On the Interplay between Interval Dimension and Dimension *

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Abstract

This paper investigates a transformation $P \to Q$ between partial orders P, Q that transforms the interval dimension of P to the dimension of Q, i.e., $idim(P) = dim(Q)$. Such a construction has been shown before in the context of Ferrer's dimension by Cogis [2]. Our construction can be shown to be equivalent to his, but it has the advantage of (1) being purely order-theoretic, (2) providing a geometric interpretation of interval dimension similar to that of Ore [15] for dimension, and (3) revealing several somewhat surprising connections to other order-theoretic results.

For instance, the transformation $P \to Q$ can be seen as almost an inverse of the well-known split operation, it provides a theoretical background for the influence of edge subdivision on dimension (e.g., the results of Spinrad [17]) and interval dimension, and it turns out to be invariant with respect to changes of P that do not alter its comparability graph, thus providing also a simple new proof for the comparability invariance of interval dimension.

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1 Introduction

An *extension* Q of a partial order P is an order on the same elements that contains all the ordered pairs of P, i.e., $x < y$ in P implies $x < y$ in Q. A family $\{Q_1, \ldots, Q_k\}$ of extensions of P is said to *realize* P or to be a *realizer* of P iff $P = Q_1 \cap \ldots \cap Q_k$, i.e., $x < y$ in P iff $x < y$ in Q_i for each i, $1 \leq i \leq k$. If we restrict the Q_i to belong to a special class of orders and seek for a minimum size realizer we come up with a concept of dimension with respect to the special class.

A *linear extension* of P is an extension which is a chain. Dushnik and Miller [5] defined the *dimension* of a partial order P, denoted $dim(P)$, as the smallest integer k for which there exist k linear extensions realizing P. Let the realizer $\{L_1, \ldots, L_k\}$ of an order P with $|P| = n$ be made by the linear extensions $L_i = (x_{i1} < x_{i2} < \ldots < x_{in})$. With every $x \in P$ we then associate the vector $(x^1, \ldots, x^k) \in \mathbf{R}^k$, where x^i gives the position (coordinate) of x in L_i. This mapping of the points of P to points of $\mathbf{R}^k$ embeds P into the componentwise ordering of $\mathbf{R}^k$. Ore [15] defined $dim(P)$ as the minimum k such that P embeds into $\mathbf{R}^k$ in this way. The projections of such an embedding on each coordinate yield a realizer of P. The extensions of this realizer need not be linear extensions of P, but it is straightforward to transform them into linear extensions. Therefore, Ore dimension and Dushnik-Miller dimension are equivalent concepts.

A partial order P is an *interval order* if it can be represented by assigning to each element $x \in P$ an open interval $I_x = (a_x, b_x)$ of the real line, such that $x < y$ in P iff $b_x \leq a_y$. Such a collection of intervals is called an *interval representation* of P. Fishburn [7] characterized interval orders as those orders that do not contain a 4–point subset forming two disjoint 2–chains, i.e., that contain no **2+2**. Interval dimension, denoted $idim(P)$, is defined by using interval extensions instead of linear extensions. Since linear orders are interval orders we obtain the trivial inequality

$$idim(P) \quad \leq \quad dim(P) \tag{1}$$

It is well known that interval orders of large dimension exist (see [16, 1, 9]). Hence, the gap between $idim(P)$ and $dim(P)$ may be arbitrarily large.

Let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be an interval realizer of P and fix an open interval representation of each interval order I_j. Let (a_x^j, b_x^j) be the interval corresponding to $x \in P$ in the representation of I_j. We now define a *box embedding* of P to $\mathbf{R}^k$. With $x \in P$ we associate the box $\prod_j (a_x^j, b_x^j) \subseteq \mathbf{R}^k$. Each of these boxes is uniquely determined by its *upper extreme corner* $u_x = (b_x^1, \ldots, b_x^k)$ and its *lower extreme corner* $l_x = (a_x^1, \ldots, a_x^k)$. Obviously $x < y$ in P iff $u_x \leq l_y$ componentwise. The projections of a box embedding onto each coordinate yield an interval realizer, so the concepts of box embeddings and interval realizers are equivalent. For interval dimension the box embeddings thus play the role of the above-mentioned point embeddings into $\mathbf{R}^k$ introduced by Ore for dimension.

A box embedding depends not only on the realizer $\mathcal{I}$ of P, but also on the representations of the I_j. Now we define the *partial order* $\mathcal{B}(\mathcal{I})$ *of extreme corners* associated with a box embedding or, equivalently, with an interval realizer $\mathcal{I}$ of P. The vertices of $\mathcal{B}(\mathcal{I})$ are the at most $2n$ different lower resp. upper extreme corners of elements of P. The order relation of $\mathcal{B}(\mathcal{I})$ is given by the componentwise order in $\mathbf{R}^k$.

By definition we have an embedding of $\mathcal{B}(\mathcal{I})$ in $\mathbf{R}^k$, so $dim\,\mathcal{B}(\mathcal{I}) \leq k = idim(P)$. The starting point of our investigations was the following question concerning the interplay between dimension and interval dimension:

$$\text{Is } dim\,\mathcal{B}(\mathcal{I}) = idim(P)?$$

In Section 2 we define a transformation $P \to B(P)$ such that $idim(P) = dim\,B(P)$. We provide two interpetations of this transformation, a combinatorial one and a geometrical one. In the combinatorial interpretation the elements of $B(P)$ are subsets of P. In the geometrical interpretation $B(P)$ is the poset $\mathcal{B}(\mathcal{I}^*)$ of extreme corners of $\mathcal{I}^*$. Here $\mathcal{I}^*$ is a box embedding obtained from an arbitrary box embedding $\mathcal{I}$ by a normalizing procedure. From the proofs we obtain an affirmative answer to the question above.

Section 3 investigates several consequences to and relations with other order-theoretic results. First, we study the transformation $P \to B(P)$ on special partial orders of height 1. In particular we show that the standard example S_n of a n-dimensional order is an (almost) fixed point of the transformation $P \to B(P)$. Therefore, $dim(S_n) = idim(S_n)$.

Second, we investigate the relationship with the split operation. This has a surprising consequence for the iterated transformation $P \to B(P) \to B^2(P) \to$

$\ldots B^k(P) \to \ldots$. For every n there are partial orders P such that $0 \leq dim(P) - dim\, B^k(P) \leq 2$ for all $k \leq n$ but $dim(P) - dim\, B^{n+1}(P) \geq m$, where m is arbitrary.

Third, we relate the interval dimension of subdivisions of P to the dimension of P, thus providing a theoretical framework for the examples of Spinrad [17].

Finally, we show the comparability invariance of the transformation $P \to B(P)$, which, as a consequence, gives another proof that the interval dimension is a comparability invariant. Some remarks concerning the recognition-complexity of special classes of orders and graphs close the paper.

2 The main result

In the last section we defined the partial order $\mathcal{B}(\mathcal{I})$ of extreme corners associated with a box embedding of P in $\mathbf{R}^k$. With the next lemmas we show that $\mathcal{B}(\mathcal{I})$ inherits some structure which is independent of the realizer $\mathcal{I}$ leading to the box embedding.

Lemma 1 *Let $\mathcal{B}(\mathcal{I})$ be the partial order of extreme corners of a box representation of P.*

a) *If the lower extreme corners of x and y are comparable in $\mathcal{B}(\mathcal{I})$, e.g., $l_x \leq l_y$, then the predecessor sets of x and y in P are ordered by inclusion, i.e., $Pred_P(x) \subseteq Pred_P(y)$.*

b) *If the upper extreme corners of x and y are comparable in $\mathcal{B}(\mathcal{I})$, e.g., $u_x \leq u_y$, then the successor sets of x and y in P are ordered by (reversed) inclusion, i.e., $Succ_P(x) \supseteq Succ_P(y)$.*

c) *If the lower extreme corner of x and the upper extreme corner of y are related by $l_x \leq u_y$ then $Pred_P(z) \supseteq Pred_P(x)$ for all $z \in Succ_P(x)$ or, equivalently, $Pred_P(x) \subseteq \bigcap_{z \in Succ_P(y)} Pred_P(z)$.*

d) *If $u_x \leq l_y$ then $\bigcap_{z \in Succ_P(x)} Pred_P(z) \subseteq Pred_P(y)$.*

Proof. a) From $l_x \leq l_y$ we obtain $a_x^j \leq a_y^j$ for all j. Therefore, in each I_j, x has less predecessors than y, i.e., $Pred_j(x) \subseteq Pred_j(y)$. The claim now follows from

$Pred_P(x) = \bigcap_j Pred_j(x)$ since the I_j realize P.
b) The proof of this part is symmetric to part a.
c) From $l_x \leq u_y$ we have $a_x^j \leq b_y^j$. If $z \in Succ(y)$, then necessarily $a_z^j \geq b_y^j \geq a_x^j$. Hence, $Pred_j(z) \supseteq Pred_j(x)$ for all j. The claim follows.
d) From $u_x \leq l_y$ we immediately obtain $x \leq y$, i.e., $y \in Succ(x)$, therefore, $Pred(y) \supseteq \bigcap_{z \in Succ(x)} Pred(z)$. □

All statments except the conclusion part of b use only the sets $Pred_P(x)$ and $\bigcap_{z \in Succ_P(x)} Pred_P(z)$. This irregularity is resolved with the next lemma.

Lemma 2 $\bigcap_{z \in Succ(x)} Pred(z) \supseteq \bigcap_{z \in Succ(y)} Pred(z)$ *if and only if* $Succ(x) \subseteq Succ(y)$.

Proof. The 'if' direction is trivial. We now prove the 'only if' direction. Let $z \in Succ(x)$ and note that $y \in \bigcap_{z \in Succ(y)} Pred(z)$. From the assumed inclusion we obtain $y \in Pred(z)$, hence $z \in Succ(y)$ □

Definition 1 *With each vertex x of a partial order P we associate the* lower set $L(x) = Pred_P(x)$ *and the* upper set $U(x) = \bigcap_{z \in Succ_P(x)} Pred_P(z)$. *The case $x \in Max(P)$ is settled by the convention $U(x) = P$.*
Define $B(P) = \{L(x), U(x) : x \in P\}$ ordered by setinclusion.

Note that this construction is in fact equivalent with Cogis' construction in the context of Ferrers dimension [2]. Cogis also uses $L(x)$, but replaces $U(x)$ by the equivalent set $\{z \in P : Succ(x) \subseteq Succ(z)\}$. He also proves Theorem 2, but in a different way and without the geometrical interpretation that our approach is based on.

The preceding lemmas prove that $l_x \to L(x)$ and $u_x \to U(x)$ together form an order preserving mapping from $B(\mathcal{I})$ to $B(P)$, hence

$$idim(P) \geq dim\, \mathcal{B}(\mathcal{I}) \geq dim\, B(P). \quad (2)$$

To get more structure into interval realizers we now introduce a procedure that transforms a interval extension $I = \{(a_x, b_x) : x \in P\}$ of P into its P-*normalization* $I^* = \{(a_x^*, b_x^*) : x \in P\}$.

- In the first step of the P-normalization we update left endpoints:
  $a_x^* = max\{b_z : z \in Pred(x)\}$ if x is not minimal,
  $a_x^* = min\{a_z : z \in Min(P)\}$ if x is minimal.
- In the second step we update right endpoints:
  $b_x^* = min\{a_z^* : z \in Succ(x)\}$ if x is not maximal,
  $b_x^* = max\{b_z : z \in Max(P)\}$ if x is maximal.

Note that the interval order I^* need not be isomorphic to I. In general I^* is a suborder of I and a minimal interval extension of P if all the a_x, b_x were different.

If P is realized by $\mathcal{I} = \{I_1, \ldots, I_k\}$ then $\mathcal{I}^* = \{I_1^*, \ldots, I_k^*\}$ realize P as well. We call the box embedding corresponding to $\mathcal{I}^*$ the *normalized* box embedding of $\mathcal{I}$. For an example see Figure 1.

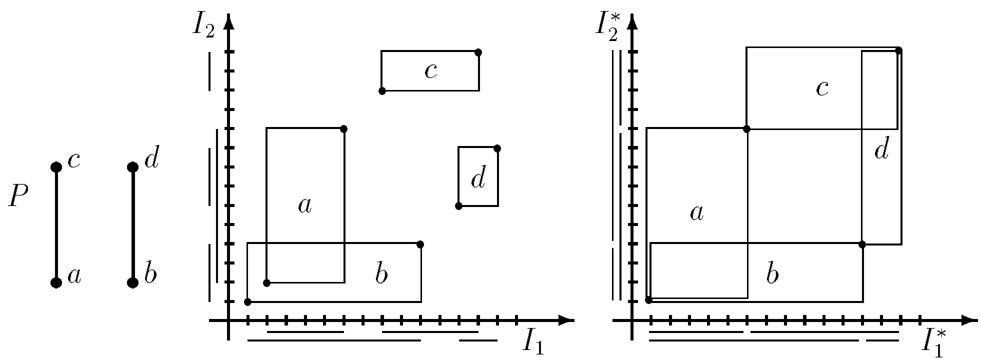

Figure 1: P, an interval realizer of P and its normalization

After normalizing we have a realizer $\mathcal{I}^* = \{I_1^*, \ldots, I_k^*\}$ of P, interval representations (a_x^{j*}, b_x^{j*}) and an associated partial order of extreme corners $\mathcal{B}(\mathcal{I}^*) = \{l_x^*, u_x^* : x \in P\}$. The next theorem shows that the geometrically defined $\mathcal{B}(\mathcal{I}^*)$ and the combinatorially defined $B(P)$ are isomorphic.

Theorem 1 *If $\mathcal{I}^*$ is a normalized realizer of P then $\mathcal{B}(\mathcal{I}^*) = B(P)$.*

Proof. First observe that both partial orders have a least element generated by $x \in Min(P)$ as l_x^* and $L(x)$, respectively, and a greatest element generated by $x \in Max(P)$ as u_x^* and $U(x)$, respectively.

Moreover, $l_x^* \to L(x)$ and $u_x^* \to U(x)$ defines an order preserving mapping by the remarks preceeding (2). To show the converse we distinguish four cases:

$U(x) \subseteq L(y)$. We know that $x \in U(x)$, so $x \in Pred(y)$. Since I_j^* is an interval extension of P, we obtain $b_x^{j*} \leq a_y^{j*}$ for all j. Hence $u_x^* \leq l_y^*$.

$L(x) \subseteq L(y)$. Remember that $a_x^{j*} = max\{b_z^j : z \in Pred(x)\}$ and $a_y^{j*} = max\{b_z^j : z \in Pred(y)\}$. By assumption $Pred(x) \subseteq Pred(y)$, so $a_x^{j*} \leq a_y^{j*}$ and $l_x^* \leq l_y^*$.

$U(x) \subseteq U(y)$. By Lemma 2, this is equivalent to $Succ(x) \supseteq Succ(y)$. Now $u_x^* \leq u_y^*$ follows symmetrically with the second case.

$L(x) \subseteq U(y)$. Since I_j^* is normalized, there are $z_0 \in Pred(x)$ and $z_1 \in Succ(y)$ with $a_x^{j*} = b_{z_0}^j$ and $b_y^{j*} = a_{z_1}^{j*}$. The hypothesis provides $z_0 \leq z_1$, hence $a_x^{j*} = b_{z_0}^j \leq b_{z_0}^{j*} \leq a_{z_1}^{j*} = b_y^{j*}$. The validity of this inequality for all j again gives $l_x^* \leq u_y^*$. □

We are ready now, to prove our main theorem about interval dimension and dimension.

Theorem 2 $dim\, B(P) = idim(P)$.

Proof. As Inequality (2) we allready have obtained $dim\, B(P) \leq idim(P)$. For the converse we need two arguments. We first show that a linear extension L of $B(P)$ induces an interval extension I_L of P. Secondly, we prove that if $L_1, \ldots, L_k$ is a realizer of $B(P)$, then the induced interval extensions $I_{L_1}, \ldots, I_{L_k}$ form an interval realizer of P.

Let $L = M_1, M_2, \ldots, M_r$ be a linear extension of $B(P)$. For each $x \in P$ there are $i, j \in \{1, \ldots, r\}$ such that $M_i = L(x)$ and $M_j = U(x)$. From $L(x) \subseteq U(x)$ and $x \notin L(x)$, $x \in U(x)$ we obtain that $i < j$. So we can associate with x a unique interval (a_x, b_x) which is defined to be (i, j). We now show that the interval order I_L induced by the interval representation $\{\ (a_x, b_x)\ : x \in P\}$ is an extension of P. If $x < y$ in P, then $U(x) \subseteq L(y)$, and thus, with $M_i = U(x)$ and $M_j = L(y)$, $b_x = i \leq j = a_y$, which implies $x < y$ in I_L.

Let $\{L_1, \ldots, L_k\}$ be a realizer for $B(P)$ and let $I_j = I_{L_j}$, for $1 \leq j \leq k$. The family $\{I_1, \ldots, I_k\}$ of interval extensions of P is an interval realizer iff all incomparabilities $x||y$ of P are realized. If $x||y$ in P, then $U(x) \not\subseteq L(y)$ since $x \in U(x)$ but $x \notin L(y)$. Therefore, $L(y)$ precedes $U(x)$ in some L_j, which gives

$a^j_y < b^j_x$. The symmetric argument yields an i with $a^i_x < b^i_y$. Both inequalities together give $x||y$ in $\bigcap_j I_j$. □

This theorem together with Inequality (2) shows that $dim\,\mathcal{B}(\mathcal{I})$ is independent of the interval realizer $\mathcal{I}$.

3 Consequences

In the previous section we introduced the operation $P \to B(P)$ mapping partial orders to partial orders. We will now investigate several connections to other order-theoretical topics and results. Note that $B(P)$ always has a greatest and a least element. We adopt the convention of calling orders with this property *bounded* and denote by $Q \to \widehat{Q}$ the bounding of a partial order Q, i.e., $\widehat{Q}$ is the order resulting from Q by adjoining a new greatest and a new least element.

We first look at the effect of the operator B applied to special classes of orders.

3.1 Transformation rules for special posets

Let S_n denote the standard poset of dimension n, i.e., the set of all 1-element and (n–1)-subsets of an n element set ordered by setinclusion. Then

$$B(S_n) = \widehat{S_n} \tag{3}$$

Let C_r denote the r-cycle. The r-cycle is the 3–dimensional poset on $2r$ elements $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r\}$ with comparabilities.

$$x_1 < y_1, y_1 > x_2, x_2 < y_2, \ldots x_r < y_r, y_r > x_1$$

$$B(C_r) = \widehat{C_r} \tag{4}$$

Let the Hasse diagram of T be a tree. The *truncation* of T, denoted by $tr(T)$, is the induced tree on the non-leaf vertices of T. Then

$$B(T) = \widehat{tr(T)} \tag{5}$$

In particular (3) shows that the standard example S_n of a n-dimensional order is (up to closures) a fixed point of the operation $P \to B(P)$, thus showing again that, for every $n \geq 3$, there is are orders P with $dim(P) = idim(P) = n$.

3.2 B as an inverse of the split operation

We now turn to the natural question, whether, for every bounded partial order Q there is some P with $Q = B(P)$. The next theorem answers this question affirmatively. Moreover, it turns out that the operation $P \to B(P)$ is an almost left inverse of the *split* operation S which has applications in different branches of poset theory (see, e.g., [21, 8, 6]). The split $S[P]$ of a partial order P is the order of height one with minimal elements $\{x' : x \in P\}$ maximal elements $\{x'' : x \in P\}$ and ordered pairs $x' < y''$ iff $x \leq y$ in P.

Theorem 3 $B(S[P]) = \widehat{P}$.

Proof. For $x \in P$ let $Pred[x] = Pred(x) \cup \{x\}$. Obviously, P is isomorphic to the setsystem $\{Pred[x] : x \in P\}$ ordered by inclusion. We will show that the elements of $B(S[P])$ are just the 'primed' sets $Pred[x]$, i.e., $(Pred[x])' = \{x' : x \in Pred[x]\}$, together with the greatest element $\{x', x'' : x \in P\}$ and the least element $\emptyset$. We have

$L(x') = \emptyset$, and

$U(x') = \bigcap_{z'' \in Succ(x')} Pred(z'') = \bigcap_{z \in Succ[x]} (Pred[z])'$. Since $x \in Succ[x]$, $U(x') \subseteq (Pred[x])'$.

On the other hand $Pred[x] \subset Pred[z]$ for $z \in Succ(x)$. Together, this gives $U(x') = (Pred[x])'$. Similarly, we obtain

$L(x'') = Pred(x'') = (Pred[x])'$. Finally,

$U(x'') = \{x', x'' : x \in P\}$ by definition since $Succ(x'') = \emptyset$. □

It is easy to verify that

$$B(\widehat{P}) \quad = \quad \widehat{B(P)}. \tag{6}$$

So we may generalize the theorem to

$$B^n(S^n[P]) \quad = \quad \left.\widehat{\vdots}\right\}^{n \text{ boundings}} \hskip-4em \widehat{P} \tag{7}$$

Investigations on the effect of iterated splitting to the dimension [19] lead to the inequality

$$dim\, P \leq dim\, S^n[P] \quad \leq \quad 2 + dim\, P \quad \text{for all } n \tag{8}$$

As a consequence of (7) and (8) we obtain that, for every n there is are partial orders P such that

$$dim\, P - dim\, B^k(P) \leq 2 \quad \text{for all } k \leq n.$$

(Just take $P = S^n[Q]$ for some order Q). If we choose Q, however, to be an m dimensional interval order we obtain a large difference in dimension with the next iteration, i.e.,

$$dim\, P - dim\, B^{n+1}(P) \geq m - 1. \tag{9}$$

3.3 The interval dimension of subdivisions

With the next theorem we relate the interval dimension of subdivisions of P to the dimension of P. Spinrad [17] showed that the dimension of a subdivision of a partial order can be an arbitrary multiple of its dimension, thus answering Trotter's Problem 4 in [22]. With our result we establish a theoretical framework for his examples.

In this context, partial orders and their diagrams are regarded as directed graphs whose edges (x, y) correspond to ordered pairs and cover pairs $x \prec y$ of P, respectively. An edge (x, y) is *subdivided* by placing a new vertex z in the 'middle' of the edge, i.e., (x, y) is replaced by (x, z) and (z, y). In the case of partial orders we then have to ensure transitivity, i.e., all edges (a, z) with $a \in Pred[x]$ and (z, b) with $b \in Succ[y]$ are also added.

The *complete diagram subdivision* $DS(P)$ is the subdivision of all edges of the diagram of P. The *complete subdivision* $CS(P)$ is the transitive closure of the subdivision of all the edges of P. Let E be any set of edges of P, we denote the order obtained by subdividing the edges of E with $Sub(P, E)$. Since P is an induced suborder of each subdivision $Sub(P, E)$, and since $Sub(P, E)$ is an induced suborder of $CS(P)$, we obtain

$$dim(P) \leq dim\, Sub(P, E) \leq dim\, CS(P). \tag{10}$$

With the next theorem we give an upper bound for $idim\, Sub(P, E)$.

Theorem 4 *$idim\, Sub(P, E) \leq idim\, DS(P) = idim\, CS(P) = dim(P)$.*

Proof. Take any embedding of P into $\mathbf{R}^k$ with $k = dim(P)$ and grow the points to obtain an embedding by 'miniboxes'. An interval embedding of a subdivision $Sub(P, E)$ is then obtained by adding the box with lower extreme corner u_x and upper extreme corner l_y for the point z subdividing the edge $(x, y) \in E$ – see Figure 2. This gives $idim\, Sub(P, E) \leq dim(P)$, independent of the choice of E.

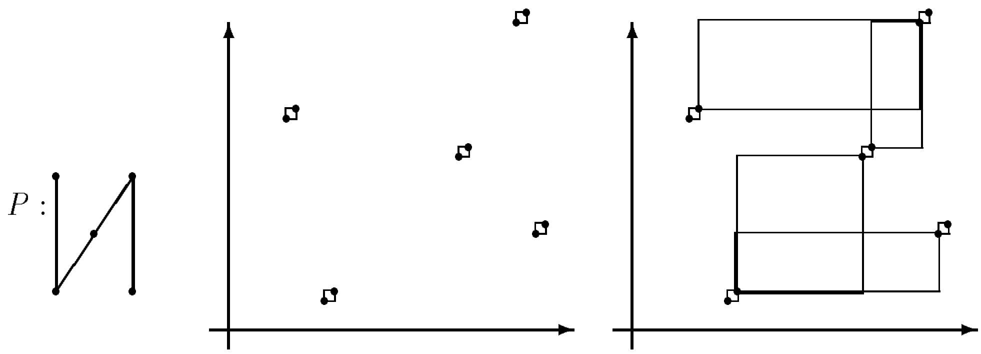

Figure 2: P, a minibox embedding of P and the box embedding of $DS(P)$

To prove that $idim\, DS(P) = idim\, CS(P) = dim(P)$ we will show that P can be embedded in $B(DS(P))$. This will give $dim(P) \leq dim\, B(DS(P))$. From Theorem 2 we know $idim\, DS(P) = dim\, B(DS(P))$. Together, we obtain $dim(P) \leq idim\, DS(P)$.

To show that P can be embedded in $B(DS(P))$ we apply the normalizing procedure to the box embedding $\mathcal{I}$ of $DS(P)$ constructed in the first paragraph of the proof. Because of the construction of the box embedding of $DS(P)$, the only changes that occur in normalization are shifts of the left endpoints of intervals corresponding to elements in $Min(P)$ and the right endpoints of elements in $Max(P)$. We then embed P into the lower extreme corners of the miniboxes of the normalized representation $\mathcal{I}^*$. This gives $dim(P) = k = idim(\mathcal{I}^*) = idim(DS(P))$ □

Note that we obtained, in fact, a slightly stronger result: If a set E of edges of P contains the edges of the diagram, then $B(Sub(P, E)) = VS(P)$, where $VS(P)$ denotes the *vertical split* of P, i.e., the order obtained from P by substituting each vertex by a 2–chain. In [20] a distinct proof for $dim(P) = idim\, VS(P)$ has been given.

3.4 Comparability invariance

For the definition and basic facts on comparability invariance see [10]. Let $Comp(P)$ be the comparability graph of P. We will show that $Comp(B(P))$ is a comparability invariant of P in the sense that if $Comp(P) = Comp(Q)$ then $Comp(B(P)) = Comp(B(Q))$. Together with Theorem 2 and the known fact that dimension is a comparability invariant, this gives an alternative proof of the comparability invariance of interval dimension in the finite case. The comparability invariance of interval dimension was first shown in [11].

A subset of elements A of P is called *autonomous* if the relation of elements in A to an element outside A is independent of the element of A. More formally, if for any $x \notin A$, whenever $x < a$, $x > a$ or $x||a$ holds for some $a \in A$, then it holds for all $a \in A$. If A is an autonomous subset of P, then $Pred(A)$ shall denote the predecessors outside A of any and hence every element of A.

Theorem 5 *$Comp(B(P))$ is a comparability invariant of P.*

Proof. Let A be an autonomous subset of P. It is enough to show (see, e.g.,[11]), that $Comp(B(P)) = Comp(B(P^A_{A^d}))$, where $P^A_{A^d}$ denotes the order resulting from substituting A by its dual A^d in P.

Note first that $B(A) = \{L(a), U(a) : a \in A\}$ is a closed suborder of $B(P)$. Let $\widetilde{B(A)}$ be $B(A)$ without its greatest element $1_{B(A)}$ and its least element $0_{B(A)}$. Our claim is that $\widetilde{B(A)}$ is autonomous in $B(P)$. To see this, observe first that, for each $a \in A$, we can decompose $Pred(a)$ into $Pred(a) = Pred(A) \cup Pred_A(a)$, hence, the same is valid for all elements of $\widetilde{B(A)}$. On the other hand, the elements of $B(P) \setminus B(A)$ either contain all of A or their intersection with A is empty. Now if $M \in B(P) \setminus B(A)$ contains all of A then it also contains $Pred(A)$ and M is above all sets in $\widetilde{B(A)}$. If $M \subseteq Pred(A)$ then M is below all sets in $\widetilde{B(A)}$. In all the other cases M is unrelated to all of $\widetilde{B(A)}$. This gives the claim.

To settle the theorem we need that $B(A^d)$ and $B(A)^d$ are isomorphic orders. To see this, consider a normalized box embedding of A in $\mathbf{R}^k$. Its extreme corners are an embedding of $B(A)$ into $\mathbf{R}^k$. Flip the embedding, i.e., reverse the relations, this gives an embedding of A^d and the extreme corners form an embedding of $B(A)^d$. □

As we have seen, autonomous sets in P induce autonomous sets in $B(P)$. The converse, however, is far from being true. Take as P a prime interval order, then $B(P)$ is a chain. Hence P has none but $B(P)$ has $\binom{|B(P)|}{2} - 1$ nontrivial autonomous sets.

3.5 Remarks on related results and computational compexity

The transformation $P \to B(P)$ obviously can be carried out in polynomial time. The degree of the polynomial of this reduction is not important if we want to decide if P has interval dimension ≥ 3, since the poset dimension and interval dimension problems have been shown to be NP-complete for $k \geq 3$ by Yannakakis [23]. It is, however, a crucial point if we want to decide if $idim(P) \leq 2$.

Dagan Golumbic and Pinter [4] addressed questions about the comparability invariance of interval dimension and (fast) recognition of interval dimension at most 2. As remarked before, the first problem has been settled affirmatively first in [11]. The recognition problem has been solved independently by several authors. Habib and Möhring [12] use the subposet of $B(P)$ defined by $B'(P) = \{L(x) : x \in P\}$ together with an algorithm for transitive orientation with side constraints to derive an $O(n \cdot n^a)$ algorithm, where $O(n^a)$ is the best known time for matrix multiplication. Currently a is about 2.37.

Cheah [3] proposed an algorithm in complexity $O(n^3)$. The same complexity is claimed by Langley [13]. Langley calles the poset $B(P)$ of the present paper the 'predecessor-successor order' of P and shows without the aid of the geometric construction that finding an interval realizer of P is equivalent to finding a linear realizer of $B(P)$.

Spinrad [18] has shown that recognition of 2-dimensional orders only requires $O(n^2)$ operations. Therefore, the bottleneck with the $P \to B(P)$ approach for the recognition of interval dimension 2 is the computation of the order $B(P)$. With a careful implementation, $B(P)$ can be computed in $O(n^a)$, where again $O(n^a)$ is the best known time for matrix multiplication.

Ma and Spinrad [14] have found an approach which allows to avoid matrix multiplication and leads to a complexity bound of $O(n^2)$. Since this is the best known result we outline the ideas behind their algorithm.

First, they construct the open split $S(P)$ of the partial order P, for which they want to decide if $idim(P) \leq 2$. The elements of the open split $S(P)$ are the same as of the split $S[P]$ defined above, the ordered pairs of $S(P)$ are , however, given by the irreflexive relation defining P, i.e., $x' < y''$ in $S(P)$ iff $x < y$ in P. They prove that the co-chain covering number of $S(P)$ equals the interval dimension of P. A theorem of Yannakakis [23] shows that the co-chain covering number of $S(P)$ and the interval dimension of $S(P)$ coincide. Hence, $idim(P) = idim(S(P))$ and we can reduce attention to the recognition of interval dimension 2 for bipartite orders, i.e., to orders of height one.

The transformation of the interval dimension 2 problem for bipartite orders to the dimension 2 problem is done in two steps. In the first step it is checkued whether the bipartite order Q has a chordal bipartite comparability graph. A chordal bipartite graph is a bipartite graph without any induced cycle C_n, $n \geq 6$. If $Comp(Q)$ is not chordal bipartite, then Q has to contain a crown and, therefore, $idim(Q) \geq 3$. Otherwise, the second step is started. In this step Q is transformed into an order P_Q which can be obtained by contracting autonomous chains in $Stack(Q)$ and, hence, has the same dimension as $Stack(Q)$. The stack operation has been introduced by Trotter [21]. He proves that for bipartite posets $dim\, Stack(Q) = idim(Q)$. As the construction of $B(Q)$, the construction of P_Q and $Stack(Q)$ requires information about the containment relation of neighborhoods in Q. For chordal bipartite graphs this information can be computed in $O(n^2)$ using a technique called doubly lexical ordering. Since after passing the first step we know that Q is chordal bipartite, we can construct P_Q and apply Spinrad's dimension 2 algorithm, all in $O(n^2)$.

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