3-INTERVAL IRREDUCIBLE PARTIALLY ORDERED SETS

STEFAN FELSNER

Abstract. In this paper we discuss the characterization problem for posets of interval dimension at most 2. That is, we attempt to compile the minimal list of forbidden posets for interval dimension 2. Members of this list are called 3-interval irreducible posets. The problem is related to a series of characterization problems which have been solved earlier. These are: The characterization of planar lattices, due to Kelly and Rival [KeRi75], the characterization of posets of dimension at most 2 (3-irreducible posets) which has been obtained independently by Trotter and Moore [TrMo76] and by Kelly [Ke77] and the characterization of bipartite 3-interval irreducible posets due to Trotter [Tr81].

We show that every 3-interval irreducible poset is a reduced partial stack of some bipartite 3-interval irreducible poset. Moreover, we succeed in classifying the 3-interval irreducible partial stacks of most of the bipartite 3-interval irreducible posets. Our arguments depend on a transformation $P \to B(P)$, such that $Idim\, P = dim\, B(P)$. This transformation has been introduced in [FHM91].

1. Introduction and Basics

An *extension* of a poset $P = (X, <_P)$ is a partial order $Q = (X, <_Q)$ on the same set, that contains all the relations of P, i.e., $x <_P y$ implies $x <_Q y$. A family $\{Q_1, \dots, Q_k\}$ of extensions of P is said to *realize* P if for all $x, y \in X$ we have $x <_P y$ exactly if $x <_{Q_i} y$ for all $1 \leq i \leq k$. If we restrict the Q_i to belong to a special class of orders and seek for a minimum size realizer we come up with a concept of dimension.

A *linear extension* of $P = (X, <)$ is an extension $L = (X, <_L)$ of P which is a chain, i.e., $x <_L y$ or $y <_L x$ for all $x, y \in X$ with $x \neq y$. Dushnik and Miller [DuMi41] defined the *(order) dimension* of a poset P, denoted $dim(P)$, as the smallest integer k for which there exist k linear extensions realizing P.

A partial order P is an *interval order* if it can be represented by assigning an interval $I_x = (a_x, b_x)$ on a chain C to each element $x \in P$, such that $x < y$ in P iff $b_x < a_y$ in C. Fishburn [Fi85] characterized interval orders as the posets with no 4–point

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subset forming two disjoint 2–chains, i.e., no **2+2**. The *interval dimension* of P, denoted $Idim(P)$, is the smallest k for which there exist k interval extensions of P which realize P. Since linear orders are interval orders we obtain the trivial inequality $Idim(P) \leq dim(P)$.

If $P' = (X', <)$ is a suborder of $P = (X, <)$, then $dim(P') \leq dim(P)$ and $Idim(P') \leq Idim(P)$. Therefore, for every integer k there is a minimal list of posets of dimension (interval dimension) $k + 1$, such that every poset which does not contain a subposet from this list has dimension (interval dimension) at most k. Members of this list then are called $(k+1)$*-irreducible* ($(k+1)$*-interval irreducible*) posets. Note that a $(k+1)$-irreducible ($(k+1)$-interval irreducible) poset P has $dim(P) = k+1$ ($Idim(P) = k+1$) but the removal of any element lowers its dimension (interval dimension).

The list of 2-irreducible posets only contains the 2 element antichain and the list of 2-interval irreducible posets only contains the **2+2**. The list of 3-irreducible posets is of a much higher complexity. Counting only one of P and P^d, the dual of P, the list consists of 10 isolated examples together with 7 infinite families (see Figure 1). This list has been obtained independently by Trotter and Moore [TrMo76] and by Kelly [Ke77]. Since this paper depends on the ideas of Kelly's argument we now give a brief (and in some details not even correct) outline of his proof.

Let $\mathbf{L}(P)$ denote the completion by cuts of a poset P, i.e., $\mathbf{L}(P)$ is the smallest lattice containing P as a suborder. Kelly's characterization of the 3-irreducible posets is mainly based on two results:

(1) $dim\, P = dim\, \mathbf{L}(P)$. (A theorem of Baker).
(2) The complete list $\mathcal{L}$ of lattices of dimension 3 with the property that every proper sublattice is of lower dimension. (Let us remark that a lattice is planar exactly if it contains no sublattice from $\mathcal{L}$). The list $\mathcal{L}$ had been obtained by Kelly and Rival [KeRi75].

Suppose that P is a 3-irreducible poset. It follows that $dim\mathbf{L}(P) \geq 3$, therefore, $\mathbf{L}(P)$ contains one or more of the lattices in $\mathcal{L}$ as sublattices. The key idea for the proof is to consider the lattices in $\mathcal{L}$ in a specific ordering, *Kelly order*. At an intermediate step in the argument we assume that $\mathbf{L}(P)$ contains a lattice Q from this list, but does not contain any lattice preceding it in the Kelly order. The details of each individual case center around chosing a copy of a 3-irreducible subposet R of Q and then one by one showing how those points of R which do not belong to P can be replaced by points of P.

A poset $P = (X, <)$ is called *bipartite* if there are two sets $X_1, X_2 \subseteq X$, such that $x < y$ implies $x \in X_1$ and $y \in X_2$. Trotter found a transformation, which associates with a bipartite poset P a poset $Stack(P)$, such that, $Idim(P) = dim\, Stack(P)$. This fact, together with the existing list of 3-irreducible posets lead to the characterization of bipartite 3-interval irreducible posets in [Tr81].

FIGURE 1. The 3 irreducible orders (part 1)

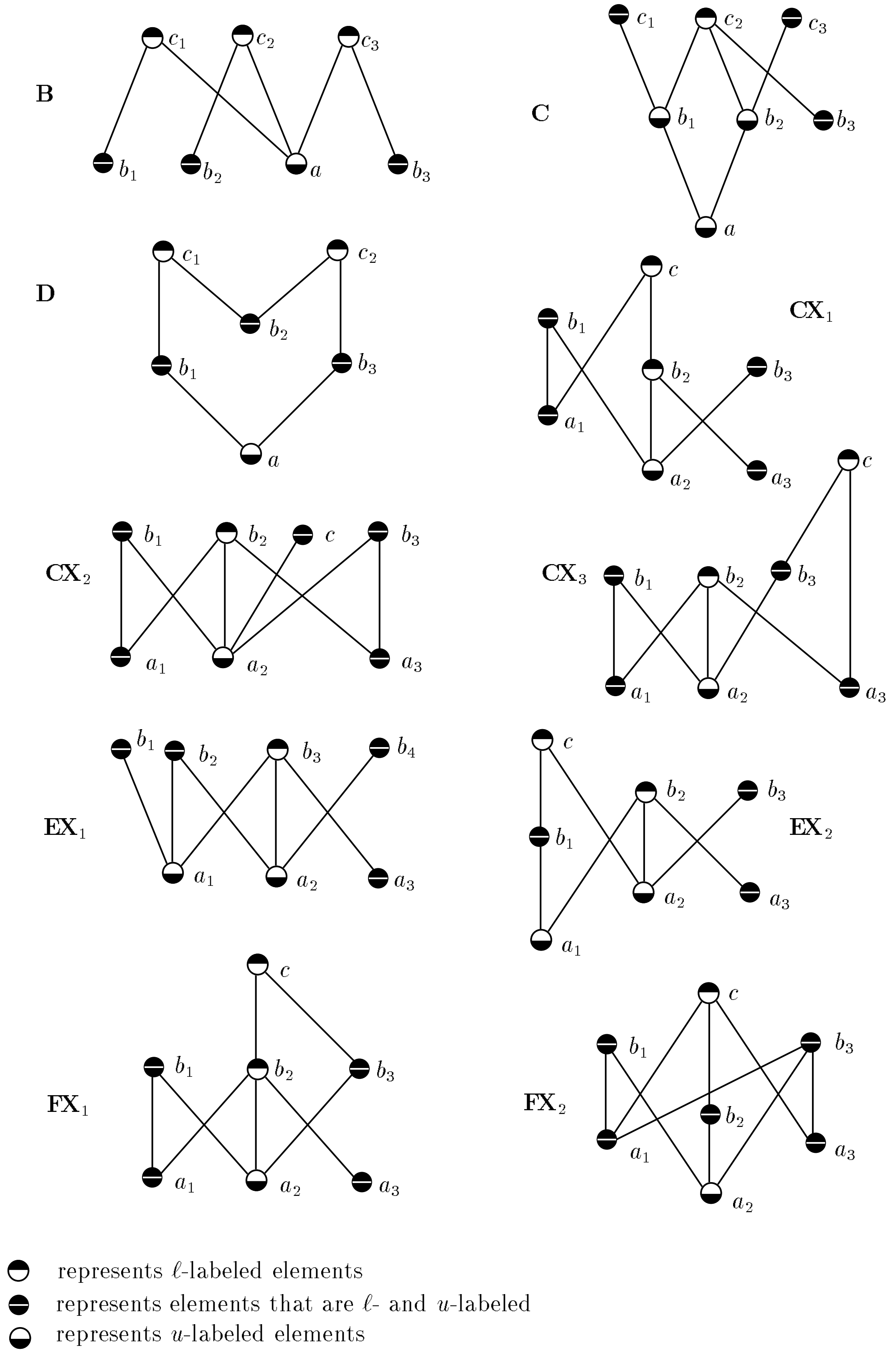


Figure 1. The 3 irreducible orders (part 2)

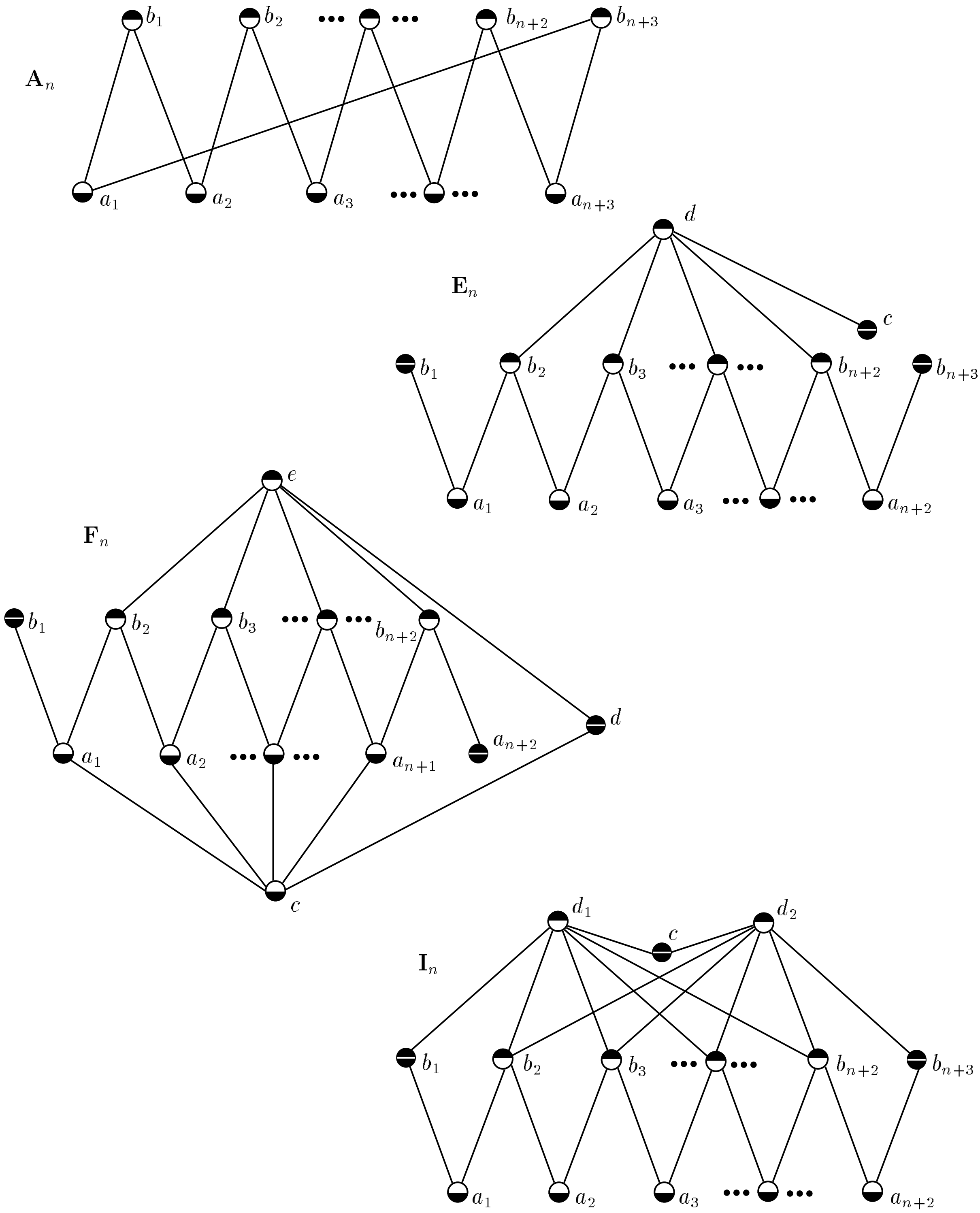
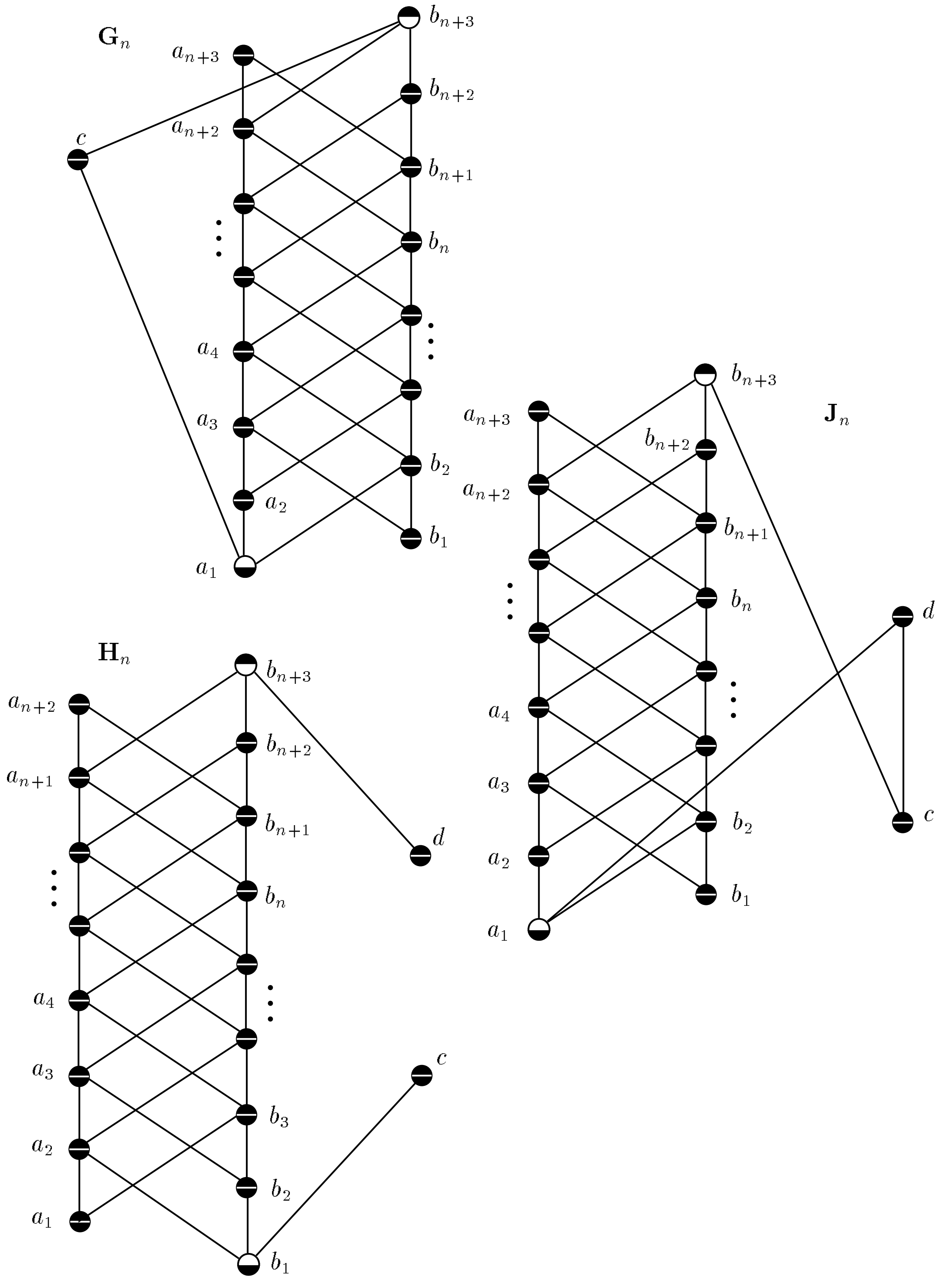


FIGURE 1. The 3 irreducible orders (part 3)



Trotter's proof uses Kelly's approach for the characterization of the 3-irreducible posets as a road map. Since the stack of a bipartite 3-interval irreducible poset P contains one (or more) of the 3-irreducible posets we can argue about the implications for the original poset. Again, a specific ordering of the 3-irreducible posets and corresponding assumptions about the 3-irreducible posets contained in $Stack(P)$ organize the proof.

In [FHM91] we have introduced a transformation $P \to B(P)$, such that $Idim\ P = dim\ B(P)$ for arbitrary posets P. In this paper we use this operator $P \to B(P)$, to approach the characterization of 3-interval irreducible posets. The argument is again based on a Kelly ordering of the 3-irreducible posets.

Let $P = (X, <)$ be a poset, for $x \in X$ we denote the predecessor set $\{ y \in X : y < x \}$ of x with $Pred(x)$, the successor set $\{ y \in X : y > x \}$ is denote by $Succ(x)$. The closed predecessor set is $Pred[x] = \{ y \in X : y \leq x \}$, i.e., $Pred[x] = Pred(x) \cup \{ x \}$, similarly $Succ[x] = Succ(x) \cup \{ x \}$. We now review the definition and some properties of the transformation $P \to B(P)$.

Definition 1. *For each element x of a poset $P = (X, <)$ let*

$L(x) = Pred(x)$,

$U(x) = \bigcap_{z \in Succ(x)} Pred(z)$, *if* $Succ(x) = \emptyset$ *then let* $U(x) = X$.

With P we associate a poset $B(P) = (Y, <)$. The elements of Y are the distinct sets occurring as $L(x)$ or $U(x)$ for some $x \in X$. The ordering of $B(P)$ is given by set-inclusion.

The next definition is taken from [Mi92].

Definition 2. *For posets $P = (X, <_P)$ and $Q = (Y, <_Q)$ we say that P has an* interval representation *on Q if there are mappings $L : X \to Y$ and $U : X \to Y$, such that*

(1) *$L(x) <_Q U(x)$, i.e, $[L(x), U(x)]$ is a nondegenerate interval of Q for each $x \in X$.*

(2) *$U(x_1) \leq_Q L(x_2)$ exactly if $x_1 <_P x_2$.*

An important fact about interval representations is given with the next lemma.

Lemma 1. *If an order P has an interval representation on some poset Q, then $Idim\ P \leq \dim Q$.*

The next theorem collects properties of the transformation $P \to B(P)$.

Theorem 1. *Let $P = (X, <)$ and $B(P) = (Y, <)$.*

(1) *The mappings $L : X \to Y$ and $U : X \to Y$ define an interval representation of P on $B(P)$.*

(2) *The dimension of $B(P)$ equals the interval dimension of P.*

We now come to the stack of a bipartite poset. As mentioned, the stack played a central role in the characterization of bipartite 3-interval irreducible posets.

Definition 3. *Let $P = (X, <_P)$ be a (connected) bipartite poset and let O be an arbitrary linear order on X. Let A denote the set of minimal elements and B the set of maximal elements of P (we assume that $a, a_i \in A$ and $b, b_i \in B$). The Stack(P) is a extension of P, the relations of Stack(P) are given by*

$a < b$	*if* $a < b$ *in* P,
$b_1 < b_2$	*if* $Pred_P(b_1) \subset Pred_P(b_2)$,
$a_1 < a_2$	*if* $Succ_P(a_1) \supset Succ_P(a_2)$,
$b < a$	*if* $a \| b$ *and* $a' < b'$ *in* P *for all* $a' \in Pred_P(b)$ *and* $b' \in Succ_P(a)$,
$b_1 < b_2$	*if* $Pred_P(b_1) = Pred_P(b_2)$ *and* $b_1 <_O b_2$,
$a_1 < a_2$	*if* $Succ_P(a_1) = Succ_P(a_2)$ *and* $a_1 <_O a_2$.

We now show, that for a bipartite order P the posets $B(P)$ and $Stack(P)$ are intimately related. Let the 0,1-*closure* of P, i.e., the adjoin of a least element 0 and a greatest element 1, be denoted by $P \to \widehat{P}$.

Theorem 2. *If $P = (X, <)$ is a bipartite order, then $B(P) = \widehat{Stack(P)} \mid_\sim$, here $\sim$ is an equivalence relation on X with the properties:*

- *each class of $\sim$ is an autonomous subset of Stack(P),*
- *Stack(P) induces a chain on each class of $\sim$.*

Proof. An element $x \in X$ is mapped to two elements of $B(P)$. Since P is bipartite we either have $L(x) = \emptyset$ or $U(x) = X$. In the first case let $M(x) = U(x)$, in the second case let $M(x) = L(x)$. Also, let O be the linear order used in the construction of $Stack(P)$. Define a poset Q on X by

(1) $x <_Q y$ if $M(x) \subset M(y)$,

(2) if $M(x) = M(y)$ then $x <_Q y$ if either $x \in Min(P)$ and $y \in Max(P)$ or x precedes y in O.

Note that $B(P)$ is the 0,1-closure of the poset $(\{ M(x) \mid x \in X \}, \subset)$. Therefore $B(P) = \widehat{Q} \mid_\sim$ when $x \sim y$ iff $M(x) = M(y)$.

We claim that $Q = Stack(P)$. The proof is an easy case analysis using the definitions of $L(x)$, $U(x)$ and the definition of the *Stack*. Let $A = Min(P)$ and $B = Max(P)$.

$a < b$ in P: In $Stack(P)$ we have $a < b$ by definition. Note that $M(a) = U(a) = \bigcap_{b' \in Succ(a)} Pred(b')$ and $M(b) = Pred(b)$. Since $a < b$ we find $Pred(b)$ in the intersection defining $M(a)$. Hence, $M(a) \subseteq M(b)$ and $a <_Q b$.

$b_1, b_2 \in B$: We have $b_1 < b_2$ in both $Stack(P)$ and Q if either $Pred(b_1) \subset Pred(b_2)$ or $Pred(b_1) = Pred(b_2)$ and b_1 precedes b_2 in O.

$a_1, a_2 \in A$: Recall from Lemma 2 of [FHM91] that $U(x) \subseteq U(y)$ iff $Succ(x) \supseteq Succ(y)$. With this remark, the present case is dual to the preceding case.

$a||b$ in P: Note that $a \notin M(b) = Pred(b)$ but $a \in M(a) = U(a)$, therefore, if a and b are comparable in Q then $b <_Q a$. The comparability $b <_Q a$ is in Q exactly if all $a' \in Pred(b)$ are elements of $Pred(b')$ for all $b' \in Succ(a)$. This, however, is the defining condition for $b < a$ in $Stack(P)$. □

The idea for our treatment of 3-interval irreducible posets will be the following. Suppose that $P = (X, <_P)$ is 3-interval irreducible, then, by Theorem 1 $dim\ B(P) = 3$. Hence, $B(P) = (Y, <)$ contains some poset Q from the list of 3-irreducible orders. Assuming that $B(P)$ contains a specified 3-irreducible poset Q, we then derive informations about the functions $L : X \to Y$ and $U : X \to Y$. Since these two functions define an interval representation of P on $B(P)$, information about L and U translates back to information about P.

This vague outline may motivate the study of properties that have to be required for two functions $L : X \to Y$ and $U : X \to Y$ from a set X to a closed order $Q = (Y, <_Q)$, such that, there is a poset P on X with $B(P) = Q$ and L and U as given. First recall that

(1) $L(x) <_Q U(x)$ for all $x \in X$,
(2) $Y = Im(L) \cup Im(U)$.

With the next lemma we give a less trivial property.

Lemma 2. *Let y_1, y_2 are incomparable elements of $Q = B(P)$*
(1) *if $Pred(y_1) \subseteq Pred(y_2)$ then $y_1 \in U(X)$,*
(2) *if $Succ(y_1) \subseteq Succ(y_2)$, then $y_1 \in L(X)$.*

Proof. This is an immediate consequence of the next Theorem. □

Theorem 3. *Let $P = (X, <_P)$ and $B(P) = (Y, <_Q)$. The following statements are equivalent:*
(1) *$y_1 \not\leq y_2$ in $B(P)$*
(2) *there is an element $x \in X$ with $U(x) \leq y_1$ but $U(x) \not\leq y_2$*
(3) *there is an element $x \in X$ with $L(x) \geq y_2$ but $L(x) \not\geq y_1$*

Proof. The equivalence of (1) and (2) is an easy consequence of the following

$$y = \{\, x \in X : U(x) \subseteq y \,\}, \text{ for all elements } y \text{ of } B(P).$$

First, suppose that $y = L(x_0)$ for some $x_0 \in X$. If $x \in y$, i.e., $x \in L(x_0) = Pred(x_0)$, then $x < x_0$. From $x_0 \in Succ(x)$ it follows, that $U(x) = \bigcap_{x' \in Succ(x)} Pred(x') \subseteq Pred(x_0) = y$. Conversely, let $U(x) \subseteq y$. From $x \in U(x)$ we obtain $x \in y$.

Now suppose that $y = U(x_0)$ for some $x_0 \in X$. If $x \in y$, i.e., $x \in U(x_0) = \bigcap_{x' \in Succ(x_0)} Pred(x')$, then $x < x'$ for all $x' > x_0$. Therefore, $Succ(x) \supseteq Succ(x_0)$ and hence $U(x) \subseteq U(x_0) = y$. Conversely, let $U(x) \subseteq y$. From $x \in U(x)$ we again obtain $x \in y$.

The equivalence of (1) and (3) is obtained by duality. The argument is based on the following observation: There is an isomorphism between $B(P^d)$ and $B(P)^d$, such that, the interval representation of P^d on $B(P)^d$ is obtained from the interval representation of P on $B(P)$ by interchanging L and U. □

Remark. Let an interval representation $L, U : P \to Q$ of P on Q be called a *proper interval representation* if for all $y_1, y_2 \in Im(L) \cup Im(U)$ the 3 statements of Theorem 3 are equivalent. If $L, U : P \to Q$ is a proper interval representation, then the ordering of Q restricted to $Im(L) \cup Im(U)$ is isomorphic to $B(P)$ and the interval representations of P on the restriction of Q and on $B(P)$ are equal with respect to this isomorphism. Hence, $B(P)$ can be characterized as the unique minimal poset which admits a proper interval representation of P.

1.1. Bipartite 3-Interval-Irreducible Orders. We are ready to show, how the list of bipartite 3-interval-irreducible orders given by [Tr81], see also [TrMo76], can be predicted. Although this list is known for more than 15 years, we think, that the method used here is new and interesting.

We start introducing some notation.

Definition 4. *Let $P = (X, <_P)$ be a poset and $B(P) = (Y, <)$. An element $y \in Y$ is called* low-made *if $y = L(x)$ for some $x \in X$, it is called* up-made *if $y = U(x)$ for some $x \in X$.*

Let $Q = (Y, <)$ be a poset. Label an element $y \in Y$ with ℓ if there is a $y' \in Y$ such that $y||y'$, i.e., y and y' are incomparable, and $Succ(y) \subseteq Succ(y')$ An element $y \in Y$ is labeled with u if there is a $y' \in Y$ such that $y||y'$ and $Pred(y) \subseteq Pred(y')$. This gives the (ℓ, u)*-labeling* of $Q = (Y, <)$. Note that $y \in Y$ is labeled ℓ/u in this labeling, if y is forced, by Lemma 2, to be low-made/up-made, whenever $Q = B(P)$ for some P

Now suppose, that $Q = (Y, <)$ is 3-irreducible. Recall that a *critical pair* is a pair (y_1, y_2) with $Pred(y_1) \supseteq Pred(y_2)$ and $Succ(y_1) \subseteq Succ(y_2)$. It is easy to see (cf. [KeTr82, Tr91]), that the 3-irreducibility of Q implies that every $y \in Y$ is contained in some critical pair. Hence, every element $y \in Y$ will receive at least one label in the (ℓ, u)-labeling of Q. (See Figure 1 for the (ℓ, u)-labelings of the 3-irreducible posets).

If every element of a poset $Q = (Y, <)$ has received a label in the (ℓ, u)-labeling, then a minimal bipartite order P with $B(P) = \widehat{Q}$ is easily obtained: For every u-labeled element $y \in Y$ take a minimal element y^u and for every ℓ-labeled element $y \in Y$ take a maximal element y^ℓ. The relations are given by $y^u < z^\ell$ if and only if $y \leq z$ in Q (for an example see Figure 2).

If Q is 3-irreducible then, in view of the preceeding remarks, the order P obtained by this process may be considerd as a candidate for the list of bipartite 3-interval-irreducible posets. To check the irreducibility we construct $B(P_x)$ for every point

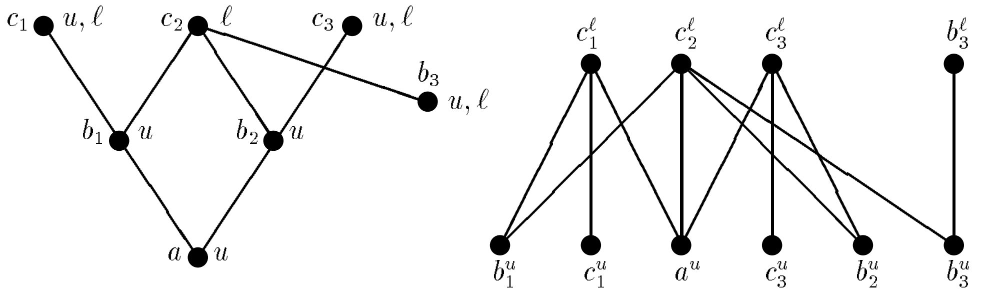


FIGURE 2. The 3 irreducible order $\mathbf{C}$ and the bipartite order $\mathcal{O}_2$ corresponding to the (ℓ, u)–labeling of $\mathbf{C}$.

deleted suborder P_x of P and compute *dim* $B(P_x)$. If *dim* $B(P_x) = 2$ for all x, then P is indeed a 3-interval-irreducible poset. The results are given in the list below.

Our notation and labelling for the collection of all 3 irreducible orders (see Figure 1) was introduced by Kelly [Ke77], it is also used in [Tr81] and [Tr91]. For bipartite 3-interval irreducible posets we use the notation introduced in [TrMo76] (see also [Tr81]).

$Q = \mathbf{A}_n$: Here we obtain the family $\mathcal{A}_n$.
$Q = \mathbf{B}$: Here we obtain $\mathcal{O}_1$.
$Q = \mathbf{C}$: Here we obtain $\mathcal{O}_2$.
$Q = \mathbf{D}$: Here we obtain $\mathcal{I}_0$.
$Q = \mathbf{EX}_2$: Here we obtain $\mathcal{O}_3$.
$Q = \mathbf{CX}_1$: The order P obtained here is not irreducible.
Removing b_1^u from P we have $B(P_{b_1^u})$ is the dual $\widehat{\mathbf{C}}^d$ of $\widehat{\mathbf{C}}$, i.e., $Idim(P_{b_1^u}) = 3$.
$Q = \mathbf{CX}_2$: The order P obtained here is not irreducible.
Removing b_1^u and b_3^u from P we have $B(P_{b_1^u,b_3^u}) = \widehat{\mathbf{C}}^d$.
$Q = \mathbf{CX}_3$: The order P obtained here is not irreducible.
Removing b_1^u and a_1^ℓ from P we have $B(P_{b_1^u,a_1^\ell}) = \widehat{\mathbf{D}}$.
$Q = \mathbf{EX}_1$: The order P obtained here is not irreducible.
Removing b_2^u from P we have $B(P_{b_2^u}) = \widehat{\mathbf{E}_0}$.
$Q = \mathbf{FX}_1$: The order P obtained here is not irreducible.
Removing a_1^ℓ from P we have $B(P_{a_1^\ell}) = \widehat{\mathbf{F}_0}$.
$Q = \mathbf{FX}_2$: The order P obtained here is not irreducible.
Removing a_1^ℓ and b_3^u from P we have $B(P_{a_1^\ell,b_3^u}) = \widehat{\mathbf{F}_0}$.
$Q = \mathbf{E}_n$: Here we obtain the family $\mathcal{E}_n$.
$Q = \mathbf{F}_n$: Here we obtain the family $\mathcal{F}_n$.
$Q = \mathbf{I}_n$: Here we obtain the family $\mathcal{I}_{n+1}$.
$Q = \mathbf{G}_n$: Here we obtain the family $\mathcal{G}_n$.
$Q = \mathbf{H}_n$: Here we obtain the family $\mathcal{H}_n$.
$Q = \mathbf{J}_n$: The order P obtained here is not irreducible.
Removing c^ℓ and d^u from P we have $B(P_{c^\ell,d^u}) = \widehat{\mathbf{G}_n}$.

Let $Q = (Y, <)$ be a 3-irreducible poset and let P be the bipartite order corresponding to the (ℓ, u)-labeling of Q. A label $\rho \in \{ \ell, u \}$ at an element $y \in Y$ is called *essential* if $dim\ B(P_{y^\rho}) = 2$. Note that, P is 3-interval irreducible exactly if every label of Q is essential.

Above we have compiled a list of bipartite 3-interval-irreducible posets. To prove that this list is indeed complete, i.e., consists of all bipartite 3-interval-irreducible posets, however, a large amount of work remains. This work has been accomplished by Trotter [Tr81]. In the next section we give a modified version of his argument to prove our Theorem 4.

Remark. The split of a poset $P = (X, <)$ is the bipartite poset with maximal elements $\{ x^\ell : x \in X \}$ and minimal elements $\{ x^u : x \in X \}$ with $x^u < y^\ell$ in $Split(P)$ iff $x \leq y$ in P. Note that, as a consequence of Trotter's result we can state that every bipartite 3-interval irreducible poset is a subposet of the split of some 3-irreducible poset.

2. General 3-Interval-Irreducible Posets

The program for dealing with the general (non-bipartite) case is the following: In the first part, we use the techniques developed in [Tr81] for the characterization the bipartite 3-interval irreducible orders to show that every 3-interval irreducible order contains a *partial stack* or a *reduced partial stack* of a bipartite 3-interval irreducible order. In the second part, we then attempt to characterize the 3-interval irreducible orders among the candidate posets obtained in the first part.

2.1. Partial Stacks and Reduced Partial Stacks.

Definition 5. *Let $P = (X, <_P)$ be a (connected) bipartite poset with minimal elements X_1 and maximal elements X_2. A* partial stack *of P is an order $Q = (X, <_Q)$ such that for all $x_1 \in X_1$ and $x_2 \in X_2$ we have $x_1 <_P x_2$ exactly if $x_1 <_Q x_2$.*

A partial stack Q of P may contain parallel elements. The reduced partial stack *corresponding to Q is obtained by contracting each set of pairwise parallel elements to a single point.*

Remark. Let $P = (X, <_P)$ be a (connected) bipartite poset with minimal elements X_1 and maximal elements X_2. Let $\mathbb{E}_P$ be the set of all extensions Q of P satisfying $x_1 <_Q x_2$ iff $x_1 <_P x_2$, i.e., $\mathbb{E}_P$ is the set of all partial stacks of P. Define a poset on $\mathbb{E}_P$ by $Q_1 \leq Q_2$ if Q_2 is an extension of Q_1. If Q is any maximal element of this poset, then Q is isomorphic to $Stack(P)$.

The main theorem of the present section is.

Theorem 4. *If P is a 3-interval irreducible poset then P contains a reduced partial stack of some bipartite 3-interval irreducible poset.*

To prove the theorem we go along Trotter's, 'Stacks and Splits' [Tr81] and rewrite the important lemmas and theorems. We omit the assumption that P is a height one poset, we replace $Stack(P)$ by $B(P)$ and instead of dealing with elements in $Stack(P)$ corresponding to maximal or minimal elements in P we deal with low-made and up-made elements of $B(P)$. As a first example note that Lemma 3 implies an analogue of Trotter's Lemma 4.

Theorem 5 (Trotter's Theorem 6). *If P is a poset and $B(P)$ contains a crown $\boldsymbol{A}_n$ for some $n \geq 0$, then there exists an integer m with $0 \leq m \leq n$ so that P contains $\boldsymbol{A}_m$.*

Proof. Choose the smallest integer $k \geq 0$ for which $B(P)$ contains $\mathbf{A}_k$. We will then show that P contains $\mathbf{A}_k$.

Of all copies of $\mathbf{A}_k$ contained in $B(P)$, choose one for which the integer $t = |\{\, b_i : 1 \leq i \leq k+3,\ b_i \text{ is low-made } \}| + |\{\, a_i : 1 \leq i \leq k+3,\ a_i \text{ is up-made } \}|$ is as large as possible. If $t = 2k+6$, then for $0 \leq i \leq k+3$ there are elements a_i', b_i' in P with $L(b_i') = b_i$ and $U(a_i') = a_i$. These elements form a copy of $\mathbf{A}_k$ in P.

Assume that $t < 2k+6$ and without loss of generality let b_{k+2} be a non low-made element. Since $a_1 || b_{k+2}$ in $B(P)$, it follows from Theorem 3, that there is an element $x \in X$ so that $L(x) \geq b_{k+2}$ and $L(x)||a_1$. If $L(x)||a_i$ for each $i = 2, 3, \dots, k+1$, then x can replace b_{k+2} in this copy of $\mathbf{A}_k$. So we must have $L(x) \geq a_i$ for some i with $2 \leq i \leq k+1$. Let i_0 be the smallest such integer. Then it follows that the subposet of $B(P)$ generated by $\{\, a_1, a_2, \dots, a_{i_0}, a_{k+3} \,\} \cup \{\, b_1, b_2, \dots, b_{i_0-1}, L(x), b_{k+3} \,\}$ is $\mathbf{A}_{i_0-2}$. But $i_0 - 2 < k$, this is a contradiction. □

Theorem 6 (Trotter's Theorem 7). *Let P be a poset and suppose that P does not contain a crown $\boldsymbol{A}_n$ for any $n \geq 0$. If y is an up-made (low-made) element of $B(P)$ and F is a connected subposet of $\mathrm{Inc}(y)$, then there exists a low-made (up-made) element x in $B(P)$ so that $x \geq y$ ($y \geq x$) and $F \subseteq \mathit{Inc}(x)$.*

Proof. We prove the theorem when y is up-made. If y is also low-made, then $x = y$ and we are done.

Suppose that y is not low-made. Let $y' > y$ be low-made and note that the existence of a low-made element $y'' > y$ with $y''||y'$ is guaranteed by Theorem 3. Therefore, there are at least two incomparable low-made elements in $Succ_{B(P)}(y)$

If there is a $z \in F$ with $x \geq z$ for all low-made elements $x > y$, then $y \geq z$ by Theorem 3. A contradiction.

We therefore may assume, that there are elements $x, x' \in Succ_{B(P)}(y)$ and $z, z' \in F$ such that $z < x$ and $z' < x'$ but $x||z'$ and $x'||z$. Now let $a_1, \dots, a_n$ be a fence in F from z' to z. It follows that $\{\, x, y, x', a_1, \dots, a_n \,\}$ contains a crown. A contradiction. □

Let $P = (X, <_P)$ be a 3-interval irreducible poset. Since the dimension of $B(P)$ is 3, $B(P)$ contains a 3-irreducible poset. Suppose first, that $B(P)$ contains a crown $\mathbf{A}_n$ for some $n \geq 0$. Then, it follows from Theorem 5, that P also contains a crown $\mathbf{A}_m$ for some $0 \leq m \leq n$. Since P is irreducible, we conclude, that $P = \mathbf{A}_n$. We may, therefore, assume in the remainder of the argument, that $B(P)$ does not contain a crown $\mathbf{A}_n$. The remainder of the argument is divided into a sequence of cases.

When discussing the case of a 3-irreducible order $Q = (Y, <_Q)$ contained in $B(P)$, we assume, that a copy of Q in $B(P)$ has been chosen. We refer to the elements of this copy of Q via the labeling of [Ke77] and [Tr81], see Figure 1. If y is an element of Q, then we say *there is an element* y^ℓ *in* P if we are sure about the existence of a low-made element $y' \geq y$ with $Pred_{B(P)}[y'] \cap Y = Pred_Q[y]$. The element y^ℓ then is any preimage of y' under L, i.e., $L(y^\ell) = y'$. Dually, we say *there is an element* y^u *in* P if we are sure about the existence of an up-made element $y'' \leq y$ with $Succ_{B(P)}[y''] \cap Y = Succ_Q[y]$. The element y^ℓ then is any element with $U(y^u) = y''$.

We will make extensive use of the following easy consequence of Theorem 6.

Lemma 3. *Let* $P = (X, <_P)$ *be a crown-free poset and suppose that a 3-irreducible poset* $Q = (Y, <_Q)$ *is contained in* $B(P)$. *If* y *is an element of* Q *such that* $Inc_Q(y)$ *is connected and for all* y' *in* Q *with* $y' \notin Pred_Q[y]$ *(*$y' \notin Succ_Q[y]$*) there is a* $y'' \in Inc_Q(y)$ *with* $y' \geq y''$ *(*$y' \leq y''$*), then there is an element* y^ℓ *(*y^u*) in* P.

In each case we choose a particular 3-irreducible poset $Q = (Y, <_Q)$ contained in $B(P)$ and suppose that $B(P)$ does not contain the 3-irreducible posets treated in the previous cases. These assumptions together with Theorem 3, Theorem 6 and Lemma 3 will allow us to show that there is an element y^ℓ (y^u) for every essential ℓ-labeled (u-labeled) element y of Q. Let $X^u \subseteq X$ be the set of these up-made elements and $X^\ell \subseteq X$ be the set of these low-made elements. The bipartite order $(X^u \times X^\ell) \cap <_P$ is isomorphic to a bipartite 3-interval irreducible poset R. Therefore, P contains a partial stack of R.

Case 1. $B(P)$ contains $\mathbf{D}$.
We apply Lemma 3 to obtain elements $b_1^\ell, b_2^\ell, b_3^\ell, c_1^\ell, c_2^\ell$ and a^u, b_1^u, b_2^u, b_3^u in P. The bipartite order $(X^u \times X^\ell) \cap <_P$ forms a copy of $\mathcal{I}_0$ in P. Hence, P contains a partial stack of $\mathcal{I}_0$.

Case 2. $B(P)$ contains $\mathbf{C}$.
From Lemma 3 we may assume, that there are elements $c_1^\ell, c_3^\ell, b_3^\ell$ and a^u, b_3^u, c_1^u, c_3^u.

Suppose, that there is no element c_2^ℓ, then by Theorem 6, there are elements $c' > c_2$ and $c'' > c_2$ in $B(P)$ with $c_1 \in Inc(c'')$ but $c'' > c_3$ and $c_3 \in Inc(c')$ but $c' > c_1$. This implies, that $\{ a, c_1, c'', b_3, c', c_3 \}$ form a copy of $\mathbf{D}$ in $B(P)$. The contradiction allows us to assume an element c_2^ℓ.

Assume, that b_1 is low-made, then choose an up-made element $b_1' < b_1$ with $b_2, c_3 \subseteq Inc(b_1')$. If $b_1' < b_3$, then $\{ c_2, b_2, a, c_1, b_1', b_3 \}$ form a copy of $\mathbf{D}^d$ in $B(P)$. We conclude, that $b_3 \in Inc(b_1')$, hence, any U-preimage of b_1' is b_1^u. Symmetrically, we may assume an element b_2^u. The bipartite order with maximal elements $c_1^\ell, c_2^\ell, c_3^\ell, b_3^\ell$ and minimal elements $a^u, b_1^u, b_2^u, b_3^u, c_1^u, c_3^u$ forms a copy of $\mathcal{O}_2$ in P. Hence, P contains a partial stack of $\mathcal{O}_2$.

Case 3. $B(P)$ contains $\mathbf{CX}_3$.
From Lemma 3 we may assume that there are elements $a_3^\ell, b_1^\ell, b_3^\ell, c^\ell$ and a_1^u, a_3^u, b_3^u. Choose a low-made element $b_2' > b_2$ with $b_3, c \in Inc(b_2')$. If $b_2' > b_1$, then $\{ a_2, b_1, b_2', a_3, c, b_3 \}$ form a copy of $\mathbf{D}$ in $B(P)$. Therefore, we may assume an element b_2^ℓ in P.

If there is no element a_2^u, then there are up-made elements $a_2' < a_2$ and $a_2'' < a_2$ with $a_1 || a_2'$ but $a_2' < a_3$ and $a_3 || a_2''$ but $a_2'' < a_1$. However, it follows, that $\{ b_2, a_1, a_2'', b_3, a_2', a_3 \}$ form a copy of $\mathbf{D}^d$. The contradiction shows that we may assume an element a_2^u.

The bipartite order corresponding to these elements y^ℓ and y^u as maximals and minimals forms a copy of $\mathcal{I}_0$ in P.

Case 4. $B(P)$ contains $\mathbf{CX}_2$.
From Lemma 3 we obtain elements $a_1^\ell, a_3^\ell, c^\ell$ and a_1^u, a_3^u, c^u.

The same argument used in Case 3, allows us to assume an up-made element a_2^u. Next, choose a low-made element $b_3' \geq b_3$ with $\{ b_1, b_2, a_1 \} \in Inc(b_3')$. If $b_3' > c$, then $B(P)$ contains $\mathbf{CX}_3$. Therefore, we assume an element b_3^ℓ. By symmetry, we may also assume an element b_1^u. Finally, choose a low-made element $b_2' > b_2$ with $c \in Inc(b_2')$. Obviously, $b_1 \not< b_2'$ and $c_2 \not< b_2'$. Let x be a preimage of b_2' under L. The bipartite order with maximal elements $a_1^\ell, a_3^\ell, b_1^\ell, x,\ b_3^\ell, c^\ell$ and minimal elements a_1^u, a_2^u, a_3^u, c^u forms a copy of $\mathcal{O}_2^d$ in P.

Case 5. $B(P)$ contains $\mathbf{CX}_1$.
If b_1 is not low-made, then choose a low-made element $b_1' > b_1$ with $\{ a_3, b_2, c \} \subseteq Inc(b_1')$. If $b_1' > b_3$, then $\{ a_2, b_3, b_1', a_1, c, b_2 \}$ form a copy of $\mathbf{D}$. Therefore, we may assume that there is an element b_1^ℓ in P.

If there is no element a_2^u, then there are up-made elements $a_2' < a_2$ and $a_2'' < a_2$ with $a_1 || a_2'$ but $a_2' < a_3$ and $a_3 || a_2''$ but $a_2'' < a_1$. It follows, that $\{ c, a_1, a_2'', b_3, a_2', a_3 \}$ form a copy of $\mathbf{D}^d$. The contradiction shows, that we may assume an element a_2^u in P.

If b_2 is not low-made, then choose a low-made element $b_2' > b_2$ with $a_1, b_1 \subset Inc(b_2')$. If $b_2' > b_3$, then $\{ a_1, a_2, a_3, b_1, c, b_3, b_2' \}$ form a copy of $\mathbf{CX}_3$. We, therefore, assume an element b_2^ℓ in P.

From Lemma 3 we obtain elements $a_1^\ell, a_3^\ell, b_3^\ell$ and a_1^u, a_3^u, b_3^u. We also may choose a low-made element $c' > c$ with $b_3 \in Inc(c')$. Let x be a preimage of c' under L.

The bipartite order with maximal elements $a_1^\ell, a_3^\ell, b_1^\ell, b_2^\ell, b_3^\ell, x$ and minimal elements $a_1^u, a_2^u, a_3^u, b_3^u$ forms a copy of $\mathcal{O}_2^d$ in P.

Case 6. $B(P)$ contains $\mathbf{EX}_2$.
Suppose, that b_2 is not low-made and there is no low-made element $b_2' > b_2$ with $c, b_3 \in Inc(b_2')$. Then it follows, that there exist points $b_4, b_5 > b_2$ with $b_3 \in Inc(b_4)$ and $c, b_1 \in Inc(b_5)$. This implies, that $b_5 > b_3$. If $b_4 > c$, then $\{ a_2, b_1, c, a_3, b_5, b_3 \}$ generate a copy of $\mathbf{D}$. So we must have $b_4 || c$, and thus $b_4 > b_1$. However, this implies that $\{ c, b_1, b_4, b_2, b_3, a_2, a_3 \}$ generate a copy of $\mathbf{CX}_1$ in $B(P)$. The contradiction allows us to assume an element b_2^ℓ in P. Since $\mathbf{EX}_2 = \mathbf{EX}_2^d$ we obtain a_2^u from the dual argument.

If c is not low-made, then choose an element $c' > c$ with $a_3, b_2 \in Inc(c')$. If $c' > b_3$, then $\{ c', b_3, a_2, b_2, a_1, b_1 \}$ form a copy of $\mathbf{D}^d$. We, therefore, may assume an element c^ℓ. Dually, we may assume a_1^u.

From Lemma 3 we obtain elements $a_3^\ell, b_1^\ell, b_3^\ell$ and a_3^u, b_1^u, b_3^u. The bipartite order with maximal elements $a_3^\ell, b_1^\ell, b_2^\ell, b_3^\ell, c^\ell$ and minimal elements $a_1^u, a_2^u, a_3^u, b_1^u$ forms a copy of $\mathcal{O}_3$ in P.

Case 7. $B(P)$ contains $\mathbf{FX}_1$.
If b_2 is not low-made, then choose a low-made element $b_2' > b_2$ with $b_2' || b_3$. If $b_2' > b_1$, then $B(P)$ contains $\mathbf{D}$. We, hence, assume an element b_2^ℓ in P.

If there is no element a_1^u, then there are up-made elements $a_1' < a_1$ and $a_1'' < a_1$ with $a_2, b_3 \in Inc(a_1')$ but $a_1' < a_3$ and $a_3 \in Inc(a_1'')$ but $a_1'' < b_3$. It follows, that $\{ c, b_3, a_1'', b_1, a_1', a_3 \}$ form a copy of $\mathbf{D}^d$. The contradiction shows, that we may assume an element a_1^u in P.

If there is no element a_2^u, then there are up-made elements $a_2' < a_2$ and $a_2'' < a_2$ with $a_1 || a_2'$ but $a_2' < a_3$ and $a_3 || a_2''$ but $a_2'' < a_1$. It follows, that $\{ b_2, a_1, a_2'', b_3, a_2', a_3 \}$ form a copy of $\mathbf{D}^d$. The contradiction shows, that we may assume an element a_2^u.

From Lemma 3 we obtain low-made elements $a_3^\ell, b_1^\ell, b_3^\ell, c^\ell$ and up-made elements a_3^u, b_1^u, b_3^u. The bipartite order with maximal elements $a_3^\ell, b_1^\ell, b_2^\ell, b_3^\ell, c^\ell$ and minimal elements $a_1^u, a_2^u, a_3^u, b_1^u, b_3^u$ forms a copy of $\mathcal{F}_0$ in P.

Case 8. $B(P)$ contains $\mathbf{EX}_1$.
Choose a low-made element $b_2' > b_2$ with $a_3, b_3 \in Inc(b_2')$. If $b_2' > b_1$ and $b_2' > b_4$, then $\{ b_2', b_1, a_1, b_3, a_2, b_4 \}$ generate $\mathbf{D}^d$. If $b_2' > b_1$ and $b_2' || b_4$, then replacing b_2 by b_2' we obtain $\mathbf{EX}_2$. If $b_2' > b_4$ and $b_2' || b_1$, then the same elements also generate $\mathbf{EX}_2$. Hence, we may assume an element b_2^ℓ in P.

Choose a low-made element $b_3' > b_3$, such that $b_2 \in Inc(b_3')$. If $b_3' > b_1$, then $\{ a_1, a_2, a_3, b_1, b_2, b_3, b_3' \}$ forms a copy of $\mathbf{FX}_1$ in $B(P)$. If $b_3' > b_4$, then $\{ a_1, a_2, a_3, b_2, b_3, b_3', b_4 \}$ generates $\mathbf{FX}_1$. We may, therefore, assume that there is an element b_3^ℓ in P.

If there is no element a_1^u, then choose an up-made element $a_1' < a_1$ with $a_2, b_4 \in$

$Inc(a_1')$. If $a_1' < a_3$, then $B(P)$ contains a copy of $\mathbf{FX}_1^d$. Thus, we may assume $a_3||a_1'$, i.e., we may assume an element a_1^u in P. By symmetry, we may also assume an element a_2^u in P.

From Lemma 3 we obtain elements $a_3^\ell, b_1^\ell, b_4^\ell$ and a_3^u, b_1^u, b_4^u. The bipartite order $(X^u \times X^\ell) \cap <_P$ is a copy of $\mathcal{E}_0$ in P.

Case 9. $B(P)$ contains $\mathbf{FX}_2$.
If there is no low-made element $c' > c$ with $b_1, b_3 \in Inc(c')$, then choose $c_1 > c$ and $c_2 > c$ with $b_3||c_1$ but $b_1 < c_1$ and b_1‘$||c_2$ but $b_3 < c_2$. It follows, that $B(P)$ contains a copy of $\mathbf{D}$. We, therefore, may assume an element c^ℓ. Dually, we also assume an element a_2^u in P.

Now, choose a low-made element $b_3' \geq b_3$ with $c, b_2 \in Inc(b_3')$. If $b_3' > b_1$, then $B(P)$ contains $\mathbf{D}$. We, therefore, assume an element b_3^ℓ in P. Dually, we also assume an element a_1^u.

From Lemma 3 we may assume elements $a_3^\ell, b_1^\ell, b_2^\ell$ and a_3^u, b_1^u, b_2^u in P. The bipartite order generated by these elements is a copy of $\mathcal{F}_0$ in P.

In cases 10 to 13 the next lemma will find repeated applications.

Lemma 4. *Let P be a poset such that $B(P)$ contains no $\boldsymbol{D}$, $\boldsymbol{CX}_3$, nor their duals or a crown $\boldsymbol{A}_n$. Let $a_1, b_1, a_2, b_2, a_3 \ldots, a_n, b_n$ be a fence in $B(P)$ and $x > b_i$ for some $i \in \{1, \ldots, n\}$.*

(1) *If j_1 is the minimal, j_2 the maximal integer with $x > a_{j_1}$ and $x > a_{j_2}$, then $x > a_j$ when $j_1 \leq j \leq j_2$.*
(2) *If j_1 is the minimal, j_2 the maximal integer with $x > b_{j_1}$ and $x > b_{j_2}$, then $x > b_j$ when $j_1 \leq j \leq j_2$.*

Proof. If $x||a_j$ for some $j_1 \leq j \leq j_2$, then $B(P)$ contains a crown.

If $x||b_j$ but $x > b_{j-1}$ and $x > b_{j+1}$, then $B(P)$ contains $\mathbf{D}^d$. Finally, suppose that $x > b_{j_*}$ and $x > b_{j^*}$ and $j_*+2 < j^*$ but $x||b_j$ for all $j_* < j < j^*$. Then from the first part we know that $x > a_{j_*+1}, a_{j_*+2}$ and the elements of $\{ a_{j_*}, a_{j_*+1}, a_{j_*+2}, b_{j_*}, b_{j_*+1}, b_{j_*+2}, x \}$ generate a copy of $\mathbf{CX}_3$ in $B(P)$. □

Note that the dual of this lemma is also true, i.e, if x is below some element a_i, then the elements of the fence comparable with x are intervals of a's and b's.

Case 10. $B(P)$ contains $\mathbf{F}_n$.
We also assume that $B(P)$ does not contain $\mathbf{F}_m$, $\mathbf{E}_m$ or $\mathbf{E}_m^d$ when $0 \leq m < n$.

Now suppose, that $2 \leq i \leq n+2$. Choose a low-made element x with $x \geq b_i$ and $x||d$.

Suppose, that $x < e$, it follows that $x||b_1$. Let j_1 be the minimal, j_2 the maximal integer with $x > a_{j_1}$ and $x > a_{j_2}$. If $j_2 = j_1 + 1$, then there is an element b_i^ℓ in

P. Otherwise, $\{ a_1, \ldots, a_{j_1}, a_{j_2}, \ldots, a_{n+2} \} \cup \{ b_1, \ldots, b_{j_1}, x, b_{j_2+1}, \ldots, b_{n+2} \} \cup \{ c, d, e \}$ generates $\mathbf{F}_m$ for some $m < n$.

Now suppose, that $x||e$. If $x||a_{n+2}$ then $\{ a_{j_2}, \ldots, a_{n+2} \} \cup \{ x, b_{j_2+1}, \ldots, b_{n+2} \} \cup \{ c, d, e \}$ generates $\mathbf{F}_m$ for some $m < n$. We, henceforth, may assume that $x > a_{n+2}$. If $x > b_1$, then $\{ c, d, e, a_{n+2}, x, b_1 \}$ form a copy of $\mathbf{D}$. Now assume, that $x||b_1$ and $j_1 > 1$, then $\{ a_1, \ldots, a_{j_1} \} \cup \{ b_1, \ldots, b_{j_1}, x \} \cup \{ d, e \}$ generate $\mathbf{E}_m$ for some $m < n$. Finally, if $b_1, e \in \mathit{Inc}(x)$ and $x > a_1$, $x > a_{n+2}$, then $\{ a_1, a_{n+2}, b_1, c, d, e, x \}$ form a copy of $\mathbf{FX}_1^d$. We may, therefore, assume an element b_i^ℓ in P.

By symmetry and duality we may also assume elements a_i^u, when $1 \leq i \leq n+1$. From Lemma 3 we obtain $a_{n+2}^\ell, b_1^\ell, e^\ell, d^\ell$ and $a_{n+2}^u, b_1^u, c^u, d^u$ in P. The bipartite order corresponding to these elements form a copy of $\mathcal{F}_n$ in P.

Case 11. $B(P)$ contains $\mathbf{E}_n$.
We also assume that $B(P)$ does not contain $\mathbf{F}_m$, $\mathbf{E}_m$ or $\mathbf{E}_m^d$ when $0 \leq m < n$.

Now suppose, that $1 \leq i \leq n+1$. Choose an up-made element a' with $a' \leq a_i$ and $a'||b_{i+2}$. Assume, that $a' < c$ and let j_0 be the minimal index j with $a' < a_j$. If $j_0 > 1$ then $\{ a_{j_0-1}, a_{j_0}, \ldots, a_{i+1} \} \cup \{ b_{j_0}, b_{j_0+1}, \ldots, b_{i+1} \} \cup \{ a', c \}$ forms a copy of $\mathbf{E}_m^d$ where $m < n$. If $j_0 = 1$ then $\{ a_1, a_2, \ldots, a_{i+1} \} \cup \{ b_1, b_2, \ldots, b_{i+1} \} \cup \{ a', c, d \}$ forms a copy of $\mathbf{F}_m^d$ where $m < n$. The contradiction allows us to conclude $a'||c$. Let j_1 (j_2) be the minimal (maximal) index j with $a' < b_j$. If $j_2 \neq j_1 + 1$, then $\{ a_1, \ldots, a_{j_1-1}, a', a_{j_2+1} \ldots, a_{n+2} \} \cup \{ b_1, \ldots, b_{j_1}, b_{j_2}, \ldots, b_{n+3} \} \cup \{ c, d \}$ forms a copy of $\mathbf{E}_m^d$ where $m < n$. We may, therefore, assume elements $a_1^u, \ldots, a_{n+1}^u$ and by symmetry also a_{n+2}^u.

Next choose an integer i with $2 \leq i \leq n+2$ and a low-made element $x \geq b_i$ with $x||c$. Suppose, that $x||d$. If $x > a_1$ and $x > a_{n+2}$, then if $b_1, b_{n+3} \in \mathit{Inc}(x)$, a copy of $\mathbf{EX}_1$ is generated by $\{ b_1, x, b_{n+3}, d, a_1, a_{n+2}, e \}$. The same point set generates $\mathbf{EX}_2$ if only one of b_1 and b_{n+3} is incomparable with x. It contains $\mathbf{D}^d$ if $b_1, b_{n+3} < x$. We now assume, without loss of generality, that $x||a_{n+2}$. Let j_0 be the maximal index j with $x > a_{j_0}$, it follows that $\{ a_{j_0}, \ldots, a_{n+2} \} \cup \{ x, b_{j_0+1}, \ldots, b_{n+3} \} \cup \{ c, d \}$ generates $\mathbf{E}_m^d$ where $m < n$. Now suppose, that $x < d$. Thus $b_1, b_{n+3} \in \mathit{Inc}(x)$. Note, that if $x > a_j$ for some $j \neq i-1, i$, then $B(P)$ contains $\mathbf{E}_m^d$ for some $m < n$. Therefore, we may assume the existence of elements $b_2^\ell, b_3^\ell, \ldots, b_{n+2}^\ell$ in P.

Finally, from Lemma 3 we obtain elements $b_1^\ell, b_{n+3}^\ell, c^\ell, d^\ell$ and b_1^u, b_{n+3}^u, c^u. The bipartite order corresponding these elements forms a copy of $\mathcal{F}_n$ in P.

Case 12. $B(P)$ contains $\mathbf{B}$.
Suppose, that a is not up-made. If there is an up-made element $a' < a$ with $|\mathit{Succ}(a') \cap \{ b_1, b_2, b_3 \}| = 1$ then $B(P)$ contains $\mathbf{E}_0^d$. Otherwise, there are up-made elements $a_1, a_2, a_3 \leq a$ with $a_i||b_i$ and $a_i||a_j$ for $i = 1, 2, 3$ and $j \neq i$. However, $\{ a_1, a_2, a_3, b_1, b_2, b_3 \}$ generate a crown $\mathbf{A}_3$ in this case. We, therefore, may assume an

element a^u.

Next, choose a low-made element $c_1' \geq c_1$ with $b_2, c_2 \in Inc(c_1')$. If $c_1' > c_3$, then $B(P)$ contains $\mathbf{C}^d$. So we assume $c_1' || c_3$. If $c_1' > b_3$, then $B(P)$ contains $\mathbf{CX}_1$, so we may assume that $c_1' || b_3$. This gives an element c_1^ℓ in P. By symmetry, we may also assume elements c_2^ℓ and c_3^ℓ.

From Lemma 3 we obtain elements $b_1^\ell, b_2^\ell, b_3^\ell$ and b_1^u, b_2^u, b_3^u. The bipartite order corresponding to these elements forms a copy of $\mathcal{O}_1^d$ in P.

Case 13. $B(P)$ contains $\mathbf{I}_n$.
We also assume that $B(P)$ does not contain a copy of $\mathbf{I}_m$ when $m < n$.

Choose an integer i with $2 \leq i \leq n+2$ and a low-made element $x \geq b_i$ with $x || c$. Let j_1 (j_2) be the minimal (maximal) integer with $x > a_{j_1}$ (a_{j_2}) and suppose that $j_2 \neq j_1 + 1$.

Assume, that $x < d_1$ and $x < d_2$. It follows that $b_1, b_{n+3} \in Inc(x)$ and $B(P)$ contains a copy of $\mathbf{I}_m$ for $m < n$.

If $x || d_2$ and $j_2 < n+2$, then $\{ a_{j_2}, \dots, a_{n+2} \} \cup \{ x, b_{j_2+1}, \dots, b_{n+3} \} \cup \{ c, d_2 \}$ is a copy of $\mathbf{E}_m$ for some $m < n$. We, therefore, assume that $j_2 = n+1$. If $x < d_1$, then $\{ a_{n+2}, b_{n+3}, d_2, c, d_1, x \}$ generate a copy of $\mathbf{D}$ in $B(P)$. Otherwise, if $x || d_1$, then, by symmetry, we may also assume $x > a_1$. If $x \not> b_1$, then $\{ a_{n+2}, a_1, b_1, c, d_1, d_2, x \}$ generate a copy of $\mathbf{FX}_2$ in $B(P)$. Again, by symmetry, we now assume $x > b_1$ and $x > b_{n+3}$, but then $\{ b_1, d_1, c, d_2, b_{n+3}, x \}$ generates a crown $\mathbf{A}_3$ in $B(P)$. Hence, we may assume elements b_i^u for $2 \leq i \leq n+2$ in P.

Now, let i be an integer with $1 \leq i \leq n+2$. Choose an up-made element $a_i' \leq a_i$ with $c \in Inc(a_i')$. If $b_1 > a_i'$ and $b_{n+3} > a_i'$, then $\{ a_i', b_{n+3}, d_2, c, d_1, b_1 \}$ generate a copy of $\mathbf{D}$ in $B(P)$. So we may assume, without loss of generality, that $a_i' || b_{n+3}$. It follows easily, that $B(P)$ contains $\mathbf{I}_m$ where $0 \leq m < n$, whenever there exists an integer j with $b_j > a_i'$ and $j \neq i, i+1$.

From Lemma 3 we obtain elements $b_1^\ell, b_{n+3}^\ell, c^\ell$ and $b_1^u, b_{n+3}^u, c^u, d_1^u, d_2^u$ in P. The bipartite order corresponding to these elements forms a copy of $\mathcal{I}_{n+1}$ in P.

Case 14. $B(P)$ contains $\mathbf{G}_n$.
We also assume, that $B(P)$ does not contain a copy of $\mathbf{G}_m$, $\mathbf{J}_m$ or $\mathbf{H}_m$ when $m < n$.

We first show, that we may assume an element a_i^ℓ when $2 \leq i \leq n+2$. Choose a low-made element $x \geq a_i$ with $b_{i-1}, b_i \in Inc(x)$. Note, that from $x || b_{i-1}$ it follows that $x \not> a_j$ and $x \not> b_j$ for all $i < j \leq n+3$.

Assume, that $x > c$ and note that this implies $x || a_j$ when $i < j \leq n+3$ and $x || b_j$ when $i - 1 \leq j < n+3$. Let $i \neq n+2$. If $x < b_{n+3}$, then $\{ a_i, a_{i+1}, \dots, a_{n+3} \} \cup \{ b_i, b_{i+1}, \dots, b_{n+3} \} \cup \{ x \}$ forms a copy of $\mathbf{G}_m$ for some $m < n$. Otherwise, if $x || b_{n+3}$, then the same set together with c generates $\mathbf{J}_m$ for some $m < n$.

Now, let $i = n+2$. If $x < b_{n+3}$, then $\{ b_{n+3}, b_{n+2}, b_{n+1}, x, a_{n+2}, a_{n+3} \}$ forms a copy of

$\mathbf{D}^d$. Otherwise, if $x||b_{n+3}$, then the same set together with c generates $\mathbf{CX}_3^d$. Therefore, we may assume, that $x||c$ and we have found an element a_i^ℓ in P.

We now show, that we may assume an element b_i^ℓ when $2 \le i \le n+2$. Choose a low-made element $x \ge b_i$ with $a_i, a_{i+1} \in \mathit{Inc}(x)$.

Suppose, that $x > c$. If $i > 2$, then $\{ a_1, \dots, a_i \} \cup \{ b_1, \dots, b_{i-1}, x \} \cup \{ c \}$ forms a copy of $\mathbf{G}_m$ for some $m < n$. Otherwise, if $i = 2$, then $\{ a_1, a_2, a_3, b_1, x, c \}$ generate $\mathbf{D}$. Therefore, $x||c$ and we have found an element b_i^ℓ in P.

We have shown, that we may assume elements a_i^ℓ and b_i^ℓ for all $2 \le i \le n+2$. By duality, we may as well assume elements a_i^u and b_i^u in P for all $2 \le i \le n+2$. Finally, from Lemma 3 we obtain elements $a_{n+3}^\ell, b_1^\ell, b_{n+3}^\ell, c^\ell$ and $a_1^u, a_{n+3}^u, b_1^u, c^u$. The bipartite order corresponding to these elements forms a copy of $\mathcal{G}_n$ in P.

Case 15. $B(P)$ contains $\mathbf{J}_n$.
We also assume, that $B(P)$ does not contain a copy of $\mathbf{G}_m$, $\mathbf{J}_m$ or $\mathbf{H}_m$ when $m < n$.

In complete analogy with Case 14, we may assume elements a_i^ℓ in P when $2 \le i \le n+2$.

We now show, that we may assume an element b_i^ℓ when $2 \le i \le n+2$. Choose a low-made element $x \ge b_i$ with $a_i, a_{i+1} \in \mathit{Inc}(x)$.

Suppose, that $x > c$. If $i > 2$ and $x > d$, then $\{ a_1, \dots, a_i \} \cup \{ b_1, \dots, b_{i-1}, x \} \cup \{ c \}$ forms a copy of $\mathbf{G}_m$ for some $m < n$. If $x||d$, then the same set together with d generates $\mathbf{J}_m$ for some $m < n$.

If $i = 2$ and $x > d$, then $\{ a_1, a_2, a_3, b_1, x, d \}$ generate $\mathbf{D}$. If $x||d$, then $\{ a_1, a_2, a_3, b_1, x, c, d \}$ generate $\mathbf{CX}_3$. The contradiction shows that $x||c$.

We have shown, that we may assume elements a_i^ℓ and b_i^ℓ for all $2 \le i \le n+2$. By duality, we may as well assume elements a_i^u and b_i^u for all $2 \le i \le n+2$ in P. Finally, from Lemma 3 we obtain elements $a_{n+3}^\ell, b_1^\ell, b_{n+3}^\ell, c^\ell$ and $a_1^u, a_{n+3}^u, b_1^u, c^u$. The bipartite order corresponding to these elements forms a copy of $\mathcal{G}_n$ in P.

Case 16. $B(P)$ contains $\mathbf{H}_n$.
We also assume that $B(P)$ does not contain a copy of $\mathbf{H}_m$ when $m < n$, nor $\mathbf{G}_m$ or $\mathbf{J}_m$ when $m \le n$.

We first show, that we may assume an element a_i^ℓ when $1 \le i \le n+2$. Choose a low-made element $x \ge a_i$ with $b_i, b_{i+1} \in \mathit{Inc}(x)$.

Assume, that $x > c$. If $i \ge 3$, then $\{ a_1, \dots a_{i-1}, x \} \cup \{ b_1, \dots, b_i \} \cup \{ c \}$ forms a copy of $\mathbf{G}_m$ for some $m < n$. Otherwise, if $i = 1, 2$, then $\{ b_1, b_2, b_3, a_1, x, c \}$ forms a copy of $\mathbf{D}$.

Assume, that $x > d$. Let $i \ne n$. If $x < b_{n+3}$, then $\{ a_i, \dots, a_{n+2} \} \cup \{ b_{i+1}, \dots, b_{n+3} \} \cup \{ x \}$ forms a copy of $\mathbf{G}_m$ for some $m < n$. Otherwise, if $x||b_{n+3}$, then the same set together with c generates $\mathbf{J}_m$ for some $m < n$.

If $i = n+1$ and $x < b_{n+3}$, then $\{ b_{n+3}, b_{n+2}, b_{n+1}, x, a_{n+1}, a_{n+2} \}$ forms a copy of $\mathbf{D}^d$. Otherwise, if $x||b_{n+3}$, then the same set together with d generates $\mathbf{CX}_3^d$.

If $i = n+2$, then $\{ b_{n+1}, b_{n+2}, b_{n+3}, d, x, a_{n+2} \}$ forms a copy of $\mathbf{D}$.

Therefore, we may assume $c, d \in Inc(x)$ and have thus found an element a_i^ℓ.

We now show, that we may assume an element b_i^ℓ when $2 \le i \le n+2$. Choose a low-made element $x \ge b_i$ with $a_{i-1}, a_i \in Inc(x)$.

Suppose, that $x > c$ and $x > d$, then $\{ b_1, a', b_{n+3}, d, x, c \}$ forms a copy of $\mathbf{D}$ where a' is one of a_{i-1} or a_i.

Suppose $x > c$ and $x||d$, it follows, that $x||a_j$ and $x||b_j$ when $j > i$. If $i < n+2$ then $\{ a_i, \dots, a_{n+2} \} \cup \{ b_i \dots .b_{n+3} \} \cup \{ x, d \}$ generates $\mathbf{H}_m$ for some $m < n$. If $i = n+2$, then $\{ a_{n+1}, a_{n+2}, b_{n+1}, b_{n+2}, b_{n+3}, d, x \}$ generates $\mathbf{CX}_3^d$.

Finally, let $x > d$ and $x||c$. If $i > 2$, then $\{ a_1, \dots, a_{i-1} \} \cup \{ b_1 \dots .b_{i-1}, x \} \cup \{ c, d \}$ generates $\mathbf{H}_m$ for some $m < n$. If $i = 2$, then depending on $x < b_{n+3}$ or $x||b_{n+3}$, a copy of either $\mathbf{CX}_1$ or $\mathbf{CX}_2$ is formed by $\{ a_1, a_{n+2}, b_1, b_{n+3}, c, d, x \}$.

We, therefore, may assume $c, d \in Inc(x)$ and have found elements b_i^ℓ for $2 \le i \le n+2$.

If b_{n+3} is not low-made, then choose a low-made element $b'_{n+3} > b_{n+3}$ with $a_{n+2}||b'_{n+3}$. If $b'_{n+3} > c$, then $\{ a_1, a_{n+2}, b_1, b_{n+3}, c, d, x \}$ generate a copy of $\mathbf{FX}_1$. Therefore, we may assume an element b_{n+3}^ℓ in P.

We have shown, that we may assume elements a_i^ℓ for $1 \le i \le n+2$ and b_i^ℓ for $2 \le i \le n+3$. Dually, we may assume elements a_i^u for $1 \le i \le n+2$ and b_i^u in P for $1 \le i \le n+2$. Finally, from Lemma 3 we obtain elements c^ℓ, d^ℓ and c^u, d^u. The bipartite order corresponding to these elements forms a copy of $\mathcal{H}_n$ in P.

2.2. 3-interval irreducible partial stacks. We begin this part with an analysis of the relation between the interval dimension of a bipartite order P and the interval dimension of partial stacks of P.

Lemma 5. *Let $P = (X, <)$ be a bipartite poset and $P' = (X, <')$ be a partial stack of P, then Idim $P' \ge$ Idim P.*

Proof. Let $I'_1, \dots, I'_k$ be a interval-realizer of P'. Now transform each of the interval orders I'_i. Extend the intervals of elements in $Min(P)$ to the left to the leftmost endpoint of an interval in I'_i, symmetrically, extend the intervals of elements in $Max(P)$ to the right to the rightmost endpoint of an interval in I'_i. From this transformation we obtain a new family $I_1, \dots, I_k$ of interval orders. This family is an interval-realizer of P. Hence, *Idim* $P' \ge$ *Idim* P as claimed. □

As a consequence we can now sharpen the result of Theorem 4.

Theorem 7. *If $P = (X, <_P)$ is a 3-interval irreducible poset then P is a reduced partial stack of some bipartite 3-interval irreducible poset.*

Proof. From Theorem 4 we know that P contains a reduced partial stack of some bipartite 3-interval irreducible poset $R = (Y, <_R)$. Therefore, there are sets $X_1, X_2 \subseteq$

X, such that $(X_1 \times X_2) \cap <_P \cong <_R$. Let R^* be the poset induced by P on $X_1 \cup X_2$. This poset is a reduced partial stack of R.

From Lemma 5 we obtain $Idim\,R^* \geq Idim\,R = 3$. On the other hand, R^* is a suborder of P, hence, $3 = Idim\ P \geq Idim\ R^*$. The irreducibility of P implies that $X = X_1 \cup X_2$ and, therefore, $P = R^*$. □

Lemma 6. *Let $P = (X, <)$ be a bipartite order, then $Idim\ P = Idim\ Stack(P)$.*

Proof. From Lemma 5 we obtain $Idim\ Stack(P) \geq Idim\ P$. As noted in the Introduction and proved in Theorems 1 and 2 $Idim\ P = dim\ Stack(P)$. Together this gives $Idim\,Stack(P) \geq dim\,Stack(P)$, the converse of this inequality is trivially valid for every order. □

Theorem 7 shows, that a classification of 3-interval irreducible posets amounts in work with partial stacks of bipartite orders. We, therefore, require a suitable notation for these objects.

Remark. An intuitive approach to partial stacks of a (connected) bipartite poset $P = (X, <_P)$ would be the following: Let $A = Min(P)$ and $B = Max(P)$. Consider the interval representation of P on $B(P) = (Y, <_Q)$. The intervals of elements $a \in A$ have $L(a) = 0$ and for $b \in B$ have $U(b) = 1$. These are called *free-ends*. This name is motivated by the observation: Suppose two functions $L', U' : X \to Y$ are given, such that, $0 \leq L'(a) < U(a)$ and $U'(a) = U(a)$ for all $a \in A$ and $L'(b) = L(b)$ and $1 \geq U'(b) > L(b)$ for all $b \in B$. The intervals $(L'(x), U'(x))$ then define an ordering $P' = (X, <'_P)$, such that, $(A \times B) \cap <_P = (A \times B) \cap <'_P$, i.e., a partial stack of P. For an example see Figure 3.

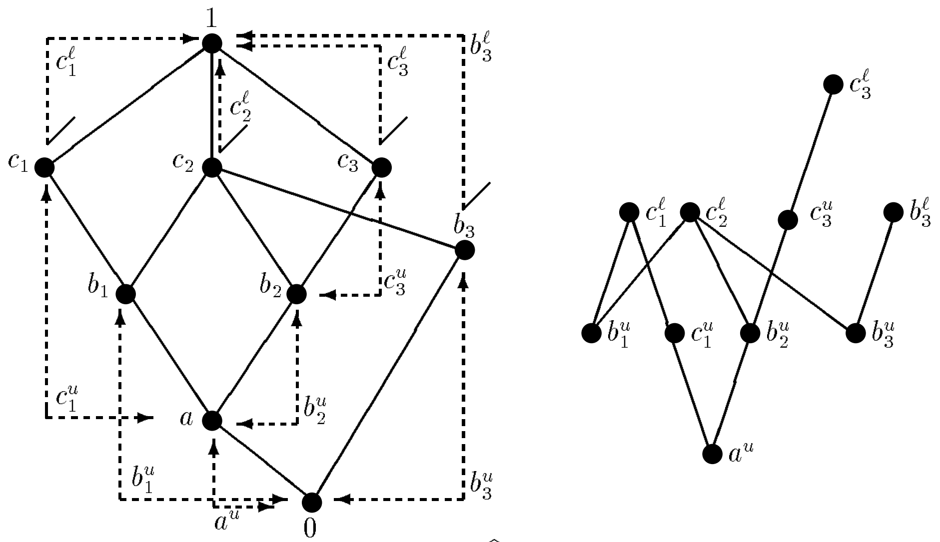


FIGURE 3. A placement of the free-ends in $\widehat{\mathbf{C}}$ and the corresponding partial stack of $\mathcal{O}_2$.

Unfortunately the above construction will not lead to all partial stacks. As an example note that there are partial stacks of $\mathcal{O}_2$ with $c_3^u > b_2^u$ but $c_3^u || a^u$. These partial stacks are not representable on $B(\mathcal{O}_2)$. Therefore we require a more formal description of partial stacks.

Let $Succ_{SP}(x)$ ($Pred_{SP}(x)$)denote the successor sets (predecessor sets) of x in $Stack(P)$. To describe a partial stack $Q = (X, <_Q)$ of a bipartite poset $P = (X, <_P)$ we will, henceforth, use sets

$Succ(x)$ with $Succ(x) \subseteq Succ_{SP}(x)$ for each $x \in Max(P)$,

$Pred(x)$ with $Pred(x) \subseteq Pred_{SP}(x)$ for each $x \in Min(P)$.

If such a family of sets is given, then the partial stack Q of P denoted by the family is the transitive closure of the union of all relations occurring in these sets together with the relations of P.

Lemma 5 and Lemma 6 might suggest that $Idim\, P = Idim\, Q$ for every partial stack Q of P. This, however, is far from truth as the next lemma shows.

Lemma 7. *For every integer n there is a bipartite interval order I, such that, every bipartite poset R on n elements is a suborder of some partial stack I_R of I.*

Proof. As I we take the interval order with $2n$ minimal elements $\{ a_1, \dots, a_{2n} \}$ and with $2n$ maximal elements $\{ b_1, \dots, b_{2n} \}$. The relations of I are: $a_i < b_j$ exactly if $i \leq j$. Now fix a bipartite n-element poset R with minimal elements $\{ x_1, \dots, x_k \}$ and maximal elements $\{ y_1, \dots, y_{n-k} \}$. A partial stack I_R containing R is given by:

$$\begin{aligned}
Pred(a_i) &= \emptyset \quad \text{for } 1 \leq i \leq k \\
Pred(a_i) &= \{ a_1, \dots, a_k \} \cup \{ b_j : x_j \leq_R y_i \} \quad \text{for } k < i \leq 2n \\
Succ(b_i) &= \{ b_{k+1}, \dots, b_{2n} \} \cup \{ a_{k+j} : x_i \leq_R y_j \} \quad \text{for } 1 \leq i \leq k \\
Succ(b_i) &= \emptyset \quad \text{for } k < i \leq 2n
\end{aligned}$$

Note, that $B(I)$ is a chain of length $2n+2$, while $B(I_R)$ consists of a chain of length $2n+3$, where the element $k+1$ has been substituted by a copy of $B(R)$ without 0 and 1. □

As a consequence we obtain, that the gap between the interval dimension of a bipartite poset P and and the interval dimension of partial stacks of P can be arbitrarily high.

We now turn to the partial stacks of $\mathcal{G}_m$ and $\mathcal{H}_m$. There will be some indications, that the complete classification of 3-interval irreducible posets among the reduced partial stacks of these two orders might be intractable.

Theorem 8. *Every bipartite poset R is a suborder of a partial stack Q_R of $\mathcal{G}_m$ ($\mathcal{H}_m$). Moreover, every bipartite 2-dimensional poset R is a suborder of an 3-interval irreducible partial stack of $\mathcal{G}_m$ ($\mathcal{H}_m$).*

Proof. We prove the theorem for partial stacks of $\mathcal{H}_m$. The idea behind the proof is the same as in the proof of Lemma 7.

Let R be a bipartite n-element poset with minimal elements $\{ x_1, \dots , x_k \}$ and maximal elements $\{ y_1, \dots , y_{n-k} \}$. Also, let a copy of $\mathcal{H}_{n-2}$ be given on the set $\{ a_1^\ell, \dots , a_n^\ell \} \cup \{ b_2^\ell, \dots , b_{n+1}^\ell \} \cup \{ c^\ell, d^\ell \} \cup \{ a_1^u, \dots , a_n^u \} \cup \{ b_1^u, \dots , b_n^u \} \cup \{ c^u, d^u \}$. The relations of $\mathcal{H}_{n-2}$ are: $x^\ell > y^u$ exactly if $x \geq y$ in $\mathbf{H}_{n-2}$. A partial stack Q_R of $\mathcal{H}_{n-2}$ containing R is given by:

$$
\begin{aligned}
Pred(a_i^u) &= \emptyset \quad \text{for } 1 \leq i \leq k \\
Pred(a_i^u) &= \{ a_j^u : 1 \leq j \leq k \} \cup \{ b_j^u : 1 \leq j \leq k \} \cup \{ a_j^\ell : x_j \leq_R y_i \} \\
&\quad \text{for } k < i \leq n \\
Succ(a_i^\ell) &= \{ a_j^\ell : k < j \leq n \} \cup \{ b_j^\ell : k < j \leq n+1 \} \cup \{ a_j^u : x_i \leq_R y_j \} \\
&\quad \text{for } 1 < i \leq k \\
Succ(a_i^\ell) &= \emptyset \quad \text{for } k < i \leq n
\end{aligned}
$$

All other elements z^ℓ have $Succ(z^\ell) = \emptyset$ and all other elements z^u have $Pred(z^u) = \emptyset$. This gives a partial stack Q_R of $\mathcal{H}_m$ and the poset generated by the elements $\{ a_i^\ell : 1 < i \leq k \} \cup \{ a_i^u : k < i \leq n \}$ is isomorphic to R.

Let $\mathbf{H}_{n-2}^+$ be isomorphic to $\mathbf{H}_{n-2}$ together with a new element e such that $Pred(e) = \{ a_1, \dots , a_k, b_1, \dots , b_k \}$ and $Succ(e) = \{ a_{k+1}, \dots , a_n, b_{k+2}, \dots , b_{n+1} \}$. The order $B(Q_R)$ is obtained by substituting e in $\mathbf{H}_{n-2}^+$ by a copy of $B(R)$ without 0 and 1.

If R is 2-dimensional, then we claim that Q_R is 3-interval irreducible. To verify this it suffices to consider the effect of point deletion in Q_R only on the essentially 3-dimensional part of $B(Q_R)$, i.e., on $\mathbf{H}_{n-2}^+$ □

Remark. Let $\mathbf{H}_m = (X, <)$ and let the completion by cuts of $\mathbf{H}_m$ be denoted by $\mathbf{L}(\mathbf{H}_m) = (X \cup Y, <)$. A large class of 3-interval irreducible reduced partial stacks of $\mathcal{H}_m$ can be characterized by the following two properties:

(1) $B(Q)$ is an order isomorphic to $\mathbf{L}(\mathbf{H}_m)$ with elements of Y substituted by (possibly empty) 2-dimensional posets.

(2) If x is ℓ-labeled (u-labeled) in $\mathbf{H}_m$, then there is a unique element $y \in Q$ with $L(y) = x$ ($U(y) = x$).

Similar properties describe a large class of 3-interval irreducible reduced partial stacks of $\mathcal{G}_m$.

However, there are more complex examples among the 3-irreducible partial stacks of $\mathcal{G}_m$ ($\mathcal{H}_m$). Here is an example.

Example.
Let $\mathcal{G}_0$ consist of the elements $\{ a_2^\ell, a_3^\ell, b_1^\ell, b_2^\ell, b_3^\ell, c^\ell \} \cup \{ a_1^u, a_2^u, a_3^u, b_1^u, b_2^u, c^u \}$ and relations

$x^\ell > y^u$ exactly if $x \geq y$ in $\mathbf{G}_0$. Let Q be the partial stack of $\mathcal{G}_0$ defined by:

$$\begin{aligned} Succ(a_2^\ell) &= \{\, a_3^u, a_3^\ell, b_3^\ell \,\} \\ Succ(b_1^\ell) &= \{\, a_3^\ell, b_2^u, b_2^\ell, b_3^\ell \,\} \\ Pred(a_3^u) &= \{\, a_2^\ell, a_2^u, a_1^u, b_1^u \,\} \\ Pred(b_2^u) &= \{\, a_1^u, b_1^u, b_1^\ell \,\} \end{aligned}$$

All other elements x^ℓ have $Succ(x^\ell) = \emptyset$ and all other elements x^u have $Pred(x^u) = \emptyset$. This gives a partial stack Q of $\mathcal{G}_0$ which does not fall into the class of the previous remark.

With these partial results we leave the classification of 3-interval irreducible reduced partial stacks of $\mathcal{G}_m$ and $\mathcal{H}_m$ as an open problem.

Problem 1. *Characterize the 3-interval irreducible reduced partial stacks of $\mathcal{G}_m$ and $\mathcal{H}_m$.*

We now turn to the partial stacks of the other bipartite 3-interval irreducible posets. With the next theorem we classify the 3-interval irreducible orders among them.

Theorem 9. *Every partial stack of $\mathcal{A}_n$, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{E}_n$, $\mathcal{I}_n$, $\mathcal{F}_0$ for every integer $n \geq 0$ and every partial stack of $\mathcal{F}_n$ $(n > 0)$ which has neither ($a_{n+2}^\ell < b_{n+2}^\ell$ and $a_{n+2}^\ell || e^\ell$) nor ($b_1^u > a_1^u$ and $b_1^u || c^u$) is a 3-interval irreducible poset.*

Proof. The argument is again divided into a series of cases. As criterion for the irreducibility of a candidate partial stack Q we use a technical lemma.

Lemma 8. *Let $P = (X, <_P)$ be a poset of interval dimension 3. Let $Q = (Y, <_Q)$ be 3-irreducible and let P have an interval representation on $\widehat{Q}$. P is 3-interval irreducible if the following three conditions are satisfied:*
(1) For every element y of Q one of three cases applies
(a) y has two essential labels.
(b) y has an essential label ℓ and a unique element y' in $\widehat{Q}$ is covered by y.
(c) y has an essential label u and a unique element y'' in $\widehat{Q}$ is covering y.
(2) Every element of Q has an essential label and every essential label of Q is satisfied by exactly one element of P.
(3) Every element of P satisfies at least one essential label of Q.

Proof. We have to show that for every $x \in X$ the interval dimension of P_x is at most 2. To do this we exhibit a poset $Q(x)$ of dimension 2 admitting an interval representation of P_x.

Let $x \in X$ and let y_x be an element of Q such that x satisfies an essential label at y_x. By duality we may suppose, that x satisfies the label ℓ at y_x. We distinguish two cases.

Suppose, that y_x has two essential labels. Let $R = (Z, <_R)$ be the bipartite 3-interval irreducible poset corresponding to the essential labels of Q. By definition, $R_{y_x^\ell}$ has interval dimension 2 and, hence, $dim\, B(R_{y_x^\ell}) = 2$. We claim that P_x has an interval representation on $B(R_{y_x^\ell})$. Let x' be any element of P_x, and denote the two endpoints of the interval of x' on $\widehat{Q}$ by $L(x') = y_1$ and $U(x') = y_2$. If $y_1 = 0$ or $y_2 = 1$, then the mappings $L^*, U^* : P_x \to B(R_{y_x^\ell})$, also map the corresponding interval end to 0 or 1. Let the essential labels at y_1, y_2 be ρ_1, ρ_2. In R we find elements $y_1^{\rho_1}$ and $y_2^{\rho_2}$ with $y_1^{\rho_1} \neq y_x^\ell$ and $y_2^{\rho_2} \neq y_x^\ell$. Now, let $L^*(x') = L(y_1^{\rho_1})$ if $\rho_1 = \ell$ and $L^*(x') = U(y_1^{\rho_1})$ if $\rho_1 = u$. Similarly, $U^*(x') = L(y_1^{\rho_1})$ if $\rho_1 = \ell$ and $U^*(x') = U(y_1^{\rho_1})$ if $\rho_1 = u$. It is easy to verify that the mappings $L^*, U^* : P_x \to B(R_{y_x^\ell})$ indeed define an interval representation of P_x.

Next, suppose that there is no essential label at y_x. Our assumptions imply, that there is a unique element y_x' covered by y_x. We claim that P_x has an interval representation on $\widehat{Q_{y_x}}$. Since Q is 3-irreducible we conclude that $dim\,\widehat{Q_{y_x}} = 2$ and, hence, that the interval dimension of P_x is 2. As mappings $L^*, U^* : P_x \to \widehat{Q_{y_x}}$ choose the mappings induced by $L, U : P \to \widehat{Q}$, except for elements x' with $U(x') = y_x$. For such an element let $U^*(x') = y_x'$. Again, it is easy to verify that the mappings $L^*, U^* : P_x \to \widehat{Q_{y_x}}$ define an interval representation of P_x. □

Case 1. P is a partial stack of $\mathcal{A}_n$.
Since $Stack(\mathcal{A}_n) = \mathcal{A}_n$ we obtain $Q = \mathcal{A}_n$ and P is 3-interval irreducible.

Case 2. P is a partial stack of $\mathcal{O}_1$.
The partial stacks of $\mathcal{O}_1$ differ only in $Succ(b_i^\ell)$ for $i = 1, 2, 3$ which is either empty or $\{ c_i^\ell \}$. An interval representation on $\widehat{\mathbf{B}}$ is obtained if we let $U(b_i^\ell) = c_i$ when $Succ(b_i^\ell) = \{ c_i^\ell \}$ and else $U(b_i^\ell) = 1$. These representations satisfy the conditions of Lemma 8 and, hence, P is 3-interval irreducible.

Case 3. P is a partial stack of $\mathcal{O}_2$.
If $Pred(c_1^u)$ is one of $\{ a^u, b_1^u \}$ or $\{ a^u \}$ or $\emptyset$ and $Pred(c_3^u)$ is one of $\{ a^u, b_2^u \}$ or $\{ a^u \}$ or $\emptyset$, then $B(P) = \widehat{\mathbf{C}}$. Moreover, the pair $(P, \mathbf{C})$ satisfies the conditions of Lemma 8 and, therefore, P is 3-interval irreducible.

If $Pred(c_1^u)$ is one of $\{ a^u, b_1^u \}$ or $\{ a^u \}$ or $\emptyset$ but $Pred(c_3^u) = \{ b_2^u \}$, then $B(P) = \widehat{\mathbf{CX}_1}^d$. Again, Lemma 8 yields the 3-interval irreducibility of P.

By symmetry, we may now assume that $Pred(c_1^u) = \{ b_1^u \}$ and $Pred(c_3^u)) = \{ b_2^u \}$. Here, $B(P) = \widehat{\mathbf{CX}_2}^d$ and again, P is 3-interval irreducible by Lemma 8.

Case 4. P is a partial stack of $\mathcal{O}_3$.
All partial stacks of $\mathcal{O}_3$, together with interval representations on $\widehat{\mathbf{EX}_2}$ can be obtained by all possible choices of $L(b_1^u) \in \{ a_1, 0 \}$, $L(b_3^u) \in \{ a_2, 0 \}$, $U(b_1^\ell) \in \{ c, 1 \}$ and $U(a_3^\ell) \in \{ b_2, 1 \}$. With Lemma 8 we then obtain the 3-interval irreducibility of P.

Case 5. P is a partial stack of $\mathcal{E}_n$.
All partial stacks of $\mathcal{E}_n$, together with interval representations on $\widehat{\mathbf{E}_n}$ can be obtained by all possible choices of $L(b_1^u) \in \{ a_1, 0 \}$, $L(b_{n+3}^u) \in \{ a_{n+2}, 0 \}$ and for $x \in \{ b_2^\ell, \ldots, b_{n+2}^\ell, c^\ell \}$ either $U(x) = d$ or $U(x) = 1$. These representations satisfy the conditions of Lemma 8 and, hence, P is 3-interval irreducible.

Case 6. P is a partial stack of $\mathcal{I}_n$.
First, suppose that $n = 0$ and let $\mathbf{D}^+$ be obtained from $\mathbf{D}$ by adding a new element d with $d > b_2$ and $d < c_1, c_2$. We claim that every partial stack of $\mathcal{I}_0$ has an interval representation on $\widehat{\mathbf{D}^+}$. It should be clear that for $i = 1, 3$ we find an appropriate point for $U(b_i^\ell)$ and $L(b_i^u)$ in $\widehat{\mathbf{D}^+}$. It remains to consider $U(b_2^\ell)$. In $\mathit{Stack}(\mathcal{I}_0)$ the successor set of b_2^ℓ is $\{ c_1^\ell, c_2^\ell \}$ and, hence, $\mathit{Succ}_P(b_2^\ell) \subseteq \{ c_1^\ell, c_2^\ell \}$. If $\mathit{Succ}_P(b_2^\ell) = \{ c_1^\ell, c_2^\ell \}$, then let $U(b_2^\ell) = d$. If $\mathit{Succ}_P(b_2^\ell) = \{ c_i^\ell \}$ with $i = 1$ or $i = 2$, then let $U(b_2^\ell) = c_i$. Finally, if $\mathit{Succ}_P(b_2^\ell) = \emptyset$, then let $U(b_2^\ell) = 1$. With a slight generalization of Lemma 8 we obtain the 3-interval irreducibility of P.

Now, let $n > 0$ and let $\mathbf{I}_n^+$ be obtained from $\mathbf{I}_n$ by adding a new element e with $e < d_1$, $e < d_2$, $e > c$ and $e > b_i$ for $i = 2, \ldots, n+2$. An analysis as in the previous case shows that every partial stack of $\mathcal{I}_n$ has an interval representation on $\widehat{\mathbf{I}_n^+}$. (Note that $\widehat{\mathbf{I}_n^+} = \mathbf{L}(\mathbf{I}_n)$). Again, the 3-interval irreducibility of P follows from the generalization of Lemma 8.

Case 7. P is a partial stack of $\mathcal{F}_0$.
If $\mathit{Pred}(b_1^u)$ is one of $\{ a_1^u, c^u \}$ or $\{ c^u \}$ or $\emptyset$ and $\mathit{Succ}(a_2^\ell)$ is one of $\{ b_2^\ell, e^\ell \}$ or $\{ e^\ell \}$ or $\emptyset$, then $B(P) = \widehat{\mathbf{F}}_0$. Moreover, the pair $(P, \mathbf{F}_0)$ satisfies the conditions of Lemma 8 and, therefore, P is 3-interval irreducible.

If $\mathit{Pred}(b_1^u)$ is one of $\{ a_1^u, c^u \}$ or $\{ c^u \}$ or $\emptyset$ but $\mathit{Succ}(a_2^\ell) = \{ b_2^\ell \}$, then $B(P) = \widehat{\mathbf{FX}_1}^d$. Again, Lemma 8 yields the 3-interval irreducibility of P.

By symmetry we may now assume that $\mathit{Pred}(b_1^u) = \{ a_1^u \}$ and $\mathit{Succ}(a_2^\ell) = \{ b_2^\ell \}$. Here, $B(P) = \widehat{\mathbf{FX}_2}^d$ and P is 3-interval irreducible by Lemma 8.

Case 8. P is a partial stack of $\mathcal{F}_n$ with $n > 0$.
Suppose that $\mathit{Succ}_P(a_{n+2}^\ell) = \{ b_{n+2}^\ell \}$. The bipartite subposet of P with minimal elements $\{ a_1^u, \ldots, a_{n+2}^u, b_1^u, d^u \}$ and $\{ b_1^\ell, \ldots, b_{n+2}^\ell, d^\ell, e^\ell \}$ as maximal elements forms a copy of $\mathcal{E}_{n-1}$. If we remove c^u and a_{n+2}^u from P to obtain P^-, then P^- is a partial stack of $\mathcal{E}_{n-1}$. Therefore, P is not irreducible.

We may now assume that $\mathit{Succ}_P(a_{n+2}^\ell)$ is either $\{ b_{n+2}^\ell, e^\ell \}$ or $\{ e^\ell \}$ or $\emptyset$. Dually, we may assume that $\mathit{Pred}_P(b_1^u)$ is $\{ a_1^u, c^u \}$ or $\{ c^u \}$ or $\emptyset$. It follows that $B(P) = \widehat{\mathbf{F}_n}$ and with Lemma 8 that P is 3-interval irreducible.

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Freie Universität Berlin, Fachbereich Mathematik, Institut für Informatik, Takustr. 9, 14195 Berlin, Germany

E-mail address: felsner@inf.fu-berlin.de