Arrangements of Approaching Pseudo-Lines

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Abstract

We consider arrangements of n pseudo-lines in the Euclidean plane where each pseudo-line ℓ_i is represented by a bi-infinite connected x-monotone curve $f_i(x)$, $x \in \mathbb{R}$, s.t. for any two pseudo-lines ℓ_i and ℓ_j with $i < j$, the function $x \mapsto f_j(x) - f_i(x)$ is monotonically decreasing (i.e., the pseudo-lines approach each other until they cross, and then move away from each other). We show that such *arrangements of approaching pseudo-lines*, under some aspects, behave similar to arrangements of lines, while for other aspects, they share the freedom of general pseudo-line arrangements. For the former, we prove that there are arrangements of pseudo-lines that are not realizable with approaching pseudo-lines. For the latter, we show: (i) There are $2^{\Theta(n^2)}$ isomorphism classes of arrangements of approaching pseudo-lines (while there are only $2^{\Theta(n \log n)}$ isomorphism classes of line arrangements). (ii) It can be decided in polynomial time whether an allowable sequence is realizable by an arrangement of approaching pseudo-lines.
Furthermore, arrangements of approaching pseudo-lines can be transformed into each other by flipping triangular cells, i.e., they have a connected flip graph, and any bichromatic such arrangement contains a bichromatic triangular cell.

1 Introduction

Arrangements of lines and, in general, arrangements of hyperplanes are paramount data structures in computational geometry, whose combinatorial properties have been extensively studied, partially motivated by the point-hyperplane duality. Many combinatorial properties of line arrangements are shared with (and actually were developed through) their generalization to pseudo-line arrangements. While pseudo-lines can be considered either as combinatorial or geometric objects, they lack certain geometric properties that may come in handy for proofs, as in the following example, which motivated the research presented here.

Consider a finite set of lines that are either red or blue, no two of them parallel and no three of them passing through the same point. Every such arrangement has a bichromatic triangular cell, i.e., an empty triangle defined both by red and blue lines. This can be shown using a distance argument similar to Kelly's proof of the Sylvester-Gallai theorem (see, e.g., [2, p. 73]). We sketch another nice proof. Suppose w.l.o.g. that no two crossings in the arrangement have the same x-coordinate (otherwise, slightly rotate the plane), and that there is a blue crossing above a red line. Continuously translate the arrangement of red lines in positive y-direction while keeping the arrangement of blue lines in place until a crossing lies on a line. Note that the crossing is monochromatic and the line has a different color. Hence, just before that event, the crossing is in the vicinity of the line and thus the three lines involved form a bichromatic triangle. However, until then, the combinatorial structure of the arrangement has not changed and thus the arrangement initially contained such a triangle.

This and all the known proofs for existence of a bichromatic triangle do not generalize to pseudo-line arrangements. Actually the question for pseudo-line arrangements is by now open for several years. The crucial property of lines used in the above argument is that shifting a subset of the lines vertically again yields an arrangement, i.e., the the shift does not introduce multiple crossings. We were wondering whether any pseudo-line arrangement can be drawn s.t. this property holds. In this abstract, we show that this is not true and that arrangements where this is possible constitute an interesting class of pseudo-line arrangements.

We define an *arrangement of pseudo-lines* as a finite family of x-monotone bi-infinite connected curves (called *pseudo-lines*) in the Euclidean plane s.t. each pair of pseudo-lines intersects in exactly one point, at which they cross. For simplicity, we consider the n pseudo-lines $\{\ell_1, \ldots, \ell_n\}$ to be indexed from 1 to n in top-bottom order at left infinity.[1] A pseudo-line arrangement is *simple* if no three pseudo-lines meet in one point; if in addition no two pairs of pseudo-lines cross at the same x-coordinate we call it *x-simple.*

An *arrangement of approaching pseudo-lines* is an arrangement of pseudo-lines where each pseudo-line ℓ_i

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[1] Pseudo-line arrangements are often studied in the real projective plane, with pseudo-lines being simple closed curves that do not separate the projective plane. All arrangements are isomorphic to x-monotone arrangements [10]. As x-monotonicity is crucial for our setting and the line at infinity plays a special role, we use the above definition.

is represented by function-graph $f_i(x)$, defined for all $x \in \mathbb{R}$, s.t. for any two pseudo-lines ℓ_i and ℓ_j with $i < j$, the function $x \mapsto f_j(x) - f_i(x)$ is monotonically decreasing (i.e., the pseudo-lines approach each other until they cross, and then they move away from each other). For most of our results, we may, as shown in the full version, consider the pseudo-lines to be *strictly approaching*, i.e., the function is strictly decreasing. For simplicity, we may sloppily call arrangements of approaching pseudo-lines *approaching arrangements*.

For two pseudo-lines, having a decreasing signed distance by increasing x-value is a property that is not maintained by a projective transformation, even if it maintains the vertical direction. We thus emphasize that approaching arrangements are defined in the Euclidean plane. This contrasts with related work that often considers pseudo-line arrangements in the projective plane. The special features of approaching arrangements are taken into account with the following notions of equivalence. Two pseudo-line arrangements are *sweep-equivalent* iff a sweep with a vertical line meets the crossings in the same order. Two pseudo-line arrangements are *vertically isomorphic* if there is an isomorphism of their face lattices (i.e., they dissect the Euclidean plane in a combinatorially equivalent way) that also respects the indexing of the pseudo-lines, i.e., their vertical order at left infinity. The reader may already have noticed that sweep-equivalence captures the allowable sequence of pseudo-line arrangements. For these also, the vertical direction plays a special role. An *allowable sequence* is a sequence of permutations in which (i) a permutation is obtained from the previous one by the reversal of one or more non-overlapping substrings, and (ii) each pair is reversed exactly once.[2] An allowable sequence is *simple* if two adjacent permutations differ by the reversal of exactly two adjacent elements. Hence, the permutations in which a vertical sweep line intersects the pseudo-lines of an arrangement gives an allowable sequence, starting with the identity permutation $I = \{1, \ldots, n\}$ of the pseudo-line's indices. The arrangement is said to *realize* that allowable sequence.

In this abstract, we identify various notable properties of approaching arrangements. In Section 2, we show how to modify approaching arrangements and how to decide whether an arrangement is sweep-equivalent to an approaching arrangement in polynomial time. In the following section, we provide arrangements without sweep-equivalent and vertically isomorphic approaching arrangements in the next section. Further, we show that there are asymptotically as many different approaching arrangements as pseudo-line arrangements.

[2] In the seminal work by Goodman and Pollack [9], "allowable sequences" have been defined as *periodic* sequences of permutations, where (i) a permutation is obtained from the previous one by the reversal of one or more non-overlapping substrings, and (ii) after the reversal of a pair ij, all other pairs are reversed before reversing i and j again [9]. This sequence is fully defined by a half-period, and we follow the frequent approach of calling that half-period an allowable sequence, as, e.g., in [4, p. 264]. Goodman and Pollack call arrangements with the same allowable sequence *combinatorially equivalent* [10].

Related work. Restricted representations of pseudo-line arrangements have been considered already at the early beginning of this concept. Goodman [8] considers their representation as *wiring diagrams*, and there are results on drawing arrangements as convex polygonal chains with few bends [6] and on small grids [5]. Any pseudo-line arrangement can be represented in these ways. Goodman and Pollack [11] consider the arrangements whose pseudo-lines are the function-graphs of polynomial functions with bounded degree. In particular, they give bounds on the degree necessary to represent all isomorphism classes of pseudo-line arrangements. Generalizing our setting to higher dimensions (by requiring that any pseudo-hyperplane can be translated vertically while maintaining that the family of hyperplanes is an arrangement) tells us that such approaching arrangements are representations of *Euclidean oriented matroids*, which occur in pivot rules for oriented matroid programming (see [4, Chapter 10]).

2 Manipulating approaching arrangements

One essential tool we use is the transformation of arrangements of general approaching pseudo-lines to pseudo-lines that are piecewise linear, similar to the transformation of pseudo-line arrangements to sweep-equivalent wiring diagrams.

Lemma 1 *For any arrangement of approaching pseudo-lines, there is a sweep-equivalent arrangement of approaching polygonal curves (starting and ending with a ray). If the allowable sequence of the arrangement is simple, then there exists such an arrangement without crossings at the bends of the polygonal curves.*

Proof. (Sketch.) We can place a vertical 'helper-line' at every crossing of the arrangement. Connect the intersection points of each pseudo-line with adjacent helper-lines by segments. (If the initial curves were approaching, these segments are as well.) This results in a sweep-equivalent arrangement of approaching polygonal curves. To complete the construction, we appropriately add rays in negative and positive x-direction. ☐

Actually we could have extended the segments from the first and the last vertical slab, respectively, as these segments are approaching and thus the crossings of the supporting lines of the segments in the, say, first vertical interval are not to the left of the first vertical helper-line. For the segments in the slabs, only the relative order of their slopes is relevant to maintain the "approaching" property.

Consider again the construction in the previous proof. After fixing the intersection points with the helper-lines, we may change the x-coordinate of these; as long as we maintain their relative order, the arrangement remains sweep-equivalent. Further, we may shift all the points on a helper-line up or down without alternating the combinatorial structure. We use this freedom for our next result, where we show that the intersection points with the helper lines can be obtained by a linear program. Asinowski [3] defines a *suballowable sequence* as a sequence obtained from an allowable sequence by removing an arbitrary number of permutations from it. An arrangement thus realizes a suballowable sequence if we can obtain this suballowable sequence from its allowable sequence.

Theorem 2 *Given a suballowable sequence, we can decide in polynomial time whether there is an arrangement of approaching pseudo-lines with such a sequence.*

Let us emphasize that deciding whether an allowable sequence is realizable by a line arrangement is an $\exists\mathbb{R}$-hard problem [13], and thus not even known to be in NP. While we do not have a polynomial-time algorithm for deciding whether there is a vertically isomorphic approaching arrangement for a given pseudo-line arrangement, Theorem 2 tells us that the problem is in NP, as we can give the order of the crossings encountered by a sweep as a certificate for a realization. The corresponding problem for lines is also $\exists\mathbb{R}$-hard [14].

The following observation is the main property that makes approaching pseudo-lines interesting.

Observation 1 *Given an arrangement of strictly approaching pseudo-lines, any vertical translation of any pseudo-line results again in an arrangement of strictly approaching pseudo-lines.*

Lemma 3 *For any simple approaching arrangement that is not x-simple, there exists an approaching arrangement that is sweep-equivalent apart from the crossings sharing the x-coordinate.*

For pseudo-line arrangements, Ringel's homotopy theorem [15] tells us that one can be transformed to any other by homeomorphisms of the plane and so-called *triangle flips*, where a pseudo-line is moved over the crossing between two other ones. For the subset of arrangements of approaching pseudo-lines, the result can be specialized in a way that the intermediate arrangements are also approaching. We first show the following specialization of Ringel's isotopy result [15].

Lemma 4 *Two sweep-equivalent arrangements of approaching pseudo-lines can be transformed into each other by a homeomorphism of the plane s.t. all intermediate arrangements are sweep-equivalent and consist of approaching pseudo-lines.*

Theorem 5 *Given two simple arrangements of approaching pseudo-lines, one can be transformed to the other by homeomorphisms of the plane and triangle swaps s.t. all intermediate arrangement are approaching.*

We note that the relative position of the crossings may change during this process. Also, our proof requires the arrangement to be simple.

Vertically translating pseudo-lines now allows us to prove a restriction of our motivating question.

Theorem 6 *Any simple arrangement of approaching red and blue pseudo-lines contains a triangular cell that is bounded by both a red and a blue pseudo-line.*

3 Properties

Considering the freedom one has in constructing approaching arrangements, one may wonder whether actually all pseudo-line arrangements are sweep-equivalent to approaching arrangements. However, this is not true, as we will see in this section. We use the following lemma, that can easily be shown using the construction for Lemma 1.

Lemma 7 *Given a simple suballowable sequence of permutations (I, π_1, π_2), where I is the identity permutation, the suballowable sequence is realizable with an arrangement of approaching pseudo-lines if and only if it is realizable as a line arrangement.*

Asinowski [3] provided such a suballowable sequence on six lines that is not realizable as a line arrangement.

Corollary 8 *There exist simple suballowable sequences that are not realizable as arrangements of approaching pseudo-lines.*

In Figure 1, we modify the construction to an arrangement not having an isomorphic approaching arrangement. The resulting object is a simple pseudo-line arrangement, and each vertically isomorphic arrangement contains Asinowski's sequence.

Corollary 9 *There are pseudo-line arrangements for which there exists no vertically isomorphic arrangement of approaching pseudo-lines.*

Aichholzer et al. [1] construct a suballowable sequence (I, π_1, π_2) on n lines s.t. all line arrangements realizing them require slope values that are exponential in the number of lines. Thus, also vertex coordinates in a polygonal representation as an approaching arrangement are exponential in n (but their size is not).

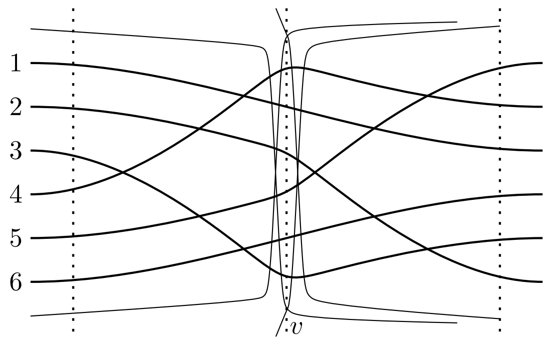

Figure 1: A part of a six-element pseudo-line arrangement (bold) whose suballowable sequence (indicated by the vertical lines) is non-realizable (adapted from [3, Fig. 4]). Adding the four thin pseudo-lines crossing in the vicinity of vertical v enforces that the allowable sequence of any vertically isomorphic arrangement contains the six-element permutation indicated by v.

Even though we have non-realizability results for approaching arrangements, their number is much larger than the number of arrangements of lines.

Theorem 10 *There exist $2^{\Theta(n^2)}$ isomorphism classes of simple arrangements of n approaching pseudo-lines.*

As there are only $2^{\Theta(n \log n)}$ isomorphism classes of simple line arrangements [12], we see that we have way more arrangements of approaching pseudo-lines. While the number of allowable sequences is $2^{\Theta(n^2 \log n)}$ [16], there are only at most n^{8n} combinatorially different line arrangements [12]. So arrangements of approaching pseudo-lines also differ in this setting, as the previous proof shows. However, we do not know whether our bound obtained via isomorphism classes is already asymptotically tight for allowable sequences.

Concerning further aspects where approaching arrangements are similar to line arrangements, we make the following conjecture, whose analogues hold for lines but not for pseudo-lines [7]. But so far we were unable to give a full proof.

Conjecture 1 *Every arrangement of n approaching pseudo-lines has at least $n-2$ triangular cells.*

4 Conclusion

In this paper, we introduced a type of pseudo-line arrangements that generalize line arrangements, but still retain certain geometric properties. One of the main algorithmic open problems is deciding the realizability of a pseudo-line arrangement as a vertically isomorphic approaching arrangement. Further, we do not know how projective transformations influence this realizability. The concept can be generalized to higher dimensions. Apart from the properties we already mentioned in the introduction, we are not aware of further non-trivial observations. Eventually, we hope for this concept to shed more light on the differences between pseudo-line arrangements and line arrangements.

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