Arrangements of Approaching Pseudo-Lines

Stefan Felsner*[1], Alexander Pilz†[2], and Patrick Schnider[3]

[1]Institut für Mathematatik, Technische Universität Berlin,
felsner@math.tu-berlin.de
[2]Institute of SoftwareTechnology, Graz University of Technology, Austria,
apilz@ist.tugraz.at
[3]Department of Computer Science, ETH Zürich, Switzerland,
patrick.schnider@inf.ethz.ch

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Abstract

We consider arrangements of n pseudo-lines in the Euclidean plane where each pseudo-line ℓ_i is represented by a bi-infinite connected x-monotone curve $f_i(x)$, $x \in \mathbb{R}$, such that for any two pseudo-lines ℓ_i and ℓ_j with $i < j$, the function $x \mapsto f_j(x) - f_i(x)$ is monotonically decreasing and surjective (i.e., the pseudo-lines approach each other until they cross, and then move away from each other). We show that such *arrangements of approaching pseudo-lines*, under some aspects, behave similar to arrangements of lines, while for other aspects, they share the freedom of general pseudo-line arrangements. For the former, we prove:

- There are arrangements of pseudo-lines that are not realizable with approaching pseudo-lines.
- Every arrangement of approaching pseudo-lines has a dual generalized configuration of points with an underlying arrangement of approaching pseudo-lines.

For the latter, we show:

- There are $2^{\Theta(n^2)}$ isomorphism classes of arrangements of approaching pseudo-lines (while there are only $2^{\Theta(n \log n)}$ isomorphism classes of line arrangements).
- It can be decided in polynomial time whether an allowable sequence is realizable by an arrangement of approaching pseudo-lines.

Furthermore, arrangements of approaching pseudo-lines can be transformed into each other by flipping triangular cells, i.e., they have a connected flip graph, and every bichromatic arrangement of this type contains a bichromatic triangular cell.

1 Introduction

Arrangements of lines and, in general, arrangements of hyperplanes are paramount data structures in computational geometry whose combinatorial properties have been extensively

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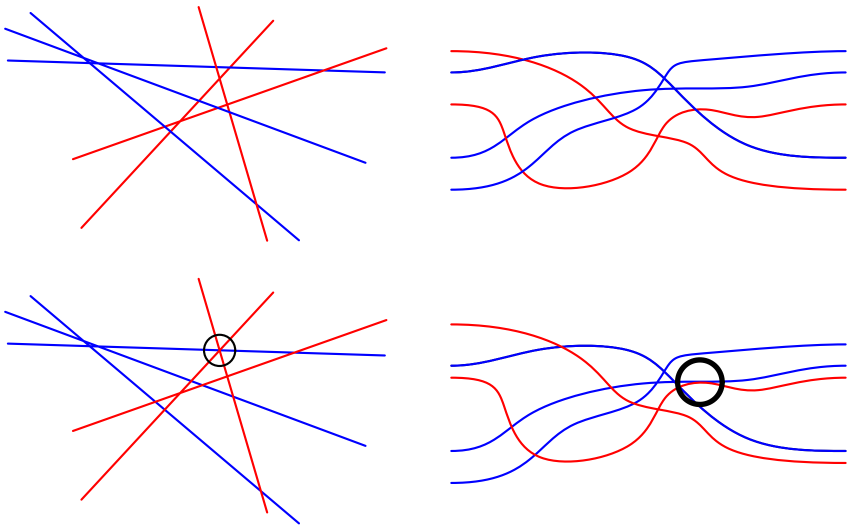

Figure 1: Vertical translation of the red lines shows that there is always a bichromatic triangle in a bichromatic line arrangement (left). For pseudo-line arrangements, a vertical translation may result in a structure that is no longer a valid pseudo-line arrangement (right).

studied, partially motivated by the point-hyperplane duality. Pseudo-line arrangements are a combinatorial generalization of line arrangements. Defined by Levi in 1926 the full potential of working with these structures was first exploited by Goodman and Pollack.

While pseudo-lines can be considered either as combinatorial or geometric objects, they also lack certain geometric properties that may be needed in proofs. The following example motivated the research presented in this paper.

Consider a finite set of lines that are either red or blue, no two of them parallel and no three of them passing through the same point. Every such arrangement has a bichromatic triangle, i.e., an empty triangular cell bounded by red and blue lines. This can be shown using a distance argument similar to Kelly's proof of the Sylvester-Gallai theorem (see, e.g., [2, p. 73]). We sketch another nice proof. Think of the arrangement as a union of two monochromatic arrangements in colors blue and red. Continuously translate the red arrangement in positive y-direction while keeping the blue arrangement in place. Eventually the combinatorics of the union arrangement will change with a triangle flip, i.e., with a crossing passing a line. The area of monochromatic triangles is not affected by the motion. Therefore, the first triangle that flips is a bichromatic triangle in the original arrangement. See Figure 1 (left).

This argument does not generalize to pseudo-line arrangements. See Figure 1 (right). Actually the question whether all simple bichromatic pseudo-line arrangements have bichromatic triangles is by now open for several years. The crucial property of lines used in the above argument is that shifting a subset of the lines vertically again yields an arrangement, i.e., the shift does not introduce multiple crossings. We were wondering whether any pseudo-line arrangement can be drawn such that this property holds. In this paper, we show that this is not true and that arrangements where this is possible constitute an interesting class of pseudo-line arrangements.

Define an *arrangement of pseudo-lines* as a finite family of x-monotone bi-infinite connected curves (called *pseudo-lines*) in the Euclidean plane such that each pair of pseudo-lines intersects in exactly one point, at which they cross. For simplicity, we consider the n pseudo-lines $\{\ell_1, \dots, \ell_n\}$ to be indexed from 1 to n in top-bottom order at left infinity.[1] A pseudo-line arrangement is *simple* if no three pseudo-lines meet in one point; if in addition no two pairs of pseudo-lines cross at the same x-coordinate we call it *x-simple*.

An *arrangement of approaching pseudo-lines* is an arrangement of pseudo-lines where each pseudo-line ℓ_i is represented by function-graph $f_i(x)$, defined for all $x \in \mathbb{R}$, such that for any two pseudo-lines ℓ_i and ℓ_j with $i < j$, the function $x \mapsto f_i(x) - f_j(x)$ is monotonically decreasing and surjective. This implies that the pseudo-lines approach each other until they cross, and then they move away from each other, and exactly captures our objective to vertically translate pseudo-lines in an arbitrary way while maintaining the invariant that the collection of curves is a valid pseudo-line arrangement (If $f_i - f_j$ is not surjective the crossing of pseudo-lines i and j may disappear upon vertical translations.) For most of our results, we consider the pseudo-lines to be *strictly approaching*, i.e., the function is strictly decreasing. For simplicity, we may sloppily call arrangements of approaching pseudo-lines *approaching arrangements.*

In this paper, we identify various notable properties of approaching arrangements. In Section 2, we show how to modify approaching arrangements and how to decide whether an arrangement is x-isomorphic to an approaching arrangement in polynomial time. Then, we show a specialization of Levi's enlargement lemma for approaching pseudo-lines and use it to show that arrangements of approaching pseudo-lines are dual to generalized configurations of points with an underlying arrangement of approaching pseudo-lines. In Section 5, we describe arrangements which have no realization as approaching arrangement. We also show that asymptotically there are as many approaching arrangements as pseudo-line arrangements. We conclude in Section 6 with a generalization of the notion of being approaching to three dimensions; it turns out that arrangements of approaching pseudo-planes are characterized by the combinatorial structure of the family of their normal vectors at all points.

Related work. Restricted representations of Euclidean pseudo-line arrangements have been considered already in early work about pseudo-line arrangements. Goodman [8] shows that every arrangement has a representation as a *wiring diagram.* More recently there have been results on drawing arrangements as convex polygonal chains with few bends [6] and on small grids [5]. Goodman and Pollack [11] consider arrangements whose pseudo-lines are the function-graphs of polynomial functions with bounded degree. In particular, they give bounds on the degree necessary to represent all isomorphism classes of pseudo-line arrangements. Generalizing the setting to higher dimensions (by requiring that any pseudo-hyperplane can be translated vertically while maintaining that the family of hyperplanes is an arrangement) we found that such approaching arrangements are representations of *Euclidean oriented matroids*, which are studied in the context of pivot rules for oriented matroid programming (see [4, Chapter 10]).

2 Manipulating approaching arrangements

Lemma 1 shows that we can make the pseudo-lines of approaching arrangements piecewise linear. This is similar to the transformation of Euclidean pseudo-line arrangements to equiv-

[1] Pseudo-line arrangements are often studied in the real projective plane, with pseudo-lines being simple closed curves that do not separate the projective plane. All arrangements can be represented by x-monotone arrangements [10]. As x-monotonicity is crucial for our setting and the line at infinity plays a special role, we use the above definition.

alent wiring diagrams. Before stating the lemma it is appropriate to briefly discuss notions of isomorphism for arrangements of pseudo-lines.

Since we have defined pseudo-lines as x-monotone curves there are two faces of the arrangement containing the points at ±infinity of vertical lines. These two faces are the *north-face* and the *south-face*. A *marked arrangement* is an arrangement together with a distinguished unbounded face, the north-face. Pseudo-lines of marked arrangements are oriented such that the north-face is to the left of the pseudo-line. We think of pseudo-line arrangements and in particular of approaching arrangements as being marked arrangements.

Two pseudo-line arrangements are *isomorphic* if there is an isomorphism of the induced cell complexes which maps north-face to north-face and respects the induced orientation of the pseudo-lines.

Two pseudo-line arrangements are *x-isomorphic* if a sweep with a vertical line meets the crossings in the same order.

Both notions can be described in terms of allowable sequences. An *allowable sequence* is a sequence of permutations starting with the identity permutation $\mathsf{id} = (1, \ldots, n)$ in which (i) a permutation is obtained from the previous one by the reversal of one or more non-overlapping substrings, and (ii) each pair is reversed exactly once. An allowable sequence is *simple* if two adjacent permutations differ by the reversal of exactly two adjacent elements.

Note that the permutations in which a vertical sweep line intersects the pseudo-lines of an arrangement gives an allowable sequence. We refer to this as *the allowable sequence* of the arrangement and say that the arrangement *realizes* the allowable sequence. Clearly two arrangements are x-isomorphic if they realize the same allowable sequence.

Replacing the vertical line for the sweep by a moving curve (vertical pseudo-line) which joins north-face and south-face and intersects each pseudo-line of the arrangement exactly once we get a notion of pseudo-sweep. A pseudo-sweep typically has various options for making progress, i.e., for passing a crossing of the arrangement. Each pseudo-sweep also produces an allowable sequence. Two arrangements are isomorphic if their pseudo-sweeps yield the same collection of allowable sequences or equivalently if there are pseudo-sweeps on the two arrangements which produce the same allowable sequence.

Lemma 1. *For any arrangement of approaching pseudo-lines, there is an x-isomorphic arrangement of approaching polygonal curves (starting and ending with a ray). If the allowable sequence of the arrangement is simple, then there exists such an arrangement without crossings at the bends of the polygonal curves.*

Proof. Consider the approaching pseudo-lines and add a vertical 'helper-line' at every crossing. Connect the intersection points of each pseudo-line with adjacent helper-lines by segments. This results in an arrangement of polygonal curves between the leftmost and the rightmost helper-line. See Figure 2. Since the original pseudo-lines were approaching, these curves are approaching as well; the signed distance between the intersection points with the vertical lines is decreasing, and this property is maintained by the linear interpolations between the points. To complete the construction, we add rays in negative x-direction starting at the intersection points at the first-helper line; the slopes of the rays are to be chosen such that their order reflects the order of the original pseudo-lines at left infinity. After applying the analogous construction at the rightmost helper-line, we obtain the x-isomorphic arrangement. If the allowable sequence of the arrangement is simple, we may choose the helper-lines between the crossings and use a corresponding construction. This avoids an incidence of a bend with a crossing. □

The construction used in the proof yields pseudo-lines being represented by polygonal curves with a quadratic number of bends. It might be interesting to consider the problem of

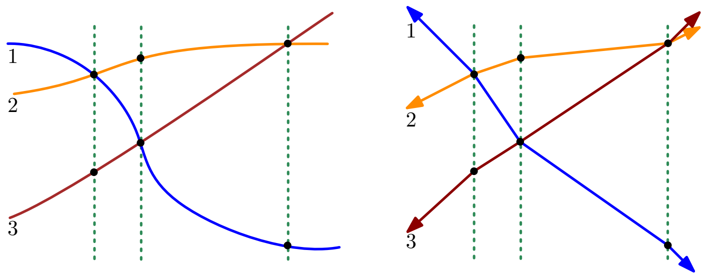

Figure 2: Transforming an arrangement of approaching pseudo-lines into an isomorphic one of approaching polygonal pseudo-lines.

minimizing bends in such polygonal representations of arrangements. Two simple operations which can help to reduce the number of bends are *horizontal stretching*, i.e., a change of the x-coordinates of the helper-lines which preserves their left-to-right order, and *vertical shifts* which can be applied a helper-line and all the points on it. Both operations preserve the x-isomorphism class.

The two operations are crucial for our next result, where we show that the intersection points with the helper-lines can be obtained by a linear program. Asinowski [3] defines a *suballowable sequence* as a sequence obtained from an allowable sequence by removing an arbitrary number of permutations from it. An arrangement thus realizes a suballowable sequence if we can obtain this suballowable sequence from its allowable sequence.

Theorem 1. *Given a suballowable sequence, we can decide in polynomial time whether there is an arrangement of approaching pseudo-lines with such a sequence.*

Proof. We attempt to construct a polygonal pseudo-line arrangement for the given suballowable sequence. As discussed in the proof of Lemma 1, we only need to obtain the points in which the pseudo-lines intersect vertical helper-lines through crossings. The allowable sequence of the arrangement is exactly the description of the relative positions of these points. We can consider the y-coordinates of pseudo-line ℓ_i at a vertical helper-line v_c as a variable $y_{i,c}$ and by this encode the suballowable sequence as a set of linear inequalities on those variables, e.g., to express that ℓ_i is above ℓ_j at v_c we use the inequality $y_{i,c} \geq y_{j,c} + 1$. Further, the curves are approaching if and only if $y_{i,c} - y_{j,c} \geq y_{i,c+1} - y_{j,c+1}$ for all $1 \leq i < j \leq n$ and c. These constraints yield a polyhedron (linear program) that is non-empty (feasible) if and only if there exists such an arrangement. Since the allowable sequence of an arrangement of n pseudo-lines consists of $\binom{n}{2} + 1$ permutations the linear program has $O(n^4)$ inequalities in $O(n^3)$ variables. Note that it is actually sufficient to have constraints only for neighboring points along the helper lines, this shows that $O(n^3)$ inequalities are sufficient. □

Let us emphasize that deciding whether an allowable sequence is realizable by a line arrangement is an $\exists\mathbb{R}$-hard problem [15], and thus not even known to be in NP. While we do not have a polynomial-time algorithm for deciding whether there is an isomorphic approaching arrangement for a given pseudo-line arrangement, Theorem 1 tells us that the problem is in NP, as we can give the order of the crossings encountered by a sweep as a certificate for a realization. The corresponding problem for lines is also $\exists\mathbb{R}$-hard [17].

The following observation is the main property that makes approaching pseudo-lines interesting.

Observation 1. *Given an arrangement A of strictly approaching pseudo-lines and a pseudo-line $\ell \in A$, any vertical translation of ℓ in A results again in an arrangement of strictly approaching pseudo-lines.*

Doing an arbitrary translation, we may run into trouble when the pseudo-lines are not strictly approaching. In this case it can happen that two pseudo-lines share an infinite number of points. The following lemma allows us replace non-strictly approaching arrangements by x-isomorphic strictly approaching arrangements.

Lemma 2. *Any simple arrangement of approaching pseudo-lines is homeomorphic to a polygonal x-isomorphic arrangement of strictly approaching pseudo-lines.*

Proof. Given an arrangement A, construct a polygonal arrangement A' as described for Lemma 1. If the resulting pseudo-lines are strictly approaching, we are done. Otherwise, consider the rays that emanate to the left. We may change their slopes such that all the slopes are different and their relative order remains the same. Consider the first vertical slab defined by two neighboring vertical lines v and w that contains two segments that are parallel (if there are none, the arrangement is strictly approaching). Choose a vertical line v' slightly to the left of the slab and use v' and w as helper-lines to redraw the pseudo-lines in the slab. Since the arrangement is simple the resulting arrangement is x-isomorphic and it has fewer parallel segments. Iterating this process yields the desired result. □

Lemma 3. *If A is an approaching arrangement with a non-simple allowable sequence, then there exists an approaching arrangement A' whose allowable sequence is a refinement of the allowable sequence of A, i.e., the sequence of A' may have additional permutations between consecutive pairs π, π' in the sequence of A.*

Proof. Since its allowable sequence is non-simple, arrangement A has a crossing point where more than two pseudo-lines cross or A has several crossings with the same x-coordinate. Let ℓ be a pseudo-line participating in such a degeneracy. Translating ℓ slightly in vertical direction a degeneracy is removed and the allowable sequence is refined. □

Ringel's homotopy theorem [4, Thm. 6.4.1] tells us that given a pair A, B of pseudo-line arrangements, A can be transformed to B by homeomorphisms of the plane and so-called *triangle flips*, where a pseudo-line is moved over a crossing. Within the subset of arrangements of approaching pseudo-lines, the result still holds. We first show a specialization of Ringel's isotopy result [4, Prop. 6.4.2]:

Lemma 4. *Two x-isomorphic arrangements of approaching pseudo-lines can be transformed into each other by a homeomorphism of the plane such that all intermediate arrangements are x-isomorphic and approaching.*

Proof. Given an arrangement A of approaching pseudo-lines, we construct a corresponding polygonal arrangement A'. Linearly transforming a point $f_i(x)$ on a pseudo-line ℓ_i in A to the point $f_i'(x)$ on the corresponding line ℓ_i' in A' gives a homeomorphism from A to A' which can be extended to the plane. Given two x-isomorphic arrangements A' and B of polygonal approaching pseudo-lines, we may shift helper-lines horizontally, so that the $\binom{n}{2} + 1$ helper-lines of the two arrangements become adjusted, i.e., are at the same x-coordinates; again there is a corresponding homeomorphism of the plane. Now recall that these arrangements can be obtained from solutions of linear programs. Since A' and B have the same combinatorial structure, their defining inequalities are the same. Thus, a convex combination of the variables defining the two arrangements is also in the solution space, which continuously takes us from A' to B and thus completes the proof. □

Theorem 2. *Given two simple arrangements of approaching pseudo-lines, one can be transformed to the other by homeomorphisms of the plane and triangle flips such that all intermediate arrangement are approaching.*

Proof. Let A_0 be a fixed simple arrangement of n lines. We show that any approaching arrangement A can be transformed into A_0 with the given operations. Since the operations are invertible this is enough to prove that if (A, B) is a pair of approaching arrangements, then A can be transformed into B with the given operations.

Consider a vertical line ℓ in A such that all the crossings of A are to the right of ℓ and replace the part of the pseudo-lines of A left of ℓ by rays with the slopes of the lines of A_0. This yields an arrangement A' isomorphic to A, see Fig. 3. This replacement is covered by Lemma 4. Let ℓ_0 be a vertical line in A_0 which has all the crossings of A_0 to the left. Now we vertically shift the pseudo-lines of A' to make their intersections with ℓ an identical copy of their intersections with ℓ_0. This yields an arrangement A'' isomorphic to A_0, see Fig. 3. During the shifting we have a continuous family of approaching arrangements which can be described by homeomorphisms of the plane and triangle flips. To get from A'' to A_0 we only have to replace the part of the pseudo-lines of A to the right of ℓ, where no crossings remain, by rays which have the same slopes of the lines of A_0. This makes all the pseudo-lines actual lines and the arrangement is identical to A_0. □

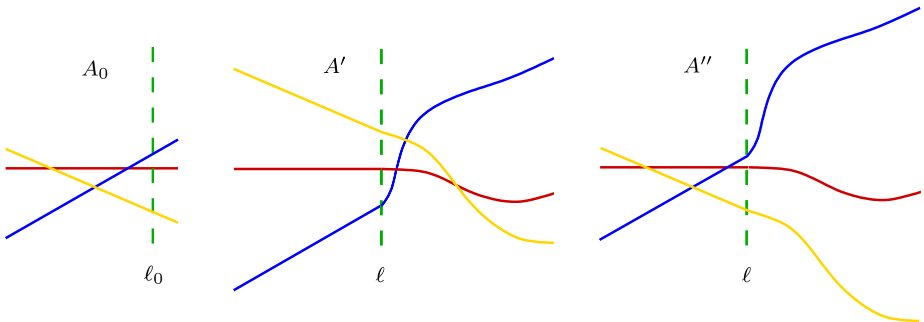

Figure 3: A line arrangement A_0 (left) and the arrangements A' and A'' used for the transformation from A to A_0.

Note that the proof requires the arrangement to be simple. Vertical translations of pseudo-lines now allows us to prove a restriction of our motivating question.

Theorem 3. *An arrangement of approaching red and blue pseudo-lines contains a triangular cell that is bounded by both a red and a blue pseudo-line unless it is a pencil, i.e., all the pseudo-lines cross in a single point.*

Proof. By symmetry in color and direction we may assume that there is a crossing of two blue pseudo-lines above a red pseudo-line. Translate all the red pseudo-lines upwards with the same speed. Consider the first moment $t > 0$ when the isomorphism class changes. This happens when a red pseudo-line moves over a blue crossing, or a red crossing is moved over a blue pseudo-line. In both cases the three pseudo-lines have determined a bichromatic triangular cell of the original arrangement.

Now consider the case that at time t parallel segments of different color are concurrent. In this case we argue as follows. Consider the situation at time $\varepsilon > 0$ right after the start of the motion. Now every multiple crossing is monochromatic and we can use an argument as in the proof of Lemma 2 to get rid of parallel segments of different colors. Continuing the translation after the modification reveals a bichromatic triangle as before. ☐

3 Levi's lemma for approaching arrangements

Proofs for showing that well-known properties of line arrangements generalize to pseudo-line arrangements often use Levi's enlargement lemma. (For example, Goodman and Pollack [9] give generalizations of Radon's theorem, Helly's theorem, etc.) Levi's lemma states that a pseudo-line arrangement can be augmented by a pseudo-line through any pair of points. In this section, we show that we can add a pseudo-line while maintaining the property that all pseudo-lines of the arrangement are approaching.

Lemma 5. *Given an arrangement of approaching pseudo-lines containing two pseudo-lines l_i and l_{i+1} (each a function $\mathbb{R} \mapsto \mathbb{R}$), consider $l' = l'(x) = \lambda l_i(x) + (1-\lambda)l_{i+1}(x)$, for some $0 \leq \lambda \leq 1$. The arrangement augmented by l' is still an arrangement of approaching pseudo-lines.*

Proof. Consider any pseudo-line l_j of the arrangement, $j \leq i$. We know that for $x_1 < x_2$, $l_j(x_1) - l_i(x_1) \geq l_j(x_2) - l_i(x_2)$, whence $\lambda l_j(x_1) - \lambda l_i(x_1) \geq \lambda l_j(x_2) - \lambda l_i(x_2)$. Similarly, we have $(1-\lambda)l_j(x_1) - (1-\lambda)l_{i+1}(x_1) \geq (1-\lambda)l_j(x_2) - (1-\lambda)l_{i+1}(x_2)$. Adding these two inequalities, we get

$$l_j(x_1) - l'(x_1) \geq l_j(x_2) - l'(x_2) \ .$$

The analogous holds for any $j \geq i+1$. ☐

The lemma gives us a means of producing a convex combination of two approaching pseudo-lines with adjacent slopes. Note that the adjacency of the slopes was necessary in the above proof.

Lemma 6. *Given an arrangement of n approaching pseudo-lines, we can add a pseudo-line $l_{n+1} = l_{n+1}(x) = l_n(x) + \delta(l_n(x) - l_{n-1}(x))$ for any $\delta > 0$ and still have an approaching arrangement.*

Proof. Assuming $x_2 > x_1$ implies

$$\begin{aligned} l_n(x_1) - l_{n+1}(x_1) &= l_n(x_1) - l_n(x) - \delta(l_n(x_1) - l_{n-1}(x_1)) = \delta(l_{n-1}(x_1) - l_n(x_1)) \\ &\geq \delta(l_{n-1}(x_2) - l_n(x_2)) = l_n(x_2) - l_{n+1}(x_2) \ . \end{aligned}$$

With $l_j(x_1) - l_n(x_1) \geq l_j(x_2) - l_n(x_2)$ we also get $l_j(x_1) - l_{n+1}(x_1) \geq l_j(x_2) - l_{n+1}(x_2)$ for all $1 \leq j < n$. ☐

Theorem 4. *Given an arrangement of strictly approaching pseudo-lines and two points p and q with different x-coordinates, the arrangement can be augmented by a pseudo-line l' containing p and q to an arrangement of approaching pseudo-lines. Further, if p and q do not have the same vertical distance to a pseudo-line of the initial arrangement, then the resulting arrangement is strictly approaching.*

Proof. Let p have smaller x-coordinate than q. Vertically translate all pseudo-lines such that they pass through p (the pseudo-lines remain strictly approaching, forming a pencil through p). If there is a pseudo-line that also passes through q, we add a copy l' of it. If q is between l_i and l_{i+1}, then we find some $0 < \lambda < 1$ such that $l'(x) = \lambda l_i(x) + (1-\lambda)l_{i+1}(x)$ contains p and q. By Lemma 5 we can add l' to the arrangement. If q is above or below all pseudo-lines in the arrangement, we can use Lemma 6 to add a pseudo-line; we choose δ large enough such that the new pseudo-line contains q. Finally translate all pseudo-lines back to their initial position. This yields an approaching extension of the original arrangement with a pseudo-line containing p and q. Observe that the arrangement is strictly approaching unless the new pseudo-line was chosen as a copy of l'. $\square$

Following Goodman et al. [14], a *spread of pseudo-lines* in the Euclidean plane is an infinite family of simple curves such that

1. each curve is asymptotic to some line at both ends,
2. every two curves intersect at one point, at which they cross, and
3. there is a bijection L from the unit circle C to the family of curves such that $L(p)$ is a continuous function (under the Hausdorff metric) of $p \in C$.

It is known that every projective arrangement of pseudo-lines can be extended to a spread [14] (see also [13]). For Euclidean arrangements this is not true because condition 1 may fail (for an example take the parabolas $(x-i)^2$ as pseudo-lines). However, given an Euclidean arrangement A we can choose two vertical lines v_- and v_+ such that all the crossings are between v_- and v_+ and replace the extensions beyond the vertical lines by appropriate rays. The result of this procedure is called the *truncation* of A. Note that the truncation of A and A are x-isomorphic and if A is approaching then so is the truncation. We use Lemma 5 to show the following.

Theorem 5. *The truncation of every approaching arrangement of pseudo-lines can be extended to a spread of pseudo-lines and a single vertical line such that the non-vertical pseudo-lines of that spread are approaching.*

Proof. Let $l_1, \ldots, l_n$ be the pseudo-lines of the truncation of an approaching arrangement. Add two almost vertical straight lines l_0 and l_{n+1} such that the slope of the line connecting two points on a pseudo-line l_i is between the slopes of l_0 and l_{n+1}. The arrangement with pseudo-lines $l_0, l_1, \ldots, l_n, l_{n+1}$ is still approaching. Initialize S with these $n+2$ pseudo-lines. For each $0 \leq i \leq n$ and each $\lambda \in (0,1)$ add the pseudo-line $\lambda l_i(x) + (1-\lambda) l_{i+1}(x)$ to S. The proof of Lemma 5 implies that any two pseudo-lines in S are approaching. Finally, let p be the intersection point of l_0 and l_{n+1} and add all the lines containing p and some point above these two lines to S. This completes the construction of the spread S. $\square$

4 Approaching generalized configurations

Levi's lemma is the workhorse in the proofs of many properties of pseudo-line arrangements. Among these, there is the so-called *double dualization* by Goodman and Pollack [10] that creates, for any arrangement of pseudo-lines, a corresponding primal generalized configuration of points.

A *generalized configuration of points* is an arrangement of pseudo-lines with a specified set of n vertices, called *points*, such that any pseudo-line passes through two points, and, at each point, $n-1$ pseudo-lines cross. We assume for simplicity that there are no other vertices in which more than two pseudo-lines of the arrangement cross.

Let $\mathcal{C} = (\mathcal{A}, P)$ be a generalized configuration of points consisting of an approaching arrangement $\mathcal{A}$, and a set of points $P = \{p_1, \ldots, p_n\}$, which are labeled by increasing x-coordinate. We denote the pseudo-line of $\mathcal{A}$ connecting points $p_i, p_j \in P$ by p_{ij}.

Consider a point moving from top to bottom at left infinity. This point traverses all the pseudo-lines of $\mathcal{A}$ in some order. We claim that if we start at the top with the identity permutation $\pi = (1, \ldots, n)$, then, when passing p_{ij} we can apply the (adjacent) transposition (i, j) to π. Moreover, by recording all the permutations generated during the move of the point we obtain an allowable sequence $\Pi_{\mathcal{C}}$.

Consider the complete graph K_P on the set P. Let c be an unbounded cell of the arrangement $\mathcal{A}$, when choosing c as the north-face of $\mathcal{A}$ we get a left to right orientation on each p_{ij}. Let this induce the orientation of the edge $\{i, j\}$ of K_P. These orientations constitute a tournament on P. It is easy to verify that this tournament is acyclic, i.e., it induces a permutation π_c on P.

- The order π corresponding to the top cell equals the left-to-right order on P. Since we have labeled the points by increasing x-coordinate this is the identity.

- When traversing p_{ij} to get from a cell c to an adjacent cell c' the two orientations of the complete graph only differ in the orientation of the edge $\{i, j\}$. Hence, π_c and π_c are related by the adjacent transposition (i, j).

The allowable sequence $\Pi_{\mathcal{C}}$ and the allowable sequence of $\mathcal{A}$ are different objects, they differ even in the length of the permutations.

We say that an arrangement of pseudo-lines is *dual* to a (*primal*) generalized configuration of points if they have the same allowable sequence. Goodman and Pollack [10] showed that for every pseudo-line arrangement there is a primal generalized configuration of points, and vice versa. We prove the same for the sub-class of approaching arrangements.

Lemma 7. *For every generalized configuration $\mathcal{C} = (\mathcal{A}, P)$ of points on an approaching arrangement $\mathcal{A}$, there is an approaching arrangement A^* with allowable sequence $\Pi_{\mathcal{C}}$.*

Proof. Let $\Pi_{\mathcal{C}} = \pi_0, \pi_1, \ldots, \pi_h$. We call (i, j) the adjacent *transposition at g* when $\pi_g = (i, j) \circ \pi_{g-1}$. To produce a polygonal approaching arrangement A^* we define the y-coordinates of the pseudo-lines $\ell_1, \ldots, \ell_n$ at x-coordinates $i \in [h]$. Let (i, j) be the transposition at g. Consider the pseudo-line p_{ij} of $\mathcal{C}$. Since p_{ij} is x-monotone we can evaluate $p_{ij}(x)$. The y-coordinate of the pseudo-line ℓ_k dual to the point $p_k = (x_k, y_k)$ at $x = g$ is obtained as $y_g(k) = p_{ij}(x_k)$.

We argue that the resulting pseudo-line arrangement is approaching. Let (i, j) and (s, t) be transpositions at g and g', respectively, and assume $g < g'$. We have to show that $y_g(a) - y_g(b) \geq y_{g'}(a) - y_{g'}(b)$, for all $1 \leq a < b \leq n$. From $a < b$ it follows that p_a is left of p_b, i.e., $x_a < x_b$. The pseudo-lines p_{ij} and p_{st} are approaching, hence $p_{ij}(x_a) - p_{st}(x_a) \geq p_{ij}(x_b) - p_{st}(x_b)$, i.e., $p_{ij}(x_a) - p_{ij}(x_b) \geq p_{st}(x_a) - p_{st}(x_b)$, which translates to $y_g(a) - y_g(b) \geq y_{g'}(a) - y_{g'}(b)$. This completes the proof. □

Goodman and Pollack use the so-called *double dualization* to show how to obtain a primal generalized configuration of points for a given arrangement A of pseudo-lines. In this process, they add a pseudo-line through each pair of crossings in A, using Levi's enlargement lemma. This results in a generalized configuration $\mathcal{C}'$ of points, where the points are the crossings of A. From this, they produce the dual pseudo-line arrangement $\mathcal{A}'$. Then, they repeat the previous process for $\mathcal{A}'$ (that is, adding a line through all pairs of crossings of $\mathcal{A}'$). The result is a generalized configuration $\mathcal{C}$ of points, which they show being the primal generalized configuration of $\mathcal{A}$. With Theorem 4 and Lemma 7, we know that both the augmentation through pairs of crossings and the dualization process can be done such that we again have approaching arrangements, yielding the following result.

Lemma 8. *For every arrangement of approaching pseudo-lines, there is a primal generalized configuration of points whose arrangement is also approaching.*

Combining Lemmas 7 and 8, we obtain the main result of this section.

Theorem 6. *An allowable sequence is the allowable sequence of an approaching generalized configuration of points if and only if it is the allowable sequence of an approaching arrangement.*

5 Realizability and counting

Considering the freedom one has in constructing approaching arrangements, one may wonder whether actually all pseudo-line arrangements are x-isomorphic to approaching arrangements. As we will see in this section, this is not the case. We use the following lemma, that can easily be shown using the construction from the proof of Lemma 1.

Lemma 9. *Given a simple suballowable sequence of permutations* $(\mathsf{id}, \pi_1, \pi_2)$*, where* id *is the identity permutation, the suballowable sequence is realizable with an arrangement of approaching pseudo-lines if and only if it is realizable as a line arrangement.*

Proof. Consider any realization A of the simple suballowable sequence with an arrangement of approaching pseudo-lines. Since the arrangement is simple, we can consider the pseudo-lines as being strictly approaching, due to Lemma 2. There exist two vertical lines v_1 and v_2 such that the order of intersections of the pseudo-lines with them corresponds to π_1 and π_2, respectively. We claim that replacing pseudo-line $p_i \in A$ by the line ℓ_i connecting the points $(v_1, p_i(v_1))$ and $(v_2, p_i(v_2))$ we obtain a line arrangement representing the suballowable sequence $(\mathsf{id}, \pi_1, \pi_2)$.

To prove the claim we verify that for $i < j$ the slope of ℓ_i is less than the slope of ℓ_j. Since A is approaching we have $p_i(v_1) - p_j(v_1) \geq p_i(v_2) - p_j(v_2)$, i.e., $p_i(v_1) - p_i(v_2) \geq p_j(v_1) - p_j(v_2)$. The slopes of ℓ_i and ℓ_j are obtained by dividing both sides of this inequality by $v_1 - v_2$, which is negative. □

Asinowski [3] identified a suballowable sequence $(\mathsf{id}, \pi_1, \pi_2)$, with permutations of six elements which is not realizable with an arrangement of lines.

Corollary 1. *There exist simple suballowable sequences that are not realizable by arrangements of approaching pseudo-lines.*

With the modification of Asinowski's example shown in Figure 4, we obtain an arrangement not having an isomorphic approaching arrangement. The modification adds two almost-vertical lines crossing in the north-cell such that they form a wedge crossed by the lines of Asinowski's example in the order of π_1. We do the same for π_2. The resulting object is a simple pseudo-line arrangement, and each isomorphic arrangement contains Asinowski's sequence.

Corollary 2. *There are pseudo-line arrangements for which there exists no isomorphic arrangement of approaching pseudo-lines.*

Aichholzer et al. [1] construct a suballowable sequence $(\mathsf{id}, \pi_1, \pi_2)$ on n lines such that all line arrangements realizing them require slope values that are exponential in the number of lines. Thus, also vertex coordinates in a polygonal representation as an approaching arrangement are exponential in n.

Ringel's Non-Pappus arrangement [19] shows that there are allowable sequences that are not realizable by straight lines. It is not hard to show that the Non-Pappus arrangement has a realization with approaching pseudo-lines. We will show that in fact the number of approaching arrangements, is asymptotically larger than the number of arrangements of lines.

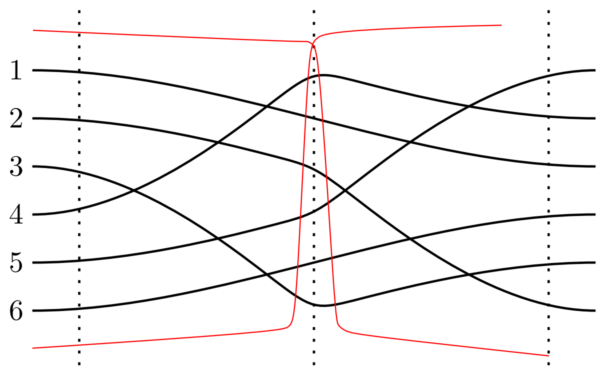

Figure 4: A part of a six-element pseudo-line arrangement (bold) whose suballowable sequence (indicated by the vertical lines) is non-realizable (adapted from [3, Fig. 4]). Adding the two thin pseudo-lines crossing in the vicinity of the vertical line crossed by the pseudo-lines in the order of π_1 and doing the same for π_2 enforces that the allowable sequence of any isomorphic arrangement contains the subsequence $(\mathsf{id}, \pi_1, \pi_2)$.

Theorem 7. *There exist $2^{\Theta(n^2)}$ isomorphism classes of simple arrangements of n approaching pseudo-lines.*

Proof. The upper bound follows from the number of non-isomorphic arrangements of pseudo-lines. Our lower-bound construction is an adaptation of the construction presented by Matoušek [16, p. 134] for general pseudo-line arrangements. See the left part of Figure 5 for a sketch of the construction. We start with a construction containing parallel lines that we will later perturb. Consider a set V of vertical lines $v_i : x = i$, for $i \in [n]$. Add horizontal pseudo-lines $h_i : y = i^2$, for $i \in [n]$. Finally, add parabolic curves $p_i : y = (x+i)^2 - \varepsilon$, defined for $x \geq 0$, some $0 < \varepsilon \ll 1$, and $i \in [n]$ (we will add the missing part towards left infinity later). Now, p_i passes slightly below the crossing of h_{i+j} and v_j at $(j, (i+j)^2)$. See the left part Figure 5 for a sketch of the construction. We may modify p_i to pass above the crossing at $(j, (i+j)^2)$ by replacing a piece of the curve near this point by a line segment with slope $2(i+j)$; see the right part of Figure 5. Since the derivatives of the parabolas are increasing and the derivatives of p_{i+1} at $j-1$ and of p_{i-1} at $j+1$ are both $2(j+i)$ the vertical distances from the modified p_i to p_{i+1} and p_{i-1} remain increasing, i.e., the arrangement remains approaching.

For each crossing $(j, (i+j)^2)$, we may now independently decide whether we want p_i to pass above or below the crossing. The resulting arrangement contains parallel and vertical lines, but no three points pass through a crossing. This means that we can slightly perturb the horizontal and vertical lines such that the crossings of a horizontal and a vertical remain in the vicinity of the original crossings, but no two lines are parallel, and no line is vertical. To finish the construction, we add rays from the points on p_i with $x = 0$, each having the slope of p_i at $x = 0$. Each arrangement of the resulting class of arrangements is approaching. We have $\Theta(n^2)$ crossings for which we make independent binary decisions. Hence the class consists of $2^{\Theta(n^2)}$ approaching arrangements of $3n$ pseudo-lines. □

As there are only $2^{\Theta(n \log n)}$ isomorphism classes of simple line arrangements [12], we see that we have way more arrangements of approaching pseudo-lines.

The number of allowable sequences is $2^{\Theta(n^2 \log n)}$ [20]. We show next that despite of the existence of nonrealizable suballowable sequences (Corollary 1), the number of allowable

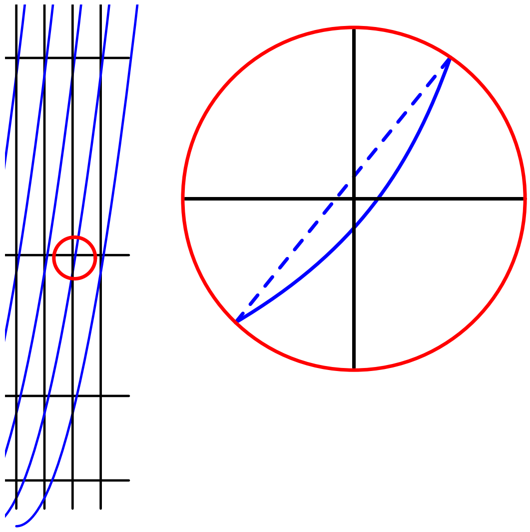

Figure 5: A construction for an $2^{\Omega(n^2)}$ lower bound on the isomorphism classes of approaching arrangements.

sequences for approaching arrangements, i.e., the number of x-isomorphism classes of these arrangements, is asymptotically the same as the number of all allowable sequences.

Theorem 8. *There are $2^{\Theta(n^2 \log n)}$ allowable sequences realizable as arrangements of approaching pseudo-lines.*

Proof. The upper bound follows from the number of allowable sequences. For the lower bound, we use the construction in the proof of Theorem 7, but omit the vertical lines. Hence, we have the horizontal pseudo-lines $h_i : y = i^2$ and the paraboloid curves $p_i : y = (x+i)^2 - \varepsilon$, defined for $x \geq 0$ and $0 < \varepsilon \ll 1$. For a parabolic curve p_i and a horizontal line h_{i+j}, consider the neighborhood of the point $(j, (i+j)^2)$. Given a small value α we can replace a piece of p_i by the appropriate line segment of slope $2(i+j)$ such that the crossing of h_{i+j} and the modified p_i has x-coordinate $j - \alpha$.

For fixed j and any permutation π of $[n-j]$ we can define values α_i for $i \in [n-j]$ such that $\alpha_{\pi(1)} < \alpha_{\pi(2)} < \ldots \alpha_{\pi(n-j)}$. Choosing the offset values α_i according to different permutations π yields different vertical permutations in the neighborhood of $x = j$, i.e., the allowable sequences of the arrangements differ. Hence, the number allowable sequences of approaching arrangements is at least the superfactorial $\prod_{j=1}^{n} j!$, which is in $2^{\Omega(n^2 \log n)}$. □

We have seen that some properties of arrangements of lines are inherited by approaching arrangements. It is known that every simple arrangement of pseudo-lines has $n-2$ triangles, the same is true for non-simple non-trivial arrangements of lines, however, there are non-simple non-trivial arrangements of pseudo-lines with fewer triangles, see [7]. We conjecture that in this context approaching arrangements behave like line arrangements.

Conjecture 1. *Every non-trivial arrangement of n approaching pseudo-lines has at least $n-2$ triangles.*

6 Approaching Arrangements in 3D

We have seen that approaching arrangements of pseudolines form an interesting class of arrangements of pseudolines. In this section we study the 3-dimensional version, this requires quite some technicalities. Therfore, before entering the detailed treatment of the subject we give an informal description of the results.

We consider pseudo-planes as functions $f : \mathbb{R}^2 \mapsto \mathbb{R}$. An arrangement of pseudo-planes is *approaching* if we can shift the pseudoplanes up and down independently and maintain the property that they form an arrangement.

Consider an arrangement of approaching pseudo-lines $f_1, f_2, \ldots, f_n$. Considering the slopes of the pseudo-lines over any point x we have $s_1 \leq s_2 \leq \ldots \leq s_n$, i.e., the point $(s_1, \ldots, s_n)$ is in the closure of the set of points with $s_1 < s_2 < \ldots < s_n$. We can sloppily state this as: The order of slopes is in the closure of the identity permutation. Now think of permutations as labeled Euclidean order types in one dimension.

In the case of arrangements of pseudo-planes we can talk about the tangent planes over a point (x, y) or, equivalently about the normals of the tangent planes. A set $n_1, n_2, \ldots, n_k$ of normals can equivalently be viewed as a labeled set of points in the plane. This set of points is an order type. It turns out that an approaching arrangement of pseudo-planes can be characterized by a non-degenerate order type χ in the sense that for every point (x, y) in the plane the order type of the normals over this point is in the closure of χ.

This ends the informal part.

An *arrangement of pseudo-planes* in $\mathbb{R}^3$ is a finite set A of function $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$ such that the intersection of any two of them projects to a pseudoline in $\mathbb{R}^2$ and the intersection of any three of them is a point. We define arrangements of approaching pseudo-planes via one of the key properties observed for arrangements of approaching pseudo-lines (Observation 1).

An *arrangement of approaching pseudo-planes* in $\mathbb{R}^3$ is an arrangement of pseudo-planes $h_1, \ldots, h_n$ where each pseudo-plane h_i is the graph of a continuously differentiable function $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$ such that for any $c_1, \ldots, c_n \in \mathbb{R}$, the graphs of $f_1 + c_1, \ldots, f_n + c_n$ form a valid arrangement of pseudo-planes. This means that we can move the pseudo-planes up and down along the z-axis while maintaining the properties of a pseudo-plane arrangement. Clearly, arrangements of planes (no two of them parallel) are approaching.

Let G be a collection of graphs of continuously differentiable functions $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$. For any point (x, y) in $\mathbb{R}^2$, let $n_i(x, y)$ be the upwards normal vector of the tangent plane of f_i above (x, y). We consider the vectors $n_i(x, y)$ as points $p_i(x, y)$ in the plane with homogeneous coordinates. (That is, for each vector we consider the intersection of its ray with the plane $z = 1$.) We call $p_i(x, y)$ a *characteristic point* and let $P_G(x, y)$ be the set of characteristic points. The Euclidean order type of the point multiset $P_G(x, y)$ is the *characteristic order type* of G at (x, y), it is denoted $\chi_G(x, y)$.

We denote by χ_G the set of characteristic order types of G on the whole plane, that is, $\chi_G = \{\chi_G(x, y) | (x, y) \in \mathbb{R}^2\}$. We say that χ_G is *admissible* if the following conditions hold:

(1) for any two points (x_1, y_1) and (x_2, y_2) in the plane, we have that if an ordered triple of characteristic points in $P_G(x_1, y_1)$ is positively oriented, then the corresponding triple in $P_G(x_2, y_2)$ is either positively oriented or collinear;

(2) for any triple p_1, p_2, p_3 of characteristic points, the set of points in the plane for which p_1, p_2, p_3 are collinear is either the whole plane or a discrete set of points (i.e, for each (x, y) in this set there is some $\varepsilon > 0$ such that the ε-disc around (x, y) contains no further point of the set);

(3) for any pair p_1, p_2 of characteristic points, the set of points in the plane for which $p_1 = p_2$ has dimension 0 or 1 (this implies that for each (x, y) in this set and each $\varepsilon > 0$ the ε-disc around (x, y) contains points which are not in the set).

From the above conditions, we deduce another technical but useful property of admissible characteristic order types.

Lemma 10. *Let χ_G be an admissible order type and $|G| \geq 3$. For any pair $p_1, p_2 \in \chi_G$ and for every point (x_0, y_0) in the plane for which $p_1 = p_2$ there is a neighborhood N such that for $V = \{p_2(x, y) - p_1(x, y) : (x, y) \in N\}$, the positive hull of V contains no line.*

Proof. Choose p_3 such that $p_3(x_0, y_0) \neq p_1(x_0, y_0) = p_2(x_0, y_0)$. In a small neighborhood N of (x_0, y_0) point p_3 will stay away from the line spanned by p_1 and p_2 (continuity). If in N the positive hull of V contains a line, then the orientation of p_1, p_2, p_3 changes from positive to negative in N, this contradicts condition (1) of admissible characteristic order types. $\square$

Theorem 9. *Let G be a collection of graphs of continuously differentiable functions $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$. Then G is an arrangement of approaching pseudo-planes if and only if χ_G is admissible and all the differences between two functions are surjective.*

Proof. Note that being surjective is a necessary condition for the difference of two functions, as otherwise we can translate them until they do not intersect. Thus, in the following, we will assume that all the differences between two functions are surjective. We first show that if χ_G is admissible then G is an arrangement of approaching pseudo-planes. Suppose G is not an arrangement of approaching pseudo-planes. Suppose first that there are two functions f_1 and f_2 in G whose graphs do not intersect in a single pseudo-line. Assume without loss of generality that $f_1 = 0$, i.e., f_1 is the constant zero function. Let $f_1 \cap f_2$ denote the intersection of the graphs of f_1 and f_2. If the intersection has a two-dimensional component, the normal vectors of the two functions are the same for any point in the relative interior of this component, which contradicts condition (3), so from now on, we assume that $f_1 \cap f_2$ is at most one-dimensional. Also, note that due to the surjectivity of $f_2 - f_1$, the intersection $f_1 \cap f_2$ is not empty. Note that if $f_1 \cap f_2$ is a single pseudo-line then for every $r \in f_1 \cap f_2$ there exists a neighborhood N in f_1 such that $f_1 \cap f_2 \cap N$ is a pseudo-segment. Further, on one side of the pseudo-segment, f_1 is below f_2, and above on the other, as otherwise we would get a contradiction to Lemma 10. In the next two paragraphs we argue that indeed $f_1 \cap f_2$ is a single pseudo-line. In paragraph (a) we show that for every $r \in f_1 \cap f_2$ the intersection locally is a pseudo-segment; in (b) we show that $f_1 \cap f_2$ contains no cycle and that $f_1 \cap f_2$ has a single connected component.

(a) Suppose for the sake of contradiction that $f_1 \cap f_2$ contains a point r such that for every neighborhood N of r in f_1 we have that $f_1 \cap f_2 \cap N$ is not a pseudo-segment. For $\varepsilon > 0$ let N_ε be the ε-disc around r. Consider ε small enough such that $f_1 \cap f_2 \cap N_\varepsilon$ consists of a single connected component. Further, let ε be small enough such that whenever we walk away from r in a component where f_2 is above (below) f_1, the difference $f_2 - f_1$ is monotonically increasing (decreasing). The existence of such an ε follows from the fact that f_1 and f_2 are graphs of continuously differentiable functions. Then $f_1 \cap f_2$ partitions N_ε into several connected components $C_1, \ldots, C_m$, ordered in clockwise order around r. In each of these components, f_2 is either above or below f_1, and this sidedness is different for any two neighboring components. In particular, the number of components is even, that is, $m = 2k$, for some natural number k. We will distinguish the cases where k is even and odd, and in both cases we will first show that at r we have $p_1 = p_2$ and then apply Lemma 10.

We start with the case where k is even. Consider a differentiable path γ starting in C_i, passing through r and ending in C_{k+i}. As k is even, f_2 is above f_1 in C_i if and only if f_2 is

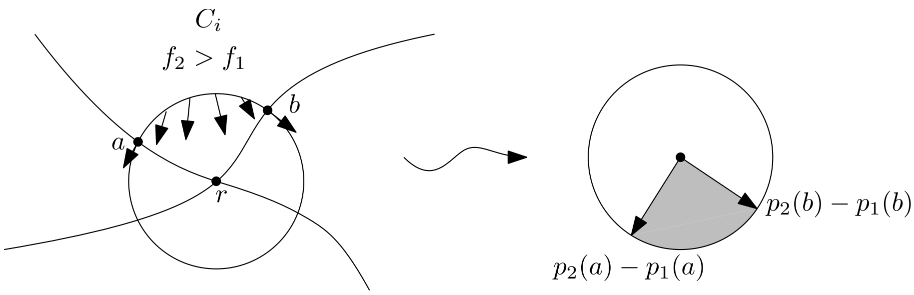

Figure 6: A component C_i induces many directions of $p_2 - p_1$.

also above f_1 in C_{k+i}. In particular, the directional derivative of $f_2 - f_1$ for γ at r is 0. This holds for every choice of i and γ, thus at r all directional derivatives of $f_2 - f_1$ vanish. This implies that at r the normal vectors of f_1 and f_2, coincide, hence $p_1 = p_2$. Now, consider the boundary of C_i. Walking along this boundary, $f_2 - f_1$ is the constant zero function, and thus the directional derivatives vanish. Hence, at any point on this boundary, $p_2 - p_1$ must be orthogonal to the boundary, pointing away from C_i if f_2 is above f_1 in C_i, and into C_i otherwise. Let now a and b be the intersections of the boundary of C_i with the boundary of N_ε. The above argument gives us two directions of vectors, $p_2(a) - p_1(a)$ and $p_2(b) - p_1(b)$, and a set of possible directions of vectors $p_2(c) - p_1(c)$, $c \in C_i$, between them. By continuity, all of these directions must be taken somewhere in C_i (see Figure 6 for an illustration). Let now C_+ be the set of all components where f_2 is above f_1, and let D_+ be the set of all directions of vectors $p_2(c) - p_1(c)$, $c \in C_+$. Further, let V_+ be the set of rays emanating from r which are completely contained in C_+. By continuity, for every small enough ε, there are two rays in V_+ which together span a line. It now follows from the above arguments, that for these ε, the directions in D_+ also positively span a line. This is a contradiction to Lemma 10.

Let us now consider the case where k is odd. Consider the boundary between C_{2k} and C_1 and denote it by γ_1. Similarly, let γ_2 be the boundary between C_k and C_{k+1}. Let now γ be the path defined by the union of γ_1 and γ_2 and consider the vectors $p_2 - p_1$ when walking along γ. Assume without loss of generality that $C_1 \in C_+$, and thus $C_{2k}, C_{k+1} \in C_-$ and $C_k \in C_+$. Analogous to the arguments in the above case, along γ the vectors $p_2 - p_1$ are orthogonal to γ, pointing from C_+ into C_-. In particular, they always point to the same side of γ. However, at r the path γ is also incident to $C_2 \in C_-$ and to $C_{k+2} \in C_+$. The same argument now shows that at r, the vector $p_2(r) - p_1(r)$ must point from C_{k+2} into C_2, that is, into the other side of γ. This is only possible if $p_2(r) - p_1(r) = 0$, and thus, as claimed, we again have $p_1 = p_2$ at r. We can now again consider the set of directions D_+, and this time, for every small enough ε, the set D_+ is the set of all possible directions (see Figure 7 for an illustration), which is again a contradiction to Lemma 10. This concludes the proof of claim (a).

(b) Suppose that the intersection $f_1 \cap f_2$ contains a cycle. In the interior of the cycle, one function is above the other, so we can vertically translate it until the cycle contracts to a point, which again leads to a contradiction to Lemma 10. Now suppose that the intersection contains two disjoint pseudo-lines. Between the pseudo-lines, one function is above the other, so we can vertically translate it until the pseudo-lines cross or coincide. If they cross, we are again in the case discussed in (a) and get a contradiction to Lemma 10. If they coincide, $f_2 - f_1$ has the same sign on both sides of the resulting pseudo-line which again leads to a contradiction to Lemma 10.

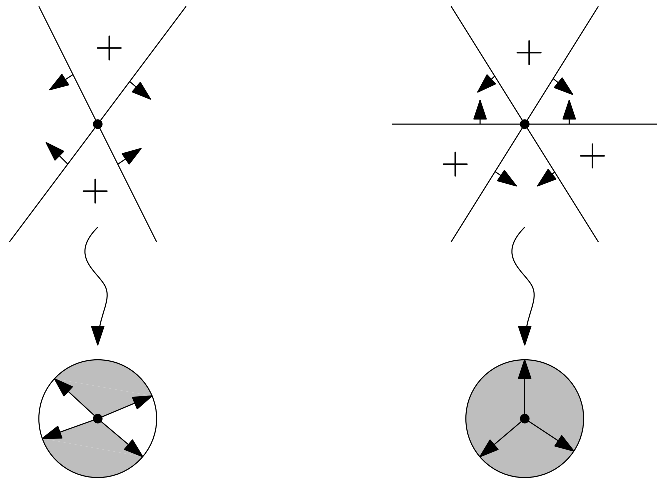

Figure 7: D_+ spans a line for k even (left) and contains all directions for k odd (right).

Thus, we have shown that if χ_G is admissible then any two pseudo-planes in G intersect in a single pseudo-line.

Now consider three functions f_1, f_2, f_3 such that any two intersect in a pseudo-line but the three do not form a pseudo-hyperplane arrangement. Then in one of the three functions, say f_1, the two pseudo-lines defined by the intersections with the other two functions do not form an arrangement of two pseudo-lines; after translation, we can assume that they touch at a point or intersect in an interval. First assume that they touch at a point. At this touching point, one normal vector of tangent planes is the linear combination of the other two: assume again without loss of generality that $f_1 = 0$. Further assume without loss of generality that the curves $f_2 \cap f_1$ and $f_3 \cap f_1$ touch at the point $(0, 0)$ and that the x-axis is tangent to $f_2 \cap f_1$ at this point. Then, as the two curves touch, the x-axis is also tangent to $f_3 \cap f_1$. In particular, the normal vectors to both f_2 and f_3 lie in the y-z-plane. As the normal vector to f_1 lies on the z-axis, the three normal vectors are indeed linearly dependent. For the order type, this now means that one vector is the affine combination of the other two, i.e., the three vectors are collinear. Further, on one side of the point the three vectors are positively oriented, on the other side they are negatively oriented, which is a contradiction to condition (1). On the other hand, if they intersect in an interval, then the set of points where the vectors are collinear has dimension greater than 0 but is not the whole plane, which is a contradiction to condition (2).

This concludes the proof that if χ_G is admissible then G is an arrangement of approaching pseudo-planes.

For the other direction consider an approaching arrangement of pseudo-planes and assume that χ_G is not admissible. First, assume that condition (1) is violated, that is, there are three pseudo-planes f_1, f_2, f_3 whose characteristic points p_1, p_2, p_3 change their orientation from positive to negative. In particular, they are collinear at some point. Assume without loss of generality that f_2 and f_3 are planes containing the origin whose characteristic points are thus constant, and assume without loss of generality that they are $p_2 = (0, 1)$ and $p_3 = (0, -1)$. In particular, the intersection of f_2 and f_3 is the x-axis in $\mathbb{R}^3$. Consider now a ε-disc B around the origin in $\mathbb{R}^2$ and let $B_<$, B_0 and $B_>$ be the subsets of B with $x < 0$, $x = 0$ and $x > 0$, respectively. Assume without loss of generality that in B the characteristic point p_1 is to the left of the y-axis in $B_<$, to the right in $B_>$, and on the y-axis in B_0. Also, assume that f_1 contains the origin in $\mathbb{R}^3$. But then, f_1 is below the (x, y)-plane everywhere in B. In particular, f_1 touches $f_2 \cap f_3$ in a single point, namely the origin. Hence, $f_1 \cap f_3$ and $f_2 \cap f_3$ is not an arrangement of two pseudo-lines in f_3.

Similar arguments show that

1. if condition (2) is violated, then after some translation the intersection of some two pseudo-planes in a third one is an interval,

2. if condition (3) is violated, then after some translation the intersection of some two pseudo-planes has a two-dimensional component,

□

On the other hand, from the above it does not follow to what extent an arrangement of approaching pseudo-planes is determined by its admissible family of characteristic order types. In particular, we would like to understand which admissible families of order types correspond to families of characteristic order types. To that end, note that for every graph in an arrangement of approaching pseudo-planes, the characteristic points define a vector field $F_i : \mathbb{R}^2 \mapsto \mathbb{R}^2$, namely its gradient vector field (a normal vector can be written as $(\mathrm{d}f(x), \mathrm{d}f(y), -1)$.) In particular, the set of all graphs defines a map $\phi(i, x, y)$ with the property that $\phi(i, \cdot, \cdot) = F_i$ and the order type of $\phi(\cdot, x, y)$ is $\chi_G(x, y)$. We call the family of vector fields obtained by this map the *characteristic field* of G. A classic result from vector analysis states that a vector field is a gradient vector field of a scalar function if and only if it has no curl. We thus get the following result:

Corollary 3. *Let $(F_1, \ldots, F_n)$ be a family of vector fields. Then $(F_1, \ldots, F_n)$ is the characteristic field of an arrangement of approaching pseudo-planes if and only if each F_i is curl-free and for each $(x, y) \in \mathbb{R}^2$, the set of order types defined by $F_1(x, y), \ldots, F_n(x, y)$ is admissible.*

Let now $G = (g_1, \ldots, g_n)$ be an arrangement of approaching pseudo-planes. A natural question is, whether G can be extended, that is, whether we can find a pseudo-plane g_{n+1} such that $(g_1, \ldots, g_n, g_{n+1})$ is again an arrangement of approaching pseudo-planes. Consider the realization of $\chi_G(x, y)$ for some $(x, y) \in \mathbb{R}^2$. Any two points in this realization define a line. Let $\mathcal{A}(x, y)$ be the line arrangement defined by all of these lines. Note that even if $\chi_G(x, y)$ is the same order type for every $(x, y) \in \mathbb{R}^2$, the realization might be different and thus there might be a point $(x', y') \in \mathbb{R}^2$ such that $\mathcal{A}(x', y')$ is not isomorphic to $\mathcal{A}(x, y)$. For an illustration of this issue, see Figure 8. (This issue also comes up in the problem of extension of order types, e.g. in [18], where the authors count the number of order types with exactly one point in the interior of the convex hull.)

We call a cell of $\mathcal{A}(x, y)$ *admissible*, if its closure is not empty in $\mathcal{A}(x', y')$ for every $(x', y') \in \mathbb{R}^2$. Clearly, if we can extend G with a pseudo-plane g_{n+1}, then the characteristic point p of the normal vector $n_{n+1}(x, y)$ must lie in an admissible cell c. On the other hand, as c is admissible, it is possible to move p continuously in c, and if all the vector fields $(F_1, \ldots, F_n)$ are curl-free, then so is the vector field F_{n+1} obtained this way. Thus, F_{n+1} is the vector field of a differentiable function f_{n+1} and by Corollary 3, its graph g_{n+1} extends G. In particular, G can be extended if and only if $\mathcal{A}(x, y)$ contains an admissible cell. As the cells incident to a characteristic point are always admissible, we get that every arrangement of approaching pseudo-planes can be extended. Furthermore, by the properties of approaching pseudo-planes, g_{n+1} can be chosen to go through any given point p in $\mathbb{R}^3$. In conclusion, we get the following:

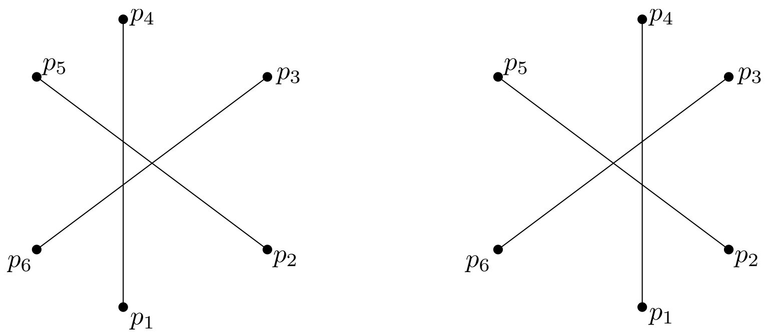

Figure 8: Two different arrangements induced by the same order type.

Theorem 10. *Let $G = (g_1, \ldots, g_n)$ be an arrangement of approaching pseudo-planes and let p be a point in $\mathbb{R}^3$. Then there exists a pseudo-plane g_{n+1} such that $(g_1, \ldots, g_n, g_{n+1})$ is an arrangement of approaching pseudo-planes and p lies on g_{n+1}.*

On the other hand, it could possible that no cell but the ones incident to a characteristic point are admissible, heavily restricting the choices for g_{n+1}. In this case, every pseudo-plane that extends G is essentially a copy of one of the pseudo-planes of G. For some order types, there are cells that are not incident to a characteristic point but still appear in every possible realization, e.g. the unique 5-gon defined by 5 points in convex position. It is an interesting open problem to characterize the cells which appear in every realization of an order type.

7 Conclusion

In this paper, we introduced a type of pseudo-line arrangements that generalize line arrangements, but still retain certain geometric properties. One of the main algorithmic open problems is deciding the realizability of a pseudo-line arrangement as an isomorphic approaching arrangement. Further, we do not know how projective transformations influence this realizability. The concept can be generalized to higher dimensions. Apart from the properties we already mentioned in the introduction, we are not aware of further non-trivial observations. Eventually, we hope for this concept to shed more light on the differences between pseudo-line arrangements and line arrangements. For higher dimensions, we gave some insight into the structure of approaching hyperplane arrangements via the order type defined by their normal vectors. It would be interesting to obtain further properties of this setting.

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