# **On Area-Universal Quadrangulations**

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We study drawings of plane quadrangulations such that every inner face 4 realizes a prescribed area. A plane graph is area-universal if for every assign-5 ment of non-negative weights to the inner faces, there exists a straight-line 6 drawing such that the area of each inner face equals the weight of the face. 7 It has been conjectured that all plane quadrangulations are area-universal. 8 We develop methods to prove area-universality via reduction to the area-9 universality of related graphs. This allows us to establish area-universality 10 for large classes of plane quadrangulations. In particular, we use our tools 11 to show area-universality of all quadrangulations with at most 13 vertices. 12

### 13 **1** Introduction

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A plane graph is a planar graph together with a crossing-free drawing. Let G be a plane graph and let F' be its set of inner faces. An *area assignment* is an assignment of a nonnegative real number to every face  $f \in F'$ , i.e., a function  $\mathscr{A}: F' \to \mathbb{R}^+_0$ . A (potentially degenerate) planar straight-line drawing D of G *realizes* the area assignment  $\mathscr{A}$  if for every  $f \in F'$  the area of f in D is  $\mathscr{A}(f)$ . A plane graph G is *area-universal* if it has a realizing drawing for every area assignment  $\mathscr{A}$ .

Ringel [18] considered straight-line drawings of plane graphs such that all faces have 20 the same area. He gave an example of a plane triangulation that has no equiareal draw-21 ing, hence, a triangulation which is not area-universal. Thomassen [19] proved that plane 22 cubic graphs are area-universal. Biedl and Velázquez [3] showed area-universality for the 23 class of plane 3-trees, also known as stacked triangulations and Apollonian networks. 24 Concerning counter examples, Kleist [12, 14] introduced a simple counting argument 25 to show that no Eulerian triangulation is area-universal. Moreover, she showed that 26 every plane graph is area-universal in the class of drawings where one bend per edge 27

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is allowed. For triangulations with special vertex orders, Kleist [13] presented a sufficient criterion for their area-universality that only requires the investigation of one area assignment. Interestingly, if the sufficient criterion applies to one plane triangulation, then all embeddings of the underlying planar graph are also area-universal. Dobbins et al. [6] investigated the complexity of deciding whether a given graph is area universal and several related problems. They conjecture that the problem is complete for the complexity class  $\forall \exists \mathbb{R}$ .

In this paper, we focus on plane bipartite graphs. Since the property of area-universality 35 is preserved under edge-deletions (see also Observation 1), we consider edge-maximal 36 plane bipartite graphs known as quadrangulations. Regarding the area-universality of 37 quadrangulations little is known. Evans et al. [9] showed that the  $m \times n$  grid is area-38 universal for all m, n > 2, even with the additional requirement that the outer face of 39 the drawing is a rectangle. Kleist [12] showed that 2-degenerate quadrangulations are 40 area-universal and that in the class of drawings where one bend per edge is allowed all 41 quadrangulations have realizing drawings for all area assignments where only half of the 42 edges have a bend. 43

The study of drawings in various drawing modes with prescribed face areas is summa-44 rized under the name cartograms. Cartograms date back to at least 1934 when Raisz [17] 45 studied rectangular population cartograms, where the US population was visualized by 46 representing the states with areas proportional to their population. This kind of vi-47 sualization is particularly useful when showing geo-referenced statistical data in order 48 to provide insight into patterns, trends and outliers in the world around us [22]. Car-49 tograms have been intensely studied for duals of triangulations and rectilinear drawings 50 with bends. The number of sides of the polygons representing a face has been improved 51 in a series of papers from 40 sides [5], to 34 sides [11], to 12 sides [2]. Finally, Alam 52 et al. [1] showed how to construct drawings with 8-sided faces, which is known to be 53 optimal. Chang and Yen [4] studied contact representations of 2-connected outerplaner 54 graphs and construct contact representations with 4-gons of prescribed area. Note that 55 in the cartogram literature the problem is usually treated in the dual setting, i.e., weights 56 are assigned to the vertices. We refer to Nusrat and Kobourov [16] for a survey of the 57 cartogram literature. 58

Area-universality has also been studied in the context of rectangular layouts, these are 59 dissections of a rectangle into rectangles with prescribed contacts between the rectangles 60 of the dissection. Eppstein et al. [8] showed that a rectangular layout is area-universal 61 if and only if it is one-sided. The key ingredient in their proof is that the weak equiv-62 alence class of any rectangular layout is area-universal. The weak equivalence class is 63 obtained by prescribing the contacts between the segments. The area-universality of the 64 weak equivalence class has been shown by different techniques [7, 10, 21]. This area-65 universality result is very special because, up to affine transformations, the rectangular 66 layout realizing a given area assignment is actually unique. 67

Our Contributions: We study area universality of plane quadrangulations. To realize non-negative face areas, we extend the set of crossing-free straight-line drawings of a plane quadrangulation by all drawings which can be obtained as the limit of a sequence

of straight-line drawings (e.g. specified by the coordinates of the vertices). In particular, 71 we allow *degenerate drawings* in which vertices and edges sharing a face may (partially) 72 coincide; if two edges partially coincide their union forms a segment. In Section 2, we 73 investigate operations that preserve area-universality. In Section 3, we use one of these 74 operations, the edge contraction, to show area-universality of grids and large classes 75 of angle graphs. In Section 4, we study strong area-universality, i.e., area-universality 76 within a prescribed outer face. Shape restrictions are also the subject of Section 5; there 77 we study convex drawings. In Section 6, we use our tools to show area-universality of 78 all quadrangulations with at most 13 vertices. In some cases the argument relies on the 79 known area universality of the class of double stacking graphs. 80

### <sup>81</sup> 2 Area-Universality Preserving Operations

<sup>82</sup> We begin with an easy observation which can also be found in [3] and [14].

<sup>83</sup> Observation 1. A subgraph of an area-universal plane graph is area-universal.

Therefore, a proof for the area-universality of plane quadrangulations, i.e., maximal plane bipartite graphs, would imply area-universality of all plane bipartite graphs. The following lemma extends Observation 1 with a new operation. A set of edge contractions in a plane graph G is *face-maintaining* if the contractions do not change the number of faces in G, i.e., for a face of degree d at most d - 3 edges are contracted.

<sup>89</sup> Lemma 1. Let G be a plane graph that can be transformed into an area-universal <sup>90</sup> plane graph G' by inserting vertices, inserting edges, and performing face-main-<sup>91</sup> taining edge contractions. Then G is area-universal.

Proof. Let  $\mathscr{A}$  denote an area assignment of G. A face f in G corresponds to a (nonempty) collection of faces  $C_f$  in G'. We define  $\mathscr{A}'$  such that for each inner face f of G it holds that  $\mathscr{A}(f) = \sum_{f' \in C_f} \mathscr{A}'(f')$ . Since G' is area-universal, there exists an  $\mathscr{A}'$ -realizing drawing D' of G'. Simply deleting all vertices and edges of G' which are not in G yields a (degenerate) drawing D of G. By definition of  $\mathscr{A}'$ , D is  $\mathscr{A}$ -realizing.

There exists a further operation that preserves area-universality and is based on decomposition. For an illustration consider Figure 1. From a plane graph G with a simple cycle C, we obtain two plane graphs  $G_I$  and  $G_E$  by *decomposing along* C:  $G_I$  is the subgraph of G consisting of C and its interior, while  $G_E$  is the subgraph of G consisting of C and its exterior. Reversely, we obtain G from  $G_I$  and  $G_E$  by identifying the outer face of  $G_I$  with the inner face of  $G_E$  whose boundary is C.

A plane graph G is strongly area-universal if for every area assignment  $\mathcal{A}$  of G and every fixed polygonal placement of the outer face of area  $\Sigma \mathcal{A}$ , there exists a realizing straight-line drawing of G within the prescribed outer face. Here we have used  $\Sigma \mathcal{A}$  to denote the sum of all assigned areas, i.e.,  $\Sigma \mathcal{A} := \sum_{f \in F'} \mathcal{A}(f)$ . Since all triangles are affine equivalent, a plane graph with a triangular outer face has a realizing drawing (if it exists) within every triangle of correct area [14]. It follows that:



Figure 1: Decomposing G along C yields two plane graphs  $G_{E}$  and  $G_{I}$ . If  $G_{E}$  is areauniversal and  $G_{I}$  strongly area-universal, then G is area-universal.

109 **Observation 2.** Plane graphs with triangular outer faces (e.g. triangulations) are 110 area-universal if and only if they are strongly area-universal.

<sup>111</sup> A similar result for quadrangulations would be a pleasant surprise.

Lemma 2. Let G be a plane graph with a simple cycle C, and  $G_I$  and  $G_E$  obtained by decomposing G along C. If  $G_E$  is area-universal and  $G_I$  is strongly area-universal, then G is area-universal.

<sup>115</sup> Moreover, if  $G_E$  is strongly area-universal, then G is also strongly area-universal.

Proof. Let A be an area assignment of G. For i ∈ {I,E}, A<sub>i</sub> denotes the induced area assignment of G<sub>i</sub>. Note that the interior of C is a face f of G<sub>E</sub>. In particular, it holds that  $A_E(f) = \Sigma A_I$ . Since G<sub>E</sub> is area-universal, we find an  $A_E$ -realizing drawing D<sub>E</sub> of G<sub>E</sub>. Since G<sub>I</sub> is strongly area-universal, we find an A<sub>I</sub>-realizing drawing D<sub>I</sub> of G<sub>I</sub> whose outer face is the polygon representing C in D<sub>E</sub>. Thus, identifying D<sub>E</sub> and D<sub>I</sub> along C yields an A-realizing drawing of G.

The ideas of this lemma have been used in [14] to show the strong area-universality of 2-degenerate quadrangulations. Recall that a graph is k-degenerate if and only if every subgraph contains a vertex of degree at most k.

<sup>125</sup> **Proposition 3** ([14]). Every 2-degenerate quadrangulation is strongly area-universal.

It is easy to see that  $K_4$  is area-universal, i.e., a vertex of degree 3 can be inserted into a triangle so that the three small triangles partition the big triangle in any prescribed ratio. This yields the following:

Lemma 4. Let T be a plane graph and  $T^+$  the plane graph where a vertex of degree 3 is inserted (stacked) into a triangle of T. Then T is area-universal if and only if  $T^+$ is area-universal.

<sup>132</sup> Since a plane 3-tree is obtained from a triangle by iteratively stacking vertices into <sup>133</sup> faces, Lemma 4 yields the result from [3]: Plane 3-trees are area-universal.

### <sup>134</sup> 3 Area-Universality via Edge Contractions

In this section, we discuss some implications of the edge contractions of Lemma 1. 135 Firstly, we show an alternative proof for the area-universality of grid graphs. The grid 136 G(m,n) is the Cartesian product  $P_m \times P_n$  of paths  $P_m$  and  $P_n$  on m and n vertices, 137 respectively. Figure 2(a) illustrates G(9,6). Area-universality of grid graphs was first 138 proved in the context of *table cartograms* where additionally the outer face is required 139 to be a rectangle [9]. Our new proof does not yield a rectangular outer face; however, 140 it is straight-forward and very simple. The reader is invited to check that Proposition 3 141 does not imply the area-universality of grid graphs, because G(m, n) is not a subgraph 142 of a 2-degenerate quadrangulation if n, m > 3. 143



Figure 2: Illustration of Proposition 5 and its proof. (a) The grid graph G(m, n) and(b) an area-universal triangulation 'containing' it.

#### <sup>144</sup> **Proposition 5.** Every grid is area-universal.

*Proof.* The idea of this proof is easy to convey by picture; see Figure 2. Contract the 145 edges of every second column of G(m, n) to super vertices that are labeled by  $1, \ldots, k$ 146 from left to right. Then, we add vertices and edges to enhance the resulting graph to 147 a triangulation G as depicted in Figure 2(b). The graph G is a stacked triangulation 148 since the interior of each triangle (i, i+1, C) with  $1 \le i \le k-1$ , and the graph induced 149 by  $A, B, C, 1, \ldots, k$  is a stacked triangulation. Thus, G is area-universal. Therefore, 150 every grid graph can be transformed into a subgraph of an area-universal graph using 151 face-maintaining edge contractions. Hence, by Lemma 1, grids are area-universal. 152

The angle graph of a plane graph G is the graph  $Q_{\rm G}$  with vertex set consisting of 153 the vertices and faces of G and edges corresponding to face-vertex incidences. If G is 154 2-connected, then  $Q_{\rm G}$  is a quadrangulation. Clearly, an angle graph is bipartite where 155 the two bipartition classes are the vertices V and the faces F of G. In the following we 156 consider angle graphs of triangulations. For a plane graph G and its angle graph Q, their 157 union (graph) G + Q, consists of the union of the vertex and edge sets of G and Q. 158 Note that the union is again a plane graph: Indeed, the vertex set of G + Q coincides 159 with the vertex set of Q. Hence, G + Q can be understood as the quadrangulation Q 160 together with the edges between the vertices of one bipartition class, namely V. 161

<sup>162</sup> **Proposition 6.** The angle graph Q of an area-universal triangulation T is area-uni-<sup>163</sup> versal.

Proof. The graph T + Q can be seen as T where a vertex of degree 3 is inserted in every face. By Lemma 4, T + Q is area-universal. Thus, Observation 1 implies that Q is area universal.

Note that the same approach shows that angle graphs of equiareal triangulations are
 equiareal. Moreover, a straightforward consequence of Proposition 6 is that angle graphs
 of stacked triangulations are area-universal.

In order for its angle graph to be area-universal, it suffices for a triangulation to 170 be close to area-universal. As shown in [12], every plane graph has an area-universal 171 subdivision. The subdivision number s(G) of a plane graph G is the minimum number 172 of subdivision vertices to be inserted into G such that it becomes area-universal. If G 173 is area-universal, then clearly s(G) = 0. To generalize Proposition 6, we introduce the 174 notion of a refined area assignment. If  $G_1$  is a subgraph of  $G_2$ , then every face f of  $G_1$ 175 corresponds to a collection of faces  $C_f$  in  $G_2$ . An area assignment  $\mathcal{A}_1$  of  $G_1$  is refined 176 by an area assignment  $\mathscr{A}_2$  of  $G_2$  if  $\mathscr{A}_1(f) = \sum_{s \in C_f} \mathscr{A}_2(s)$ . We also say  $\mathscr{A}_2$  refines  $\mathscr{A}_1$ . 177

Theorem 7. The angle graph Q of a plane triangulation T with  $s(T) \le 1$  is area-universal.

<sup>180</sup> Proof. Figure 3(a) illustrates the proof. Let e be an edge of T such that subdividing e<sup>181</sup> yields the area-universal graph  $T_{\circ}$ . The strategy is as follows: For an area assignment <sup>182</sup>  $\mathscr{A}$  of Q, we define a refining area assignment  $\mathscr{A}'$  of the union  $U := Q + T_{\circ}$ . Let  $\mathscr{A}_{\circ}$  be <sup>183</sup> the unique area assignment of  $T_{\circ}$  such that  $\mathscr{A}'$  refines  $\mathscr{A}_{\circ}$ . The drawing of  $T_{\circ}$  realizing <sup>184</sup>  $\mathscr{A}_{\circ}$  yields an  $\mathscr{A}'$ -realizing drawing of U.



(a) The graph  $U := Q + T_{\circ}$  where T is the octahedron graph.

(b) Faces in the neighborhood of the subdivided edge e.



For the definition of  $\mathscr{A}'$ , note that every face f of Q corresponds to two faces in U. Let  $q_e$  denote the face of Q that is split by the subdivided edge e in U. We arbitrarily partition the area  $\mathscr{A}(q)$  assigned to face q between the two corresponding faces in U, for all faces q in Q except the four faces  $q_1$ ,  $q_2$ ,  $q_3$ , and  $q_4$  sharing a boundary edge with  $q_e$ . Let  $f_1$  and  $f_2$  denote the two faces adjacent to e in T. For  $q_i$ ,  $i \in [4]$ , assign area  $\mathscr{A}(q_i)$  to the triangular face from U which is neither incident to  $f_1$  nor to  $f_2$  in U;

assign an area of 0 to the triangular faces incident to  $f_1$  or  $f_2$ . This defines the area 191 assignment  $\mathscr{A}'$  and  $\mathscr{A}_{\circ}$  is the area assignment of  $T_{\circ}$  that is refined by  $\mathscr{A}'$  of U. 192 Let  $D_{\circ}$  be an  $\mathscr{A}_{\circ}$ -realizing drawing of  $T_{\circ}$ . Since each vertex  $f \in F - \{f_1, f_2\}$  of U acts 193 as a vertex of degree 3 stacked into a face of  $T_{\circ}$ , we can insert f in  $D_{\circ}$  by Lemma 4 such 194 that the areas of  $\mathscr{A}'$  are realized. To obtain an  $\mathscr{A}'$ -realizing drawing of U, it remains to 195 insert  $f_1$  and  $f_2$ . We call the highlighted (thick, red) edges in Figure 3(b) incident to  $f_1$ 196 and  $f_2$ , the red edges of  $f_1$  and  $f_2$ , respectively. By definition of A', the red edges must 197 be contracted in every  $\mathscr{A}'$ -realizing drawing of U. Consequently, given a drawing  $T_{\circ}$ , we 198 easily insert  $f_1$  and  $f_2$  at the same location as the already placed vertex of the red edges, 199 respectively. This yields an  $\mathscr{A}'$ -realizing drawing of U. Since  $\mathscr{A}'$  refines  $\mathscr{A}$ , deleting the 200 edges of  $T_0$  yields an  $\mathscr{A}$ -realizing drawing of Q. 201

This ties in with a result based on an operation called *diamond addition*. Let G be a plane graph and e an edge incident to two triangular faces consisting of e and the vertices  $u_1$  and  $u_2$ , respectively. Applying a diamond addition of order k on edge e of G results in a graph G' in which the edge e is subdivided by vertices  $v_1, \ldots, v_k$  which are also adjacent to  $u_1$  and  $u_2$ , as illustrated in Figure 4. Two diamond additions are *disjoint* if the partitioned triangles are different.



Figure 4: A diamond addition on edge e.

- Theorem 8 ([15]). Let G be a graph obtained from an area-universal graph G' by (multiple disjoint) diamond addition(s) adding k vertices in total. Then,  $s(G) \leq k$ .
- As a special case of graphs obtained by diamond additions, Kleist [13] studied *double*
- <sup>211</sup> stacking graphs  $H_{\ell,k}$  that can be obtained from the plane octahedron graph. Labeling the octahedron as in Figure 5,  $H_{\ell,k}$  is obtained by applying one diamond addition of



Figure 5: A double stacking graph  $H_{\ell,k}$ .

order  $\ell - 1$  on Au and one diamond addition of order k - 1 on  $\nu w$ . The octahedron graph is the smallest graph of this class and has parameters  $\ell = k = 1$ .

<sup>215</sup> With ideas similar to the ones used to show Theorem 7, we obtain the following <sup>216</sup> results.

Theorem 9. The angle graph Q of a plane triangulation T is area-universal if one of the following holds: the following holds:

- (i) T is obtained from an area-universal triangulation T' by several disjoint dia mond additions of an arbitrary order.
- (ii) T has a set of edges that is far apart such that subdividing each of them at most once yields an area-universal subdivision  $T_o$ ; a set of edges of T is far apart if the subgraph of the dual graph induced by the duals of these edges and their vertices is a collection of stars.

The results imply the area-universality of several classes of angle graphs.

<sup>226</sup> Corollary 10. The angle graph Q of a plane triangulation T is area-universal if

• T is a stacked triangulation,

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- T is 4-connected and has at most ten vertices, or
- any (possibly a different) embedding of T is a double stacking graph  $H_{\ell,k}$ .

**Proof.** Stacked triangulations are area-universal, hence Proposition 6 implies the areauniversality of its angle graphs. Theorem 8 can be used to show that triangulations with at most nine vertices and all embeddings of  $\ell$ k-double stacking graphs have subdivision number at most 1 [15]. Consequently, Theorem 7 implies that their angle graphs are areauniversal. Moreover, 4-connected plane triangulations on ten vertices can be obtained from area-universal triangulations by at most two disjoint diamond additions. Thus, their area-universality follows from Theorem 9(i).

<sup>237</sup> It remains to show Theorem 9.

*Proof of Theorem 9.* To prove (i), we consider a diamond addition of order k applied 238 on an edge (u, w) of T'. Let T' denote the graph obtained from the triangulation T' 239 by subdividing the edge (u, w) with k additional vertices as in T; in other words the 240 edge (u, w) is replaced by a path P with k + 1 edges. Let A and B denote the two 241 common neighbors of u and w in T' such that Auw and Buw are faces in T'. Recall 242 that Q is the angle graph of T and consider the union  $U := Q + T'_0$ . Define H as the 243 restriction of U to the interior of AuBw. Figure 6 depicts H for a diamond addition of 244 order 3. 245

Given an area assignment  $\mathcal{A}$  of Q, we construct an area assignment  $\mathcal{A}'$  of T' and an area assignment  $\mathcal{A}_{U}$  of U that refines both  $\mathcal{A}$  and  $\mathcal{A}'$ . Observe that every face q of Q is either a face of U or corresponds to two faces of U.

In the latter case, we partition the prescribed area of q equally between its two faces in U and obtain the area assignment  $\mathcal{A}_{U}$  of U. The restriction of  $\mathcal{A}_{U}$  to H is denoted  $\mathcal{A}_{H}$ . We define  $\mathcal{A}'$  of T' as the area assignment refined by  $\mathcal{A}_{U}$ , where we identify the path P



Figure 6: The graph H for a diamond addition of order 3; together with  $\mathcal{A}_{H}$ .

and the edge (u, w). From an  $\mathcal{A}'$ -realizing drawing D' of T', we construct an  $\mathcal{A}_{U}$ realizing drawing of U as follows: First, we add all face vertices not adjacent to P using Lemma 4; recall that they act as vertex of degree 3 in a triangle. Next we discuss how to reinsert H.

We show the realizability of  $\mathscr{A}_H$  by considering the top and bottom half of H independently. A tent graph  $T_k$  is a plane graph with the outer face  $v, x_0, x_1, x_2, \ldots, x_{k+1}$  and inner vertices  $y_0, y_1, \ldots, y_k$  where  $y_i$  is incident to  $x_i, x_{i+1}$  and v. Figure 7(a) shows  $T_3$ . Observe that splitting H along P results in two tent graphs  $T_k$ .



Figure 7: Illustration of Lemma 11 and its proof.

Lemma 11. Every area assignment  $\mathcal{A}$  of a tent graph  $T_k$  has an  $\mathcal{A}$ -realizing drawing within each triangle that has area  $\Sigma \mathcal{A}$  and corners  $v, x_0, x_{k+1}$ . Moreover, the length of every segment  $x_i x_{i+1}$  can be made proportional to the area of the incident triangle.

<sup>264</sup> *Proof.* We denoted the assigned areas of  $T_k$  by  $a_i$  and  $b_i$  as depicted in Figure 7(b). We <sup>265</sup> position  $x_i$  on the segment  $x_0x_{k+1}$  such that

$$\|x_{i+1} - x_i\| = \frac{b_i}{\sum_i b_i} \|x_{k+1} - x_0\|.$$

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Then, in a realizing drawing the vertices  $y_i$  lie on a line  $\ell$  parallel to the segment  $x_0x_{k+1}$ 267 such that for each point x on  $\ell$  the triangle  $x_0 x_{k+1} x$  has area  $\sum_i b_i$ . Note that, by the 268 placement of  $x_i$ , each position of  $y_i$  on  $\ell$  realizes  $b_i$ . Thus, we may use the freedom 269 to realizing  $a_i$  when placing  $y_i$  on  $\ell$  with the following procedure: Defining  $y_{-1}$  as the 270 intersection of the segment  $vx_0$  with the line  $\ell$ , we suppose that  $y_{i-1}$  is placed already 271 when we consider  $y_i$  for i > 0. Move  $y_i$  rightwards on the line  $\ell$  starting at  $y_{i-1}$  and 272 observe the area of the face  $vy_{i-1}x_iy_i$ . Clearly, it starts at 0 and increases continuously. 273 The intermediate value theorem guarantees a position, where the area equals  $a_i$ . We 274 place  $y_i$  at the corresponding position and continue with  $y_{i+1}$ . Due to the correct total 275 area, the area of  $a_{k+1}$  is realized if all other face areas are correct. Thus, we obtain an 276  $\mathcal{A}$ -realizing drawing of  $T_k$ . 277

We use Lemma 11 to reinsert each of the two tent graphs of H. By definition of  $\mathcal{A}_{H}$ , the subdivision vertices on (u, w) are placed consistently when applying Lemma 11 to the two tent graphs when splitting H along P. Here we use the fact that the assigned areas of Q were split equally into two when defining  $\mathcal{A}_{U}$ . Since  $A_{U}$  refines  $\mathcal{A}$ , we obtain an  $\mathcal{A}$ -realizing drawing of Q by deleting the edges of  $T'_{o}$ , this is (i).

Now, we show (ii). First, we consider the case when the set of subdivided edges of T forms only one dual star, i.e. T has a face  $f_{\circ}$  such that subdividing each edge incident to  $f_{\circ}$  (at most once) yields an area-universal graph  $T_{\circ}$ . Theorem 7 shows the claim if exactly one edge of  $f_{\circ}$  is subdivided. In the following, we show how to deal with the case of three subdivision vertices. The case of two subdivision vertices can be handled by a slight modification which is explained afterwards.

We denote the three faces incident to subdivision vertices in  $T_{\circ}$  by  $f_{\circ}$ ,  $f_1$ ,  $f_2$ ,  $f_3$  as illustrated in Figure 8; the corresponding vertices in Q are denoted by  $f_{\circ}^{\nu}$ ,  $f_1^{\nu}$ ,  $f_2^{\nu}$ ,  $f_3^{\nu}$ . Moreover, we let  $v_1$ ,  $v_2$ ,  $v_3$  be the vertices of  $f_{\circ}$  and let  $w_i$  be the third vertex of  $f_i$  as depicted.



(a) Neighborhood of  $f_{\circ}$  in  $T_{\circ}$ .



(b) Neighborhood of  $f_{\circ}^{\nu}$  in U, where  $T_{\circ}$  is depicted in black and Q in gray.

Figure 8: Illustration of the notation for the proof of Theorem 9 (ii).

Note that two of the *w*-vertices may coincide implying that a *v*-vertex has degree 3. However, at most one *v*-vertex has degree 3; otherwise T is the complete graph on four



when  $v_3$  has degree 3.  $\mathcal{A}_{11}$  is given in orange.

Figure 9: Illustration of the proof of Theorem 9 (ii) for three subdivision vertices if there exists a vertex of degree 3.

vertices which is area-universal. Therefore, it remains to consider two cases: One v295 vertex has degree 3 and none v-vertex has degree 3. 296

We first consider the case that  $v_3$  has degree 3. Figure 9(a) illustrates the neighborhood 297 of  $f_{\circ}$  in this case. Our strategy is as follows: For an area assignment  $\mathscr{A}$  of Q, we define an 298 area assignment  $\mathcal{A}_{U}$  of the union  $Q+T_{\circ} =: U$ . This yields a unique area assignment  $\mathcal{A}_{\circ}$  of 299  $T_{\circ}$  such that  $\mathscr{A}_{U}$  of U refines  $\mathscr{A}_{\circ}$  of  $T_{\circ}$ . From an  $\mathscr{A}_{\circ}$ -realizing drawing of  $T_{\circ}$ , we construct 300 an  $\mathcal{A}_{U}$ -realizing drawing of U. Deleting the edges of  $T_{\circ}$  results in an  $\mathcal{A}$ -realizing drawing 301 of Q. 302

For a given area assignment  $\mathcal{A}$  of Q, we construct an area assignment  $\mathcal{A}_{U}$  of U such 303 that the area of every face q of Q is partitioned between the two faces of  $T_{o}$ . Let B 304 denote the edges of  $T_{\circ}$  bounding the region formed by the faces  $f_1, f_2, f_3, f_{\circ}$  in  $T_{\circ}$ ; note 305 that B has four edges since the common vertex of  $f_{\circ}$ ,  $f_2$ ,  $f_3$  has degree 3. The edges of B 306 divide four faces of Q in U. For all faces outside B (and not incident to B), the face 307 areas of Q are partitioned arbitrarily between the two faces of U. The area of each of 308 the four faces of Q divided by B is assigned to the subface outside B as indicated in 309 Figure 9(b). We define  $\mathcal{A}_{\circ}$  of  $T_{\circ}$  as the unique area assignment refining  $\mathcal{A}_{U}$ . 310

Given an  $\mathscr{A}_{\circ}$ -realizing drawing of  $T_{\circ}$ , note that every vertex of Q outside B is of 311 degree 3 and can be inserted with Lemma 4. We will redraw all vertices inside B. By 312 definition of  $\mathcal{A}_{U}$ , two incident triangular faces of  $f_{1}^{\nu}$  are supposed to be 0. Hence, we 313 must place  $f_1^{\nu}$  at the same location as  $w_1$ . Then, we place the vertex  $f_0^{\nu}$  as a vertex of 314 degree 2 in the quadrangle  $v_1 f_1^v v_2 w_2$  such that the area of the quadrangle  $v_1 f_1^v v_2 f_0^v$  is 315 correct, this is possible by Proposition 3. It remains to realize the graph G illustrated 316 in Figure 9(c). We later show that every area assignment of G is realizable within any 317 fixed outer face of correct total area: This follows from Lemma 15 and the fact that G 318 is the core of  $c(S_3)$ . 319

Now, we turn to the case that no  $v_i$  has degree 3. This implies that all  $w_i$  are distinct. 320 The resulting neighborhood of  $f_{\circ}^{\nu}$  is illustrated in Figure 10(a). 321





(a) Neighborhood of  $f_{\circ}$  in  $T_{\circ}$  and  $\mathscr{A}_{\circ}$ .

(b) Neighborhood of  $f_{\circ}$  in  $Q + T_{\circ}$  and  $\mathscr{A}_{U}$ .

# Figure 10: Illustration of the proof of Theorem 9 (ii) for three subdivision vertices of high degree.

Let  $\mathscr{A}$  be an area assignment of Q. In a first step, we define an area assignment  $\mathscr{A}_{U}$  of the union  $Q + T_{\circ} =: U$ . Note that every face of Q corresponds to two faces in U. Except for the faces incident to  $f_{\circ}^{\nu}, f_{1}^{\nu}, f_{2}^{\nu}, f_{3}^{\nu}$ , we arbitrarily partition the area  $\mathscr{A}(q)$  of a face q of Q between its two faces in U. For the faces incident to  $f_{\circ}^{\nu}, f_{1}^{\nu}, f_{2}^{\nu}, f_{3}^{\nu}$ , we assign their area as depicted in Figure 10(b).

let  $\mathcal{A}_{\circ}$  be the area assignment of  $T_{\circ}$  refining  $\mathcal{A}_{U}$ . Since  $T_{\circ}$  is area-universal there exists an  $\mathcal{A}_{\circ}$ -realizing drawing  $D_{\circ}$  of  $T_{\circ}$ . Most face-vertices of Q act as vertices of degree 3 stacked into triangles of  $T_{\circ}$ . Hence, by Lemma 4 and we can insert them in  $D_{\circ}$  such that they realize the area of  $\mathcal{A}_{U}$ . It remains to insert the vertices  $f_{1}^{\nu}, f_{2}^{\nu}, f_{3}^{\nu}$  and  $f_{\circ}^{\nu}$ . By definition of  $\mathcal{A}_{U}$ , we must place  $f_{i}^{\nu}$  as to coincide with  $w_{i}$ , i.e., such that the edge  $f_{i}^{\nu}w_{i}$ is contracted. To place  $f_{\circ}^{\nu}$  in  $D_{\circ}$ , we need some geometric considerations.

The area of  $f_{\circ}$  in  $D_{\circ}$  is 0, therefore at each  $v_i$  the two boundary edges of  $f_{\circ}$  are collinear. 333 If the slopes of the 3 supporting lines are pairwise different, then the triangle formed 334 by the three lines is in  $f_{\circ}$ , whence the triangle has area 0 which means that the three 335 lines intersect in a point p. This point is the position of the three subdivision vertices 336 in the drawing  $D_{\circ}$  and can be used for  $f_{\circ}^{\nu}$ . If two of the lines have the same slope, then 337 because they share one of the subdivision vertices they coincide. If the third line has a 338 different slope, then the intersection point of the lines is a good position for  $f_{\circ}^{\nu}$ . If all 339 three lines coincide there are many different foldings of the boundary of  $f_o$ , we leave it 340 as an exercise to show that in each case there is a position for  $f_{\circ}^{\nu}$  such that the edges to 341  $v_1, v_2, v_3$  can be drawn 'inside'  $f_{\circ}$ . 342

If there are two subdivision vertices on the boundary of  $f_{\circ}$  we use the area assignment  $\mathcal{A}_{U}$  and the corresponding  $\mathcal{A}_{\circ}$  as in the previous case. In the drawing  $D_{\circ}$  we pretend that the third edge of  $f_{\circ}$ . The considerations for the case of three subdivided edges show that there is a good position for  $f_{\circ}^{\circ}$ .

Since the set of subdivided edges is far apart, every subdivided edge belongs to a star. We handle each star separately as described above; in particular, the star consists of one, two or three edges since T is a triangulation. By the independence, for every two stars, the edges of T surrounding the regions of the stars are disjoint; these edges form a so-called *boundary cycle* of a star. For an example consider Figure 11.



Figure 11: An set of subdivided edges that is far apart.

Note that in all cases, when defining  $\mathcal{A}_{U}$  from  $\mathcal{A}$ , only the areas inside and adjacent to the boundary cycle are affected. Since these sets of faces in U are disjoint, the subdivision vertices can be handled independently. This finishes the proof.

### **4 Strongly Area-Universal Quadrangulations**

In this section, we study strongly area-universal quadrangulations. Recall that a quadrangulation is *strongly area-universal* if it is area-universal within every fixed outer face of the correct total area. A nice property of this class is that we can stack any strongly area-universal into a face of an (strongly) area-universal quadrangulation to obtain an (strongly) area-universal quadrangulation. Therefore, few small strongly area-universal quadrangulations can serve as building blocks in order to construct infinite and rich families of area-universal quadrangulations.

Note that for n > k, the area of a convex n-gon strictly exceeds the area of any contained k-gon. Therefore, we immediately obtain that the plane bipartite graph depicted in Figure 12(a) is not strongly area-universal: Fixing the outer face as a regular hexagon, there exists no drawing in which the inner 4-face covers more than 2/3 of the area.

<sup>367</sup> Observation 3. Not all plane bipartite graphs are strongly area-universal.



Figure 12: (a) A plane bipartite graph that is not strongly area-universal. (b) An illustration of the proof of Proposition 12.

In contrast, we do not know of a quadrangulation that is not strongly area-universal. Neither do we know of an area assignment that requires a convex outer face. In fact <sup>370</sup> these two questions are closely related.

**Proposition 12.** If there exists a plane quadrangulation G and an area assignment  $\mathcal{A}$ such that every realizing drawing has a convex outer face, then there exists a plane quadrangulation H that is not area-universal. Moreover, if G is 3-connected, we can ensure that H is 3-connected.

Proof. Suppose we are given a quadrangulation G and an area assignment  $\mathscr{A}$  with the described properties. Let H<sub>0</sub> denote the plane graph of K<sub>2,4</sub>, i.e., H<sub>0</sub> has three bounded faces each being a quadrangle. For each bounded face of H<sub>0</sub>, we take a copy of G and identify the outer 4-cycle of the copy of G with the boundary of the face. This yields the quadrangulation H as schematically illustrated in Figure 12(b). If G is 3-connected, then H is also 3-connected.

Assigning  $\mathscr{A}$  to each copy of G, we claim that H has no realizing drawing. Suppose, by contradiction, that there exists a realizing drawing D. Due to the positive total area of the central copy of G, either the right or the left copy of G has a non-convex boundary cycle in D. Consequently, this copy induces an  $\mathscr{A}$ -realizing drawing of G where the outer face is not convex and thus, we obtain a contradiction.

Now, we present a family of plane graphs that have the property of being strongly area-universal. The family consists of the angle graphs of wheels, which are also known as *pseudo-double wheels*. The *pseudo-double wheel*  $S_k$  has 2k + 2 vertices and consists of a cycle with vertices  $v_1, v_2, \ldots, v_{2k}$  and a vertex v adjacent to all vertices on the cycle with odd index and a vertex w adjacent to all vertices on the cycle with even index, see Figure 13. Up to the labeling the plane embedding of  $S_k$  is unique.



Figure 13: Pseudo-double wheel  $S_k$ .

The smallest pseudo-double wheel  $S_3$  is also known as the *cube graph*. In this section, we show that the cube graph – and more generally, all pseudo-double wheels – are strongly area-universal.

<sup>395</sup> Theorem 13. The pseudo-double wheel  $S_k$ ,  $k \ge 3$ , is strongly area-universal.

We can combine Lemma 2 and Theorem 13 in order to construct further strongly area-universal graphs. A graph is a *stacked pseudo-double wheel* if there exists a set of cycles such that decomposition along these cycles yields several pseudo-double wheels. A generalized stacked pseudo-double wheel can be decomposed into pseudo-double wheels and copies of the unique plane quadrangulation  $Q_5$  on five vertices. INote that  $Q_5$  is the plane graph that can be obtained by starting with a plane  $C_4$ , inserting an edge between two non-incident vertices and then subdividing this new edge. It is easy to see that  $Q_5$  is strongly area-universal [12]. Together with Lemma 2 and Theorem 13, it follows that

405 Corollary 14. Generalized stacked pseudo-double wheels are strongly area-universal.

We proceed to prove Theorem 13. To do so, we first study a subgraph of  $S_k$ , namely the plane graph  $c(S_k)$ , called the *core*, which is obtained by deleting  $v_1$ . Figure 14 illustrates the core of  $S_5$ .



(a) The core  $c(S_5)$  of  $S_5$  with an (b) The layout in case (i). (c) The layout in case (ii). area assignment.

Figure 14: Illustration of the proof of Lemma 15. The gray disks indicate that the contained vertices are placed at the same position, the center of the disk.

Lemma 15. Let  $c(S_k)$  be the core of a plane pseudo-double wheel with an area assignment A. Let q be a quadrangle of area  $\Sigma A$  containing the diagonal AB and whose corners are identified with the vertices A, w, B, v. Then,  $c(S_k)$  has a A-realizing drawing within q.

<sup>413</sup> *Proof.* We distinguish two cases. We call the faces of  $c(S_k)$  incident to w the *bottom* <sup>414</sup> *faces* and the faces incident to v the *top faces* of  $c(S_k)$ . For simplicity, we denote the <sup>415</sup> vertices  $v_2$  and  $v_{2k}$  by A and B, respectively, and the face areas by  $a_i$  for the top and by <sup>416</sup>  $b_i$  for the bottom faces. Consider also Figure 14(a).

<sup>417</sup> Case (i): If  $\sum_{i} b_i > AREA(\triangle AwB)$ , we position the even vertices on the segment AB <sup>418</sup> as illustrated in Figure 14(b). Note that adding the edges of consecutive even vertices <sup>419</sup> and Av and Bv (and deleting w) results in a tent graph. We partition the face area <sup>420</sup>  $b_i$  of each bottom face into  $b_i^1$  and  $b_i^2$  such that the ratio  ${}^{b_i^1/b_i^2}$  coincides for all i and <sup>421</sup>  $\sum_i b_i^2 = AREA(\triangle AwB)$ . By Lemma 11, the tent graph has a realizing drawing within <sup>422</sup> the triangle vAB. Due to the same ratio, the vertex placement on AB also realizes the <sup>423</sup> area for triangles incident to w. Figure 14(b) visualizes the realizing drawing of  $c(S_k)$ . <sup>424</sup> Case (ii): If  $\sum_{i} a_i > AREA(\triangle vAB)$ , we position the odd vertices on the segment AB as <sup>425</sup> illustrated in Figure 14(c). Note that the graph in the bottom triangle is a tent graph. <sup>426</sup> Therefore, we partition the area  $a_i$  of a top faces into  $a_i^1$  and  $a_i^2$  such that the ratio <sup>427</sup>  $a_i^1/a_i^2$  coincides for all i and  $\sum_i a_i^1 = AREA(\triangle vAB)$ . As in case (i), we use Lemma 11 to <sup>428</sup> find a realizing drawing of the tent graph. Figure 14(c) visualizes the realizing drawing <sup>429</sup> of  $c(S_k)$ .

430 This lemma helps us to settle three out of four cases of Theorem 13.

<sup>431</sup> Proof of Theorem 13. For an area assignment  $\mathscr{A}$  of  $S_k$ , we consider an arbitrary but <sup>432</sup> fixed quadrangle q of area  $\Sigma \mathscr{A}$  whose corners are identified with  $v_1v_2wv_{2k}$ . We distinguish <sup>433</sup> two cases depending on the shape of q. Note that q can be triangulated by the segment <sup>434</sup>  $v_2v_{2k}$  or  $v_1w$  (or both).

In case 1, the segment  $v_2v_{2k}$  lies inside q. We distinguish two subcases based on the assigned areas a and b of the faces incident to  $v_1$  relative to the area of the triangle  $v_1v_2v_{2k}$ .

<sup>438</sup> Case 1(i):  $a+b \leq AREA(v_1v_2v_{2k})$ . We can position v such that the triangles  $v_1v_2v$  and <sup>439</sup>  $v_1vv_{2k}$  realize a and b, respectively. The remaining graph corresponds to the core  $S_k$ <sup>440</sup> which we realize in the quadrangle  $vv_2wv_{2k}$  containing the segment  $v_2v_{2k}$  by applying <sup>441</sup> Lemma 15. Figure 15(a) visualizes the resulting layout.



Figure 15: Illustration of case 1 in the proof of Theorem 13 in which the segment  $v_2v_{2k}$  is contained in q. In case 1(i) the face areas incident to  $v_1$  are small; in case 1(ii) the face areas incident to  $v_1$  are big. Both cases reduce to Lemma 15.

Case 1(ii):  $a + b > AREA(v_1v_2v_{2k})$ . We position v on the segment  $v_2v_{2k}$  such that  $v_3$ and  $v_{2k-1}$  are forced to be on a line parallel to the segment  $v_2v_{2k}$ , i.e., v partitions  $v_2v_{2k}$ according to the ratio of a and b. The positions of  $v_3$  and  $v_{2k-1}$  on the line are such that the areas of the triangles  $v_2wv_3$  and  $v_{2k-1}wv_{2k}$  realize  $b_1$  and  $b_k$ . The graph induced by the vertices in the interior of  $vv_3wv_{2k-1}$  is the core of a smaller pseudo-double wheel and contains the diagonal  $v_3, v_{2k-1}$ . Consequently, Lemma 15 yields a realizing drawing, see Figure 15(b).

In case 2, the segment  $v_1w$  lies inside q. We call the faces incident to  $v_2$  the *left faces* and the faces incident to  $v_{2k}$  the *right faces*. We say the left (right) faces are *small* if their assigned area is at most the area of the triangle  $v_1, v_2, w$   $(v_1, w, v_{2k})$ . Otherwise, we call the left (right) faces *big*. Note that the left or right faces must be small. By symmetry, we assume without loss of generality, that the left faces are small. Then we can realize the left faces by triangular faces, by positioning  $v_3$  accordingly.



Figure 16: Illustration of case 2 in the proof of Theorem 13, in which the segment  $v_1w$  is contained in q. In case 2(i), the left and right faces are small; in case 2(ii), the left faces are small and the right faces are big.

<sup>455</sup> Case 2(i): the right faces are small as well; this case is illustrated in Figure 16(a). <sup>456</sup> We reduce this case to subgraphs of two stacked triangulations. To do so, we contract <sup>457</sup> the edge  $vv_1$  and realize the right faces by triangular faces with corner  $v_{2k-1}$ . Denote <sup>458</sup> the areas of the inner faces by  $x_i$  as illustrated in Figure 16(a). There exists an i with <sup>459</sup>  $4 \le i \le 2k - 1$  such that

$$a+c+\sum_{j=4}^{i-1}x_j\leq \texttt{AREA}(\triangle \nu_1\nu_2w) \quad \text{ and } \quad \sum_{j=i+1}^{2k-1}x_j+b+d\leq \texttt{AREA}(\triangle \nu_1w\nu_{2k}).$$

The exact layout depends on whether  $x_i$  is the area of a top or bottom face of the core. If it is a top face, then its unique vertex with odd index is placed at w. If  $x_i$  belongs to a bottom face, then the unique vertex with even index is placed at v. Afterwards we insert the remaining vertices. For j < i, we iteratively insert  $v_j$  such that it realizes the face area  $x_j$  by triangular face with a flat angle at  $v_{j+1}$  from left to right. For j > i, we follow the same strategy but in decreasing order.

Case 2(ii): the right faces are big. Figure 16(b) depicts this case. Recall that  $v_3$  has been fixed already. Place  $v_{2k-1}$  on  $v_1w$  such that the area of the quadrangle  $v_1v_3v_{2k-1}v_{2k}$ exceeds b but is not enough to also realize all top faces of the core, i.e., the striped faces incident to v in Figure 16(b). The remaining graph can be handled by Lemma 15: To do so, we choose  $vv_3wv_{2k-1}$  as the outer face and insert an artificial vertex on  $wv_{2k-1}$ .

### 472 **5** Quadrangulations and Convexity

A drawing of a planar graph is *convex* if each face is bounded by the boundary of a 473 convex polygon. Convexity is a visually appealing property of drawings of planar graphs 474 which has therefore been studied extensively in graph drawing. For example, Tutte's 475 spring embeddings [20] guarantee convex drawings for every 3-connected planar graph. 476 In this section, we aim for convex realizing drawings, i.e., given an area assignment  $\mathscr{A}$  of 477 a quadrangulation Q, we want to find an  $\mathscr{A}$ -realizing drawing of Q which is also convex. 478 A planar graph is *convex area-universal* if for every area assignment there exists a 479 convex realizing drawing. Although convex area-universality seems to be a very strong 480 property, there are examples of convex area-universal graphs, such as the cube graph. 481

<sup>482</sup> Proposition 16 ([12]). The cube graph is convex area-universal.

Indeed, this result can be generalized in two directions. First, the cube graph is convex area-universal for every convex outer face. Second, this holds not only for the cube graph but also for all pseudo-double wheels. We say a graph is *strongly convex area-universal* if for every area assignment  $\mathcal{A}$  and every convex drawing of the outer face with total area  $\Sigma \mathcal{A}$ , there exists a realizing drawing.

<sup>488</sup> Theorem 17. The pseudo-double wheel  $S_k$ ,  $k \ge 3$ , is strongly convex area-universal.

With Theorem 13 we have shown that pseudo-double wheels are area-universal the proof made ample use of Lemma 15. Since drawings obtained by using this lemma may contain non-convex faces we need an independent proof for Theorem 17.

*Proof.* Let  $\mathscr{A}$  be a given area assignment of  $S_k$ . We denote the areas assigned to the inner faces adjacent to the outer edges by a, b, c, d and the remaining areas by  $x_4, x_5, \ldots, x_{2k-1}$ where  $x_{2k-1} = c$  as depicted in Figure 17(a).



Figure 17: Illustration of convex realizing drawings of Theorem 17. (a) shows a pseudodouble wheel with a given area assignment. (b) and (c) illustrate the realizing drawings in case 1 and case 2 of the proof, respectively.

Let q be a convex quadrangle of area  $\Sigma \mathscr{A}$  with corners A, B, C, D which are identified with the outer vertices  $v_1, v_2, w, v_{2k}$  of  $S_k$ , respectively. Considering the diagonal AC, shows that at least one of the two following inequalities hold:  $a + b \leq AREA(\triangle ABC)$  or  $c + d \leq AREA(\triangle ACD)$ . By symmetry, we may assume that the first inequality holds. Hence, we may place  $v_3$  in the triangle ABC such that the areas a and b are realized by the triangular faces  $ABv_3$  and  $BCv_3$  where v and  $v_4$  are placed on  $Av_3$  and  $Cv_3$ , respectively. Now we distinguish two cases.

<sup>502</sup> Case 1:  $d \leq AREA(\triangle Av_3D)$ . By placing v on the segment  $Av_3$  we realize area d by <sup>503</sup> a triangular face. Split the quadrangle  $vv_3CD$  into two parts by the diagonal vC and <sup>504</sup> determine  $i \in [4, 2k - 1]$  such that

$$\sum_{j=4}^{\iota-1} x_j \leq \texttt{AREA}(\bigtriangleup \nu_3 C \nu) \quad \text{ and } \quad \sum_{j=\iota+1}^{2\kappa-1} x_j \leq \texttt{AREA}(\bigtriangleup \nu C D).$$

For all j < i, we realize the area  $x_j$  by a triangle, (namely  $uv_{j-1}v_j$  for u = v if j is even and u = w if j is odd), by placing  $v_j$  accordingly. Likewise for all j > i, we realize the area  $x_j$  by a triangle, (namely  $uv_jv_{j+1}$  for u = v if j is even and u = w if j is odd), by placing  $v_j$  accordingly. Finally, by placing  $v_i$  at v if i is odd and at w if i is even we the area  $x_i$  with the convex quadrangle  $vv_{i-1}wv_{i+1}$ . Figure 17(b) visualizes the resulting drawing.

<sup>512</sup> Case 2:  $d > AREA(\triangle Av_3D)$ . In this case we place v at  $v_3$  and  $v_4$  at w. We place  $v_{2k-1}$ <sup>513</sup> such that the area c and d are realized. In decreasing order, we place  $v_i$  such that  $x_i$  is <sup>514</sup> realized by the triangle  $uv_iv_{i+1}$  with u = v if i is even and u = w if i is odd. This yields <sup>515</sup> a realizing drawing within every convex outer face, see Figure 17(c).

### 516 5.1 Not all Quadrangulations are Convex Area-Universal

<sup>517</sup> Plane drawings of  $K_{2,n}$  have non-convex faces when  $n \ge 4$ . Tutte's spring embedding [20] <sup>518</sup> theorem, however, warrants that 3-connected quadrangulations have convex drawings. <sup>519</sup> In [12], it was asked whether all 3-connected quadrangulations are convex area-universal. <sup>520</sup> Here we answer this question in the negative.

### <sup>521</sup> Theorem 18. There is a 3-connected quadrangulation that is not convex area-universal.

<sup>522</sup> Proof. We show that the 3-connected quadrangulation Q, depicted in Figure 18(b), <sup>523</sup> has an area assignment which does not allow for a convex drawing in any (even non-<sup>524</sup> convex) outer face. The construction is based on a non-realizable area assignment  $\mathscr{A}$ <sup>525</sup> of the octahedron graph G [14]. The area assignment is such that the white faces in <sup>526</sup> Figure 18(a) have small area  $\varepsilon > 0$  and the gray faces have area 1.

Proposition 19 ([14], Theorem 1). For small enough  $\varepsilon > 0$ , the octahedron graph has no drawing where the white faces have area of at most  $\varepsilon$  and the gray faces have area of at least 1.

<sup>530</sup> We show that a convex drawing of Q induces an  $\mathscr{A}$ -realizing drawing of G, yielding a <sup>531</sup> contradiction. Let Q' be the subgraph of Q which is induced by the black vertices in <sup>532</sup> Figure 18(b). Note that Q' contains a 1-subdivision of the octahedron. We call the two <sup>533</sup> bipartition classes of Q' the *squared* and *circled* vertices.



(a) The octahedron graph G. There exists no drawing of G in which each white face has area 0 and each gray face has area 1.



(b) The quadrangulation Q and its black subgraph Q'. There exists no convex drawing of Q in which each white face has area 0 and each gray face has area 1/3.



Lemma 20. For small enough  $\varepsilon > 0$ , Q' has no drawing where the white faces have area  $\varepsilon$ , the gray faces have area 1, and each segment between two squared vertices is contained in some white face.

*Proof of Lemma 20.* Suppose, Q' has such a drawing D. Because each segment between two squared vertices is contained in some white face, these segments together with the squared vertices form a straight-line drawing D' of G where the white faces have area at most  $\varepsilon$  and the gray faces have area of at least 1. The red dotted graph in Figure 18(b) illustrates D'. This contradicts Proposition 19.

Suppose, by contradiction that for every  $\varepsilon > 0$ , Q has a convex drawing in which each white face has area  $\varepsilon/3$  and each gray face area 1/3. However, the induced drawing of Q' yields a counter example to Lemma 20. A contradiction. Consequently, Q is not convex area-universal.

Remark. Since we did not use the shape of the outer face in the proof of Theorem 18, the
quadrangulation Q does not even have an A-realizing drawing where we only require
that each inner face is convex.

Remark. The construction of Q in the proof of Theorem 18 is based on the octahedron
graph and Proposition 19. More generally, a white/gray-coloring of any Eulerian triangulation yields non-realizable area assignment [14]. This fact allows the construction of
a large family of quadrangulations that are not convex area-universal.

### **553** 6 Small Quadrangulations

<sup>554</sup> In this section we show that our methods are strong enough to prove area-universality for quadrangulations with up to 13 vertices via reductions to know area-universal graphs.

<sup>556</sup> Theorem 21. Every quadrangulation on at most 13 vertices is area-universal.

<sup>557</sup> *Proof.* First, it follows from Lemma 2 and Proposition 3 that a minimal non-area-<sup>558</sup> universal quadrangulation has minimum degree 3. Thus, the smallest quadrangulation <sup>559</sup> of interest is the cube graph on eight vertices. Figure 19 displays all quadrangulations on <sup>560</sup> up to 13 vertices with minimum degree 3. We denote them by  $Q_1, \ldots, Q_9$  as illustrated.



Figure 19: The planar quadrangulations on up to 13 vertices with min-degree 3. A single (double) checkmark indicates (strong) area-universality for all embeddings. The red edges form auxiliary area-universal triangulations.

Since all embeddings of a pseudo-double wheel  $S_k$  are equivalent, Theorem 17 proves the (strong) area-universality of pseudo-double wheels. Specifically, this shows the (strong) area-universality of  $Q_1 = S_3$ ,  $Q_2 = S_4$  and  $Q_4 = S_5$ . Moreover,  $Q_5$  is a stacked cube graph and  $Q_9$  is obtained from the cube by first adding a subdivided diagonal to a face and then stacking a cube graph in one of the two new faces. Since the cube graph is the double wheel  $S_3$ , both quadrangulations are area-universal by Corollary 14.

We reduce all remaining quadrangulations to area-universal triangulations. The quadrangulation Q<sub>3</sub> on 11 vertices is a subgraph of a stacked triangulation T, for which every embedding of T is area-universal. In Figure 19, the vertices and edges of the stacked triangulation are highlighted in red.

The three remaining quadrangulations are subgraphs of an area-universal graph family which was shown to be area-universal by Kleist [13].

Theorem ([13], Theorem 3). Any (embedding of a) double stacking graph  $H_{\ell,k}$  is areauniversal if and only if  $\ell \cdot k$  is even.

More precisely,  $Q_6$ ,  $Q_7$ ,  $Q_8$  are subgraphs of an area-universal double stacking graph 575 with some additional vertices of degree 3 stacked into triangular faces. Thus, their 576 area-universality follows from Observation 1 and the above theorem. In Figure 19 the 577 vertices which remain after iterative removal of degree-3 vertices are highlighted in red. 578 The quadrangulation  $Q_6$  on twelve vertices reduces to the double stacking graph  $H_{2,2}$ ; 579 the quadrangulations  $Q_7$  and  $Q_8$  on thirteen vertices reduce to the double stacking 580 graph  $H_{2,1}$ . The vertices in the interior of red dotted curves in Figure 19 are added by 581 diamond additions on the respective edge. 582

### **583** 7 Conclusions and Future Work

In this paper we develop several useful tools for the study of area-universality. With the help of these tools we provide a very simple proof that grid graphs are area-universal. We also prove that all quadrangulations with at most 13 vertices are area-universal. The natural question, whether all quadrangulations are area-universal remains an interesting open problem.

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