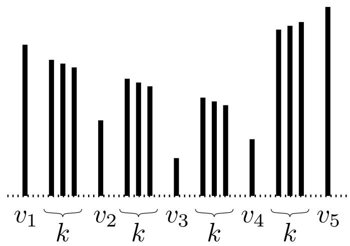

Figure 13: Example of a non-perfect SBkV: The v_i form an induced C_5.

3.2 Maximum Number of Edges

In this section, we show a tight upper bound for the maximum number of edges in an SBkV, and we characterize the bar representations of SBkVs attaining this bound.

Let us start with some edge counting. If an SBkV on $n \leq 2k+3$ vertices is given, then by Proposition 7 the tight upper bound on its number of edges is $\binom{n}{2}$. For $n = 2k+2$ and $n = 2k+3$ this bound coincides with the bound for larger n given in the following theorem:

Theorem 10. *A semi bar k-visibility graph on* $n \geq 2k+2$ *vertices has at most* $(k+1)(2n-2k-3)$ *edges.*

Proof. Think of the edges as being directed from longer to shorter bars. Since each bar has at most $2(k+1)$ longer neighbors, each vertex has at most $2(k+1)$ incoming edges. Thus we have a first upper bound of $2n(k+1)$ on the number of edges.

Let us look more closely at the longest bars: The vertex r_1 corresponding to the longest bar does not have any incoming edges, r_2 has at most one, r_3 two, and so on until reaching r_{2k+2} which has no more than $2k+1$ incoming edges. Substracting these edges that we counted too many in our first bound from $2n(k+1)$, we obtain the desired upper bound:

$$2n(k+1) - \sum_{i=1}^{2k+2} i = 2n(k+1) - \frac{1}{2}(2k+2)(2k+3) = (k+1)(2n-2k-3)$$

□

For any $k \geq 0$ and $n \geq 2k+2$, semi bar k-visibility graphs with $(k+1)(2n-2k-3)$ edges exist. See Figure 14 for a schematic bar representation of such a *maximal SBkV*: The $k+1$ leftmost bars together with the $k+1$ rightmost bars form a complete graph. We call these bars *outer bars* and the corresponding vertices *outer vertices*. Every shorter bar has $k+1$ incoming edges from either side. Here we speak of *inner bars* and *inner vertices*.

Note that any maximal SBkV has to have a bar representation with the pattern described above. The $2(k+1)$ outer bars have to induce a complete

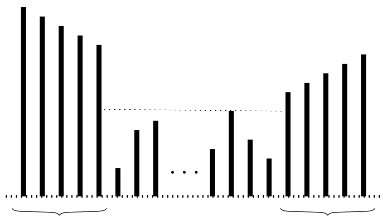

Figure 14: Structure of a maximal SBkV

graph, and every inner bar has to have $k + 1$ incoming edges from either side – thus, the outer bars need to be the longest ones. (However, there is some more freedom in the pattern than the figure might suggest, as there are other orders in which the $2k + 2$ longest bars form a complete graph).

We now have a quite good idea of the structure of maximal SBkVs: One can represent all of them by starting with $2k + 2$ bars inducing a clique and inserting bars in the middle one by one, by order of their length.

3.3 Reconstructing Semi Bar k-Visibility Representations

One of the first questions one can ask about a semi bar k-visibility graph $G = (V, E)$ is how to find a bar representation of a given graph which is known to be an SBkV. With some additional information as input, we can explicitely construct all bar representations inducing G, as we show in the following. We first need some new terminology here.

Given a bar representation $\mathcal{B}$ of G, we label the bars from left to right and denote the corresponding order of the vertices with $t_1, t_2, \ldots, t_n$ a *t-order* of V. As there might be many bar representations of G, there can also be many *valid t-orders.* Recall that in the introductory Section 1.2, we labeled vertices r_1 to r_n according to the length of the bars. This order will be called r-order in the following.

For a given valid t-order of V, there are some edges that need to be contained in the graph, because the bars in a corresponding representation automatically see each other. We call these edges the *trivial edges*:

Definition 11. *Let an SBkV $G = (V, E)$ with bar representation $\mathcal{B}$ be given. Let $V = \{t_1, t_2, \ldots, t_n\}$ in the t-order induced by $\mathcal{B}$. An edge $t_i t_j \in E$ is called a* trivial edge *if $|i - j| \le k + 1$.*

Note that the vertex represented by the shortest bar in $\mathcal{B}$ is only incident to trivial edges.

If G is given without a corresponding bar representation, knowing a valid t-order of the vertices is equivalent to knowing a valid set $\bar{E}$ of trivial edges: From $\bar{E}$ one can uniquely deduce a t-order such that $\bar{E}$ fulfills the above definition. In the following theorem, we need one of these informations as additional input.

Theorem 12. *For $k \geq 0$, let an SBkV G be given with a valid t-order of its vertices. Then an r-order of the vertices can be computed which, together with the given t-order, defines a bar representation inducing G.*

Proof. The idea is to find a bar representation by first choosing a vertex which will be represented by the shortest bar, then deleting it from the graph, and iterating this until we have defined a complete r-order of the vertices.

Let $\bar{E}$ denote the set of trivial edges defined by the given t-order. Since we know that a bar representation with this t-order exists, we know that there is at least one vertex which is only incident to edges in $\bar{E}$. The crucial observation is that this property determines exactly the vertices that can be represented by the shortest bar in a bar representation of G (see Figure 15). Thus, choose one such vertex as r_n and delete it from G.

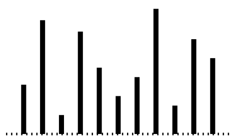

Figure 15: An SBkV G. Possible choices apart from t_3 for the shortest bar in a bar representation inducing G are t_6 and t_9.

The t-order of the remaining vertices can be used to define a new set of trivial edges. Now in each step i for i descending to 1, we use the current set of trivial edges to find candidates for r_i, choose one of them and delete it from the graph. Then we use the t-order to readjust the set of trivial edges. In the end we have an r-order of the vertices which defines a bar representation inducing G. □

Note that the above construction does not just yield one r-order inducing G with the given t-order, one can in fact find *all* such bar representations of G this way.

For the case $k = 0$, the graph G is outerplanar and one can use a dual tree to find an explicit formula counting the number of bar representations inducing G (see Chapter 2 of [13]). The construction also makes fundamental use of the given t-order. It remains an open question how a bar representation of an SBkV can be found if no t-order is given.

3.4 Connectivity

Now we want to determine the connectivity of a bar k-visibility graph $G = (V, E)$ given with bar representation $\mathcal{B}$. It is clear that G is connected, and it is not too hard to guess that its connectivity is $k+1$, since any bar can see its $k+1$ left and $k+1$ right neighbors.

Theorem 13. *Semi bar k-visibility graphs on more than $k+1$ vertices are $(k+1)$-connected.*

Proof. In order to separate the subgraph built by the trivial edges (cf. Def. 11) of G, a separating set has to contain at least $k+1$ vertices. □

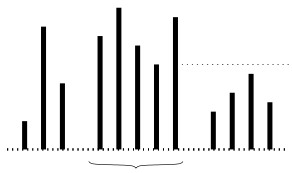

Figure 16: A bar representation containing a $(k+1)$-separator

Figure 16 shows that in general an SBkV can have a $(k+1)$-separator. If we delete the $k+1$ vertices corresponding to the bars in the middle, the graph falls apart into the two components induced by the three bars on the left and the remaining ones on the right side. There is an explicit way of characterizing the $(k+1)$-separators:

Proposition 14. *Let $G=(V,E)$ be an SBkV given with a bar representation $\mathcal{B}$ and a corresponding t-order of V. Let $S \subset V$. Then S is a $(k+1)$-separator of G if and only if $S=\{t_i, t_{i+1}, \dots, t_{i+k}\}$ for some i with $1 < i < n-k$, and left or right of $B(S)$ in $\mathcal{B}$ all bars are shorter than the shortest one in $B(S)$.*

Proof. We first show that the two conditions are necessary for S to be a $(k+1)$-separator. In order to separate the subgraph built by the trivial edges of G, the vertices of S have to be successive in the t-order. Also, they must not be located at one of its ends, else there would be nothing to separate. This proves the first condition. Now let $S=\{t_i, t_{i+1}, \dots, t_{i+k}\}$ with $1 < i < n-k$. Let $B(t_j)$ be the shortest bar in $B(S)$. Suppose that left and right of $B(S)$ there is a bar longer than $B(t_j)$. Let $B(t_\ell)$ be the rightmost such bar left of $B(S)$, and let $B(t_m)$ be the leftmost such bar right of $B(S)$. Then there is a line of sight between $B(t_\ell)$ and $B(t_m)$, since there are at most k longer bars which could cloud it. The edge $t_\ell t_m$ connects the two paths $t_1, \dots, t_{i-1}$ and $t_{i+k+1}, \dots, t_n$. Thus, S cannot be a separating set.

For the reverse direction observe that any set of vertices with the described two properties separates the vertices corresponding to the "shorter half" of the bar representation from the rest of the graph (see Figure 16), and thus is a $(k+1)$-separator. □

Using the characterization of Proposition 14 one can efficiently find all the $(k+1)$-separators of an SBkV which is given by a permutation defining a bar representation. This is done by running through the vertices following the t-order, retaining the longest already seen and the longest not yet seen bar, and checking for each set of $k+1$ successive bars if the shortest among them is longer than one of the two retained ones.

If we turn to maximal SBkVs, we get an even higher connectivity: By further exploring the structure of their bar representations one can see that they are $2k+2$-connected. (This is the highest possible connectivity of an SBkV since the shortest bar has at most that many neighbors). The proof uses the global version of Menger's Theorem and finds two $(k+1)$-bundles of independent paths between any two vertices. The idea is nice and simple – every such path is built by successively jumping $k+1$ bars to the left (right). The whole proof in its (technical) details can be found in [13].

4 Thickness of Semi Bar 1-Visibility Graphs

Let $G = (V, E)$ be an SB1V given by a bar representation, see e.g. Figure 2. In this section we present an algorithm which 2-colors the edges of G such that each color class forms a plane graph in an embedding induced by the bar representation. Consequently the thickness of an SB1V is at most 2.

Between the full class of B1Vs and the subclass of SB1Vs there is the class of bar 1-visibility graphs admitting a representation by a set of bars such that there is a vertical line stabbing all bars of the representation. Note that Theorem 2 implies that already in this intermediate class there are graphs of thickness 3.

4.1 One-Bend Drawing

A one-bend drawing of a graph is a drawing in the plane in which each edge is a polyline with at most one bend. Here we introduce a one-bend drawing of SB1Vs with some specified properties. This drawing is not planar in general, but it will be helpful for the construction and the analysis of the 2-coloring.

Enlarge the bars of each vertex v to a rectangle $B(v)$ with a uniform width. Recall that we assume that the heights of all bars are different and that $B(r_1)$, $B(r_2), \ldots, B(r_n)$ lists the bars by decreasing height. Assign the stripe between the horizontal line touching the top of bar $B(r_i)$ and the horizontal line touching the top of bar $B(r_{i-1})$ to r_i. The dotted lines in Figure 17 separate the stripes. Embed each vertex v at the midpoint of the upper boundary of $B(v)$. We think of the edges as being directed from the longer bar (its *starting bar*) to the shorter bar (its *ending bar*).

Now draw each edge $e = r_i r_j$ composed of two segments; the first segment is contained in $B(r_i)$, it connects r_i with the inflection point x_e, the second segment connects x_e with r_j within the stripe of r_j. Place x_e on the vertical boundary of $B(r_i)$ which is closer to r_j, with a height which is inside the stripe of r_j. This position of x_e bewares from crossings between edges emanating from r_i. We call the segment (r_i, x_e) the *vertical part*, the segment (x_e, r_j) the *horizontal part* of the edge. Note that the stripe associated with $B(v)$ contains the horizontal parts of the incoming edges of v. Other edges might cross this stripe, but only with their vertical parts.

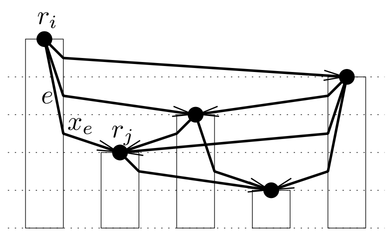

Figure 17: A one-bend drawing of an SB1V

4.2 2-Coloring Algorithm

Now we present the algorithm 2PLANAR that provides a 2-coloring of E, i.e., a partition of the edges into two planar graphs (both on the vertex set V), using the given embedding. Thus, the algorithm produces planar embeddings of the two graphs such that each edge has only one bend. We think of the partition of the edges as a coloring with blue and red.

The idea of the algorithm is the following: Given a one-bend drawing, start with r_1, color all outgoing edges, move on to r_2, and so on. The algorithm uses an auxiliary coloring of the bars to determine the color of the edges.

Algorithm 2PLANAR

1. Start with r_1. Color $B(r_1)$ and all outgoing edges of r_1 blue. Whenever such an edge traverses another bar, color that bar red.
2. For $i = 2, \ldots, n-1$
   If $B(r_i)$ is uncolored, then color this bar blue.
   For each uncolored edge $e = r_i r_j$
   (a) If e is a direct edge, it obtains the color of its starting bar $B(r_i)$.
   (b) If e is an indirect edge, check if the traversed bar has a color. If so, e obtains the other color. Otherwise, it receives the color of its starting bar $B(r_i)$, and the traversed bar gets the opposite color.

Note that 2(b) implies the following:

Invariant. Whenever an edge traverses a bar the colors of the edge and the color of the bar are different.

Theorem 15. *In the given setting, 2PLANAR produces a partition of E into two plane edge sets.*

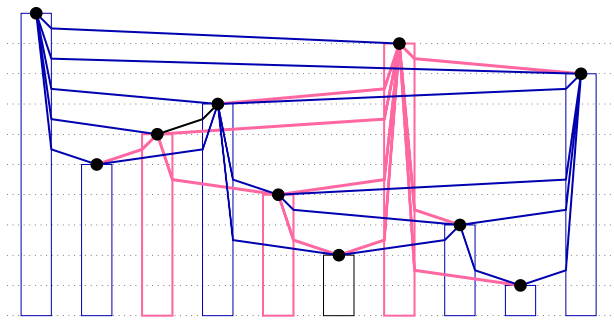

Figure 18: A coloring produced by the algorithm 2PLANAR.

Proof. We have to show that in the 2-coloring computed by 2PLANAR, any two crossing edges have different colors.

The one-bend drawing implies that crossings between edges only appear between the vertical part of one edge and the horizontal part of another edge. Consider a crossing pair e, f of edges, assume that the crossing is on the vertical part of e and the horizontal part of f. Hence, the crossing point is inside of the starting-bar of e, and the edge f is an indirect edge traversing this bar (see Figure 19).

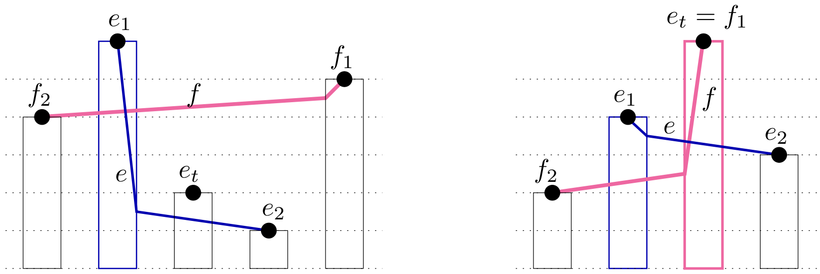

Figure 19: Two crossing edges e and f, shown in two possible configurations. Note that there can be many shorter bars between the bars depicted here.

Let the start-vertex of e be e_1 and its end-vertex e_2. Similarly, let f lead from f_1 to f_2. Suppose that the color of f is red, then the invariant implies that $B(e_1)$ is blue.

If e is a direct edge it obtains the color of its starting bar, in this case blue. Thus, assume that e is an indirect edge. Then its color depends on the color of the traversed bar, let $B(e_t)$ be this bar. (Note that $e_t = f_1$ or $e_t = f_2$ are possible.)

The key for concluding the proof lies in the following lemma:

Lemma 16. *If $B(v)$ is an arbitrary bar, then at most one longer bar can be the* *starting bar of indirect edges traversing $B(v)$.*

Proof. Assume that $x = x_1x_2$ is an edge traversing $B(v)$ with $B(x_1)$ longer than $B(v)$, such that $B(x_2)$ is the shortest ending bar among all such edges. Then we know that between $B(x_1)$ and $B(v)$ in the left-to-right-order there can be no bar longer than $B(x_2)$, else it would block the line of sight which correspond to x.

Suppose there is another edge $y = y_1y_2$ starting from a bar $B(y_1)$ which is longer than $B(v)$. The choice of x implies that the horizontal part of y is above the horizontal part of x. Since y can traverse only one bar it must connect to a bar $B(y_i)$ which is between $B(v)$ and $B(x_1)$ in the left-to-right-order. Since y is above x the bar $B(y_i)$ is longer than $B(x_2)$. This is in contradiction to the conclusion of the previous paragraph. △

Now let us first assume that $B(e_t)$ is shorter than $B(e_1)$. Then by the lemma we know that $B(e_1)$ is the only longer bar sending an edge (e.g. the edge e) through $B(e_t)$. Therefore $B(e_t)$ is still uncolored when the algorithm considers $B(e_1)$, therefore, e is colored with the color of $B(e_1)$, which is blue. This shows that the edges e and f have different colors in this case.

If $B(e_t)$ is longer than $B(e_1)$, then we can deduce $e_t = f_1$. For if a bar longer than $B(e_t)$ would be located strictly between $B(f_1)$ and $B(f_2)$ in the left-to-right-order, the line of sight corresponding to f would have to traverse two bars ($B(e_1)$ and this longer bar), which is a contradiction. In addition, we know that $B(f_2)$ is shorter than $B(e_1)$, else e and f would not cross. Thus, we have $e_t = f_1$, and $B(f_1)$ is longer than $B(e_1)$. In this case the lemma tells us that $B(f_1)$ is the only longer bar sending an edge through $B(e_1)$. It follows that $B(e_1)$ was still uncolored when algorithm considered $B(f_1)$. Therefore, the red color of f was chosen equal to the color of the bar $B(f_1)$. The invariant implies that the edge e, traversing the red bar $B(f_1)$, is blue. Hence, again e and f have different colors. □

The algorithm shows that SB1Vs have graph-theoretical thickness not more than 2, and it defines a partition of the edges into two planar graphs, providing two plane embeddings. Since the positions of each vertex coincide in these two embeddings, a natural question now is to ask whether we can also bound the geometric thickness of SB1Vs. Recall that for geometric thickness, rectilinear planar embeddings are needed (cf. Section 1.3). The following theorem shows that we can turn our one-bend drawing into a straight-line embedding. This theorem confirms a conjecture from [8].

Theorem 17. *The geometric thickness of SB1Vs is at most* 2.

Proof. Let an SB1V G with a corresponding bar representation be given and reconsider the definition of a one-bend drawing of G. We change the height of the stripes S_j, with j increasing to n. In each step, we consider the incoming edges $\hat{e} = r_ir_j$ of vertex r_j. We let S_j be so high that the rectilinear connection between r_i and r_j leaves the bar $B(r_i)$ in a point $x_{\hat{e}}$ inside of S_j. See Figure 20 as illustration.

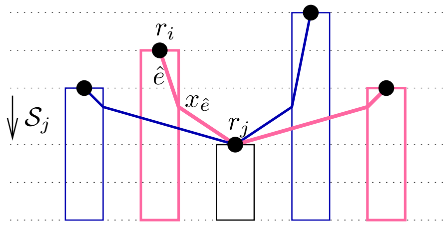

Figure 20: We let the stripe $\mathcal{S}_j$ be so high that the incoming edges of r_j can be drawn rectilinear.

Having done this for all stripes we obtain a straight-line embedding of G where each edge $\hat{e}$ falls into its "vertical" part from v_i to $x_{\hat{e}}$ and its "horizontal" part from $x_{\hat{e}}$ to r_j. The vertical part runs inside of the bar $B(r_i)$ and the horizontal part inside of the stripe S_j. This embedding is a one-bend drawing as defined in Section 4.1 (with the additional property that all edge-bends have an angle of 180 degree). Thus by Theorem 15 the algorithm 2PLANAR partitions this drawing into two plane layers. □

4.3 Structure of the Blue and Red Graph

Given a semi bar 1-visibility graph $G = (V, E)$, we can say more about the effect of the algorithm 2PLANAR on G: It partitions the edges evenly among the blue and the red graph. As *blue graph* let us define $G_{\text{blue}} := (V', E_{\text{blue}})$ by taking all blue edges on the vertex set V and deleting isolated vertices. The *red graph* is defined analogously as $G_{\text{red}} := (V'', E_{\text{red}})$. Recall that from Theorem 10 we know that an SB1V has at most $(k+1)(2n-2k-3) = 4n-10$ edges.

Proposition 18. *The blue and the red graph each contain at most $2n-3$ edges.*

Proof. We count the incoming blue edges at each vertex and claim that there are at most two of them. See Figure 21 for an illustration. Consider a bar $B(v)$ and the closest bar to the left (say) of it that is the starting bar $B(w)$ of an incoming blue edge at v. Either such an edge springs from a direct line of sight between $B(v)$ and the blue bar $B(w)$, or it is induced by an indirect line of sight traversing a red bar $B(u)$.

In the first case, any other incoming blue edge from the left has to traverse $B(w)$ and therefore is colored red. In the second case, any other such edge would have to traverse $B(w)$ as well as $B(u)$, which is not possible.

Thus there is at most one incoming blue edge from each side, which yields an upper bound of $2n$ blue edges. But the vertex r_1 represented by the longest bar has no incoming edges, and r_2 has only one. Therefore we can substract three edges, obtaining the desired result. The same argument applies to the number of red edges. □

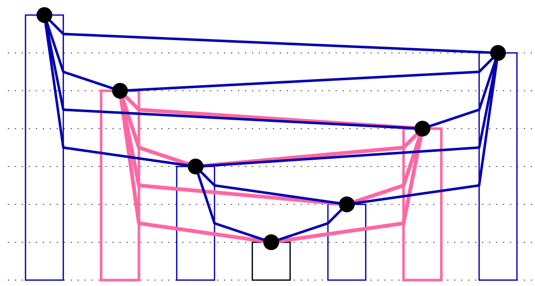

Figure 21: Example of an SB1V with $2n-3$ blue edges.

The upper bound of Proposition 18 is sharp, as the blue graph in Figure 21 shows: Any vertex except for r_1 and r_2 has two incoming edges. The pattern shown by the example in the figure can be used to construct an SB1V with $2n-3$ blue edges for arbitrary n.

The edge bound of $2n-3$ may sound familiar – it also holds for outerplanar graphs. However, the example in the figure shows that, in general, G_{blue} and G_{red} are not outerplanar: The blue graph induced by the five longest bars forms a $K_{2,3}$. In Figure 22, the red graph induced by the vertices t_2, t_3, t_5, t_6 and t_7 contains K_4 as a minor, which can be obtained by contracting the edge t_3t_5.

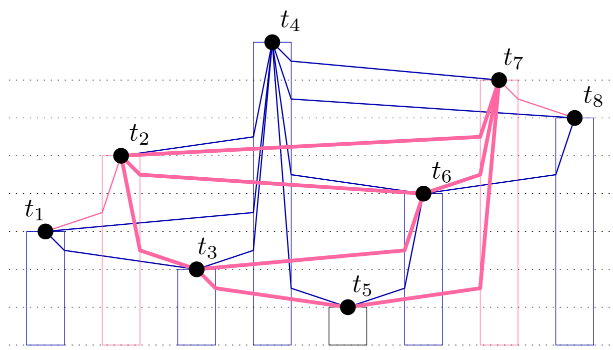

Figure 22: The red graph contains K_4 as a minor.

For the case of maximal SB1Vs, there is more structure to explore within the blue and the red graph defined by 2PLANAR. In fact it can be shown (see [13]) that in this case they are both *Laman graphs*, which implies a number of additional properties.

5 Conclusion and Open Problems

In this paper, we disproved the conjecture of Dean et al. [3] that the tight upper bound on the thickness of bar 1-visibility graphs is 2.

On the way of getting a better understanding of bar k-visibility graphs we considered semi bar k-visibility graphs which have a strong combinatorial structure. We used this structure to get tight bounds on the clique number, the chromatic number, the maximum number of edges and the connectivity of SBkVs. For the case $k = 1$, we proved the tight upper bound of 2 on the thickness of SBkVs, confirming the conjecture of Dean et al. for this subclass.

Still, many questions are left open, and new ones emerged. The following open problems may serve as a starting point for further research.

1. In [3], it is shown that the thickness of BkVs can be bounded by a function in k (proven is a quadratic one). What is the smallest such function?
2. What is the largest thickness or geometric thickness of SBkVs?
3. What is the largest chromatic number of BkVs? Dean et al. show an upper bound of $6k + 6$.
4. Hartke, Vandenbussche and Wenger [9] found some forbidden induced subgraphs of BkVs. They ask for further characterization of BkVs by forbidden subgraphs.
5. Hartke et al. also examined regular BkVs. Are there d-regular BkVs for $d \geq 2k + 3$?
6. What is the largest crossing number of BkVs?
7. What is the largest genus of BkVs?
8. How can SBkVs be characterized?
9. Can SBkVs be recognized in polynomial time?

Acknowledgments

We thank Günter Rote for a hint leading to the proof of Theorem 17.

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