1 Geometric Graphs: Turán Problems

The first part of this chapter collects rather elementary and well known material: We start with definitions of geometric and topological graphs, rush through basic notions from graph theory and report on facts about planar graphs. Beginning with Section 1.4 we discuss problems from the extremal theory for geometric graphs. That is, we deal with questions of Turán type: How many edges can a geometric graph avoiding a specified configuration of edges have?

1.1 What is a Geometric Graph?

As usual in texts dealing with graph theory, we shall begin by defining a *graph* G to be a pair (V, E) where V is the set of *vertices*, E is the set of *edges* $\{v, w\}$ each joining two vertices $v, w \in V$. To illustrate the definition, let us look at some pictures.

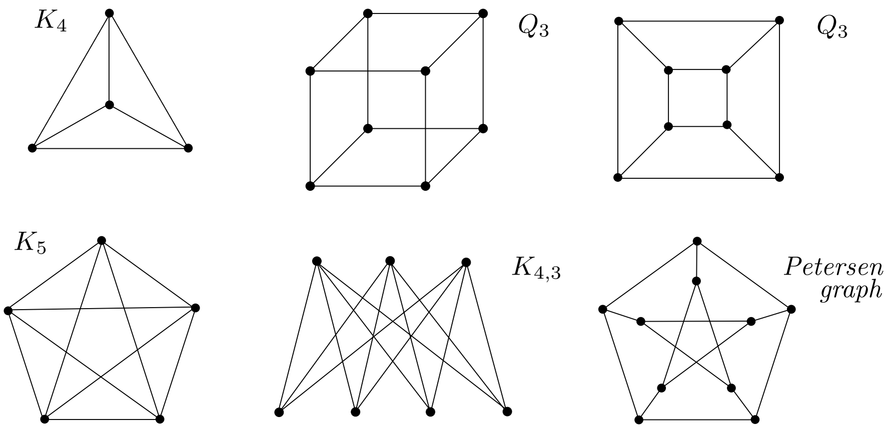

Figure 1.1 A little gallery of graphs, the cube graph Q_3 is shown with two drawings.

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane with straight edges. That is, V is a set of points in the plane and E is a set of line segments with endpoints in V. For convenience, we often assume that the points of V are in general position, i.e., no three points of V are collinear. Pictures of graphs, as Figure 1.1, display geometric graphs. Hence, the study of geometric graphs can be viewed as the study of straight line drawings of a graph.

A *topological graph* is a graph drawn in the plane such that edges are represented by Jordan curves with the property that any two of these curves share at most one point. Obviously, geometric graphs are a subclass of topological graphs

In general, a drawing of a graph G will display *crossings* of edges that are disjoint as edges of G. Many interesting questions about geometric and topological graphs are concerned with crossing patterns in one way or the other. Among the simplest and most natural questions of this type is the quest of characterizing those graphs admitting a drawing without crossing edges. These *planar graphs* have been intensely studied for more than hundred years. Before initiating our studies on geometric graphs with a section on planar graphs, we continue with a brief overview of basic concepts of graph theory.

1.2 Fundamental Concepts in Graph Theory

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $V \to V'$ such that two vertices are joined by an edge in G iff the corresponding vertices are joined by an edge in G'. When discussing the structure of a graph G the concrete set of vertices is, usually, of no relevance and G is used as an arbitrary representative of its isomorphism class. If this is not the case, then we emphasize the fact that the vertex set of G is distinguished by calling G a *labeled graph*. Two vertices joined by an edge are called *adjacent*. If vertex v is a vertex of edge e, then v and e are said to be *incident*. The *neighborhood* $N(v)$ of vertex v is the set of adjacent vertices. The size of the neighborhood is the degree $d(v) = |N(v)|$. A *complete graph* is a graph such that every pair of its vertices is an edge. With K_n we denote a complete graph (the isomorphism class of complete graphs) on n vertices.

A *path* in G is a sequence $v_1, v_2, \ldots, v_k$ of vertices of G such that $\{v_i, v_{i+1}\}$ is an edge for $1 \leq i < k$. We call $v_1, v_2, \ldots, v_k$ *a path from* v_1 *to* v_k and vertices v_1 and v_k the *endpoints* of the path. A path is a *simple path* if all its vertices are distinct. A *cycle* is a path with identical endpoints, i.e., $v_1 = v_k$. A graph without cycles is a *forest*.

A graph G is *connected* if any two vertices can be joined by a path, otherwise G is *disconnected*. The maximal connected pieces of a graph G are the *components* of G. A connected forest is a *tree*. Trees have many characterizations.

Proposition 1.1 *For a graph G with n vertices the following are equivalent:*

(a) *G is connected and has no cycles (i.e., G is a tree).*

(b) *G is connected and has $n - 1$ edges.*

(c) *G has no cycles and $n - 1$ edges.*

A *leaf* is a vertex of degree 1, every tree with at least two vertices has at least two leaves. If $G = (V, E)$ is connected and W is a set of vertices, such that removing W and all edges incident to vertices in W from E leaves a disconnected graph, then we call W a *separating set* of G. For $k \geq 2$ we say that a graph G is *k-connected* if every separating set of G has cardinality at least k. For complete graphs we make the exceptional agreement that K_{k+1} is k-connected.

A *subgraph* of $G = (V, E)$ is a graph $H = (W, F)$ with $W \subseteq V$ and $F \subseteq E$. The subgraph H of G is *spanning* if H has the full vertex set of G, i.e., $W = V$. A particularly important class of spanning subgraphs are the *spanning trees* An *induced subgraph* of G is a subgraph H containing all edges of G joining two vertices in W. The subgraph of G induced by W is denoted $G[W]$. A *clique* in $G = (V, E)$ is a set W of vertices such that $G[W]$ is a complete graph. If $G[W]$ has no edges, then we call W an *independent set*. A graph $G = (V, E)$ is a *bipartite* graph if V can be partitioned into two independent sets V_1, V_2 such that $E \subseteq V_1 \times V_2$. The graphs Q_3 and $K_{4,3}$ from Figure 1.1 are examples of bipartite graphs. Similarly, G is *k-partite* if there is a partition $V_1, V_2, \ldots, V_k$ of V such that each V_i is independent. With $K_{n_1,n_2,\ldots,n_k}$ we denote a *complete k-partite* graph: It has a k-partition $V_1, V_2, \ldots, V_k$ such that V_i contains n_i vertices and the edges are all pairs of vertices from distinct classes.

Many problems in *extremal graph theory* can be stated in the form: How many edges can a graph on n vertices satisfying a certain property P have? An important special case of this question is when P is the property of avoiding a subgraph isomorphic to some fixed graph H, graphs avoiding such a subgraph are called *H-free.* Turán initiated extremal graph theory by solving a problem of this type.

Theorem 1.2 (Turán 1941) *The number of edges of a K_{k+1}-free graph on n vertices is at most $(1 - \frac{1}{k})\frac{n^2}{2}$.*

Turán also provided a complete characterization of the extremal examples. There is -up to isomorphism- a unique K_{k+1}-free graph with a maximum number of edges, this graph, the *Turán graph* $T_{k+1}(n)$ is the complete k-partite graph with $|V_i| = \lceil n/k \rceil$ or $|V_i| = \lfloor n/k \rfloor$ for all $i = 1, \ldots, k$.

Before studying extremal problems for geometric graphs, we have a section on planar graphs.

1.3 Planar Graphs

A graph is *planar* if it can be drawn in the plane without crossing edges. A *plane graph* is a planar graph together with such a drawing. In this context we do not insist that the edges are straight, that is, a plane drawing of a planar graph shows a topological graph but not necessarily a geometric graph. The existence of a straight line drawing for every planar graph is a nontrivial fact (the next chapter contains a proof). A planar

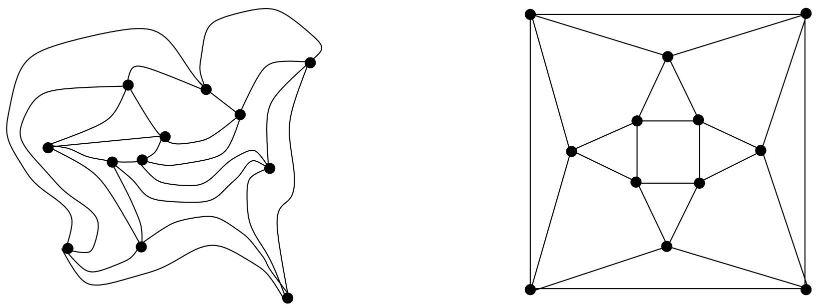

Figure 1.2 A topological and a straight plane drawing of the same graph.

drawing decomposes the plane into connected regions, one of them is the unbounded (outer) region, these regions are referred to as *faces.* To circumvent the special role taken by the outer face it is sometimes convenient to think of a planar drawing as a drawing on the sphere. A graph has a non-crossing drawing in the plane iff it has such a drawing on the sphere. This can be shown by stereographic projections: Place a sphere S on the plane and identify a point x in the plane with the point x' on the sphere where the ray from x to the north-pole N intersects S, see Figure 1.3.

One of the most fundamental and useful facts about planar graphs is Euler's Formula.

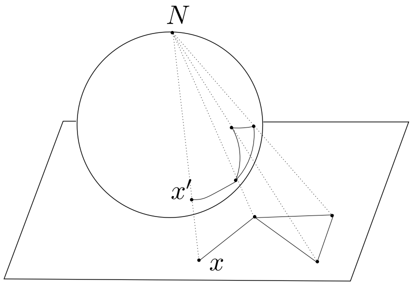

Figure 1.3 A stereographic projection.

Theorem 1.3 (Euler's Formula) *If G is a connected plane graph with v vertices, e edges and f faces, then*

$$v - e + f = 2.$$

Proof. Remove the edges of G in any order from the drawing. Upon removal of any given edge a there are two possibilities:

(1) Edge a separates two different faces. Removing a reduces the number of faces by one and leaves the number of components invariant.

(2) Edge a is incident to the same face on both sides. Removing a increases the number of components by one and leaves the number of faces invariant[1].

In the original graph we have one component and f faces. The final graph consisting of isolated vertices has v components and one face. Therefore, the number of edge removals of type (1) and (2) was $f - 1$ and $v - 1$ respectively. The total number of edge removals was e, hence, $e = (f - 1) + (v - 1)$. □

Note that the formula even holds if G has multiple edges connecting the same pair of vertices or loops, i.e, edges with a double endpoint. For the next result, however, we have to stay in the realm of (simple) graphs. This application of Euler's Formula gives the solution of an extremal problem.

Theorem 1.4 *A planar graph G with n vertices has at most $3n - 6$ edges. Moreover, if G is planar and contains no 3-cycle then it has at most $2n - 4$ edges.*

Proof. Assume that G is connected, otherwise some edges could be added such that the graph remains planar. Let f_k be the number of faces with k bounding edges, clearly, $\sum f_k = f$. Counting the bounding edges of all the faces we count every edge exactly twice, i.e., $2e = \sum k f_k$. Since every face has at least three edges on its boundary, this yields $2e \geq 3f$. Inserting this into Euler's Formula we obtain $3n - 6 = 3e - 3f \geq e$.

If G contains no 3-cycle every face has at least four edges on its boundary and $2e \geq 4f$. In this case we obtain $2n - 4 = 2e - 2f \geq e$. □

The proof shows that $e = 3n - 6$ and $f = f_3$ are equivalent. Hence, the extremal graphs are exactly those having only triangular faces, they are known with the name *planar triangulation*.

[1] This innocent statement is based on the Jordan Curve Theorem.

The count of edges alone allows the conclusion that K_5 and $K_{3,3}$ are not planar. In fact, K_5 has ten edges, one to much for a planar graph with five vertices. For $K_{3,3}$ it can be used that this graph contains no 3-cycle and $e = 9 > 8 = 2n - 4$.

Since every subgraph of a planar graph is planar we may ask for a characterization of planarity in terms of forbidden subgraphs. In a classical theorem Kuratowski has shown that the two graphs K_5 and $K_{3,3}$ are the only minimal obstructions against planarity. Let a *subdivision* of a graph G be a graph obtained from G by inserting new vertices of degree two into the edges of G. Planarity is invariant under taking subdivisions, in particular, subdivisions of K_5 or $K_{3,3}$ are not planar.

Theorem 1.5 (Kuratowski 1930) *A graph G is planar if and only if G contains no subgraph which is a subdivision of K_5 or $K_{3,3}$.*

A drawing of a planar graph is called *convex* if every face boundary (including the unbounded face) is a convex polygon. Note that the 2-connected planar graph $K_{2,4}$ has no convex drawing.

Theorem 1.6 (Tutte 1960) *Every 3-connected planar graph admits a convex drawing.*

Tutte's theorem can also be derived from a much older result from geometry, the theorem of Steinitz.

Theorem 1.7 (Steinitz 1922) *Every 3-connected planar graph is the skeleton graph, i.e., the graph of 0-dimensional faces (vertices) and 1-dimensional faces (edges), of a 3-dimensional polytope.*

Given a 3-connected planar graph G, let P be a 3-dimensional polytope with skeleton G. Place P on the plane and choose a point a below the face of contact of P and the plane. Project each point x from P to the point x' in the plane where the ray from x to a intersects the plane. If a was chosen close enough to the plane this yields a convex drawing of G.

Planar graphs have many special properties. One of the most fundamental is the existence of a dual graph. A *dual* G^* of a plane graph G is a plane graph having a vertex in each face of G. Every edge e of G has a corresponding dual edge e^* in G^*: If F and F' are the faces on the two sides of e then e^* connects the vertices of G^* in F and F'.

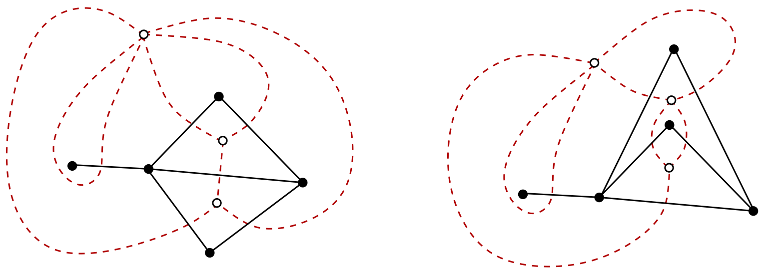

Figure 1.4 A planar graph with two non-isomorphic drawings and the corresponding duals.

Figure 1.4 illustrates that a dual may have some features which make it necessary to extend our concept of graphs. The dual G^* may have loops and multiple edges even though G is a simple graph. Having extended the definition of a graph accordingly it is almost evident from the drawings that $(G^*)^* = G$ for every connected plane graph G.

The figure also shows that a planar graph may have different drawings which have non-isomorphic duals. Whitney has characterized the well behaving planar graphs.

Theorem 1.8 (Whitney 1933) *A planar graph G has a unique dual iff G is 3-connected. Moreover, the dual is a simple graph in this case.*

Let G be a plane graph and C be the edge set of a cycle in G. This cycle C splits the plane into at least two connected regions and each of these regions contains at least one face of G. If v^* and w^* are dual vertices from different regions, then every path connecting v^* and w^* has to cross some edge of C and, hence, contain an edge from the dual set C^*. Therefore, C^* is a *cut*, i.e., a disconnecting set of edges for G^*. In fact there is a bijection between simple cycles of G and minimal cuts of G^* and vice versa.

1.4 Outerplanar Graphs and Convex Geometric Graphs

A planar graph is *outerplanar* if it has a plane drawing such that all vertices lie on the boundary of the exterior face. Equivalently, G is outerplanar if the graph obtained by adding a new vertex x joined to all vertices of G is still planar.

Using Kuratowski's Theorem and the second definition of outerplanar graphs we conclude that a graph is outerplanar if and only if it contains no subgraph which is a subdivision of K_4 or $K_{2,3}$. The maximum number of edges of an n vertex outerplanar graph is $2n-3$. The extremal graphs can be drawn as convex n-gons with a triangulation of the interior by chords.

A *convex geometric graph* is a geometric graph with the property that the vertices are the vertices of a convex polygon.

A convex geometric graph without crossings is just a maximal outerplanar, therefore, a convex geometric graph with n vertices and without crossings has at most $2n-3$ edges. The next theorem deals with a less trivial extremal problem for convex geometric graphs. A pair of edges of a geometric graph is called *disjoint* if they do not cross in the geometric representation and share no vertex.

Theorem 1.9 *Let $dc_k(n)$ be the maximum number of edges that a convex geometric graph can have without containing $k+1$ pairwise disjoint edges. If $n \geq 2k+1$ then*

$$dc_k(n) = kn.$$

Proof. Let $x_0, x_1, \dots, x_{n-1}$ be the cyclic order of the vertices of a convex geometric graph. Consider the set of all interior segments $x_i x_j$. Two segments $x_i x_j$ and $x_k x_l$, with i, j, k, l distinct are called *parallel*, if there is a t such that $k = i + t$ and $l = j - t$, where indices are taken modulo n. This equivalence relation partitions the segments into n classes of pairwise parallel segments. If G has no $k+1$ pairwise disjoint edges, then G can have at most k edges from each class. This implies $dc_k(n) \leq kn$.

To see that the bound can be attained we need appropriate graphs. The idea is to build a graph whose edges are the long segments: Let $G_k(n)$ be the graph whose edges are the segments $x_i x_{i+\lfloor \frac{n}{2} \rfloor + j}$ for $0 \leq i \leq n-1$ and $1 \leq j \leq k$. Figure 1.5 shows three examples.

Suppose that the vertices $x_0, x_1, \dots, x_{n-1}$ are in clockwise order. Think of the edge $x_i x_{i+\lfloor \frac{n}{2} \rfloor + j}$ as being oriented $x_i \to x_{i+\lfloor \frac{n}{2} \rfloor + j}$. The oriented edge has $\lceil \frac{n}{2} \rceil - j - 1$ vertices

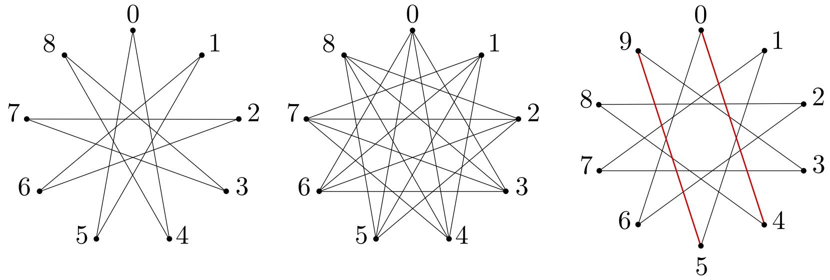

Figure 1.5 The graphs $G_1(9)$, $G_2(9)$ and $G_1(10)$.

on its right side and $\lfloor \frac{n}{2} \rfloor + j - 1$ vertices on its left side. Hence, the right side is always smaller than the left side. Therefore, all the edges in the definition of are different and $G_k(n)$ has nk edges.

Let P be a set of pairwise disjoint edges of $G_k(n)$, we want to show $|P| \leq k$. Consider the plane graph G_P consisting of the cycle $x_0, x_1, \ldots, x_{n-1}, x_0$ together with the edges of P, these edges are disjoint chords in the cycle. Therefore, G_P is an outerplanar graph. Consider the dual graph of G_P and remove the vertex corresponding to the outer face of G_P. The remaining truncated dual is a tree, hence, has at least two leaves. They correspond to faces F_1 and F_2 of G_P which have only one edge e_1 resp. e_2 of P on the boundary. The vertices of F_i, $i = 1, 2$, are the two vertices of e_i together with all vertices on one side of e_i. From the above we know that each side of each edge of $G_k(n)$ contains at least $\lceil \frac{n}{2} \rceil - k - 1$ vertices. Vertices of F_i which are not on e_i cannot be incident to edges of P. We conclude, that there are at most $n - 2(\lceil \frac{n}{2} \rceil - k - 1)$ vertices which are incident to edges of P. In general this number is $\leq 2k + 2$ and if n is odd it is $\leq 2k + 1$. Edges of P are disjoint, therefore, the same bound holds for the number of incidences between edges of P and vertices. This later number, however, is twice the number of edges of P. Hence $|P| \leq k$ for n odd.

For n odd and $n \geq 2k + 1$ we have shown that the graph $G_k(n)$ proves $dc_k(n) \geq kn$. This is also true if n and k are both even and $n \geq 2k$. For n even and k odd however, $G_k(n)$ contains sets of $k + 1$ pairwise disjoint edges. For an example look at the graph $G_1(10)$ from Figure 1.5. It is possible to modify $G_k(n)$ to show that $dc_k(n) \geq kn$ also holds for the pairs n even and k odd. □

1.5 Geometric Graphs without $(k+1)$-Pairwise Disjoint Edges

We consider the question: How many edges a geometric graph on n points can have without having $k+1$ pairwise disjoint edges? This is an old problem with a long history, see the notes at the end of the chapter. We begin with the easiest special case.

Theorem 1.10 *A geometric graph with no two disjoint edges can have at most n edges.*

Proof. Every vertex may mark an edge incident to it. For vertices of degree one there is no choice. For all other vertices mark the right edge at the largest angle at the vertex. If there remains an unmarked edge $e = uv$ we find a situation as shown in Figure 1.6. The angles α and β cannot be the largest angles at u and v. Hence, they are both less than π

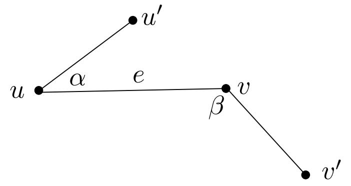

Figure 1.6 An unmarked edge e and its two disjoint adjacent edges.

and the edges uu' and vv' must be disjoint because they reach into different halfplanes of the line supporting e. We conclude: In the absence of disjoint edges there is no unmarked edge and the number of edges is at most as large as the number of vertices. □

More complex marking arguments have been used to show that $3n+1$ edges force three disjoint edges and $10n+1$ edges force four disjoint edges. The following theorem deals with the general case.

Theorem 1.11 *A geometric graph with no $k+1$ disjoint edges cannot have more then $256\,k^2 n$ edges.*

Proof. Let G be a geometric graph on n vertices containing no $k+1$ disjoint edges. Let $x(v)$ and $y(v)$ denote the x- and y-coordinate of point v. If there are two vertices with the same x-coordinate we slightly rotate the plane to make all x-coordinates different. Edge e is said to *lie above* edge e' if every vertical line intersecting both intersects e above e'. Define four relations $\prec_1, \prec_2, \prec_3$ and $\prec_4$ on disjoint pairs of edges. Let $e = v_1w_1$ and $e' = v_2w_2$ be two edges such that e is above e' and $x(v_1) < x(w_1)$ and $x(v_2) < x(w_2)$, we define:

$$
\begin{aligned}
e \prec_1 e' \quad &\text{iff} \quad x(v_1) < x(v_2) \text{ and } x(w_1) < x(w_2),\\
e \prec_2 e' \quad &\text{iff} \quad x(v_1) > x(v_2) \text{ and } x(w_1) > x(w_2),\\
e \prec_3 e' \quad &\text{iff} \quad x(v_1) < x(v_2) \text{ and } x(w_1) > x(w_2),\\
e \prec_4 e' \quad &\text{iff} \quad x(v_1) > x(v_2) \text{ and } x(w_1) < x(w_2).
\end{aligned}
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The definitions are illustrated in Figure 1.7. It is important to note that each of these

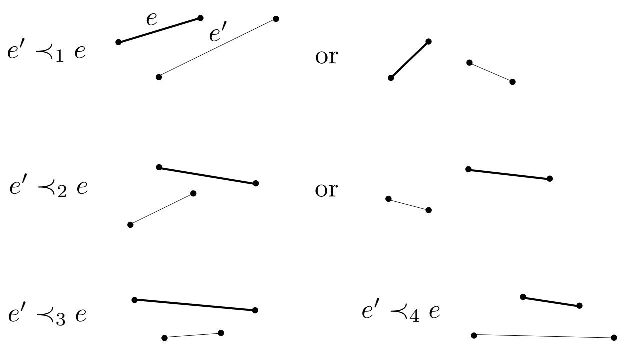

Figure 1.7 The four orders defined on disjoint edges.

relations is transitive, i.e., an order relation.

For any vertex v_i we partition the edges incident to v_i into *left edges*, those edges v_i, v_j with $x(v_j) < x(v_i)$, and *right edges*, those edges v_i, v_j with $x(v_i) < x(v_j)$. The *left degree* l_i is the number of left edges at v_i and the *right degree* r_i is the number of right edges at vertex v_i.

For every vertex v define two linear orders on the right edges of v. $R_s(v)$ is the order by decreasing slope and $R_x(v)$ is the order by increasing x-coordinate. The intersection $R_s(v) \cap R_x(v)$ is a partial order, actually a two dimensional order.

The following theorem is a corollary from the Greene-Kleitman duality theory:

Theorem 1.12 *Let $P = (X, <)$ be a partial order on n elements, then there is a family $\mathcal{C}$ of at most $\sqrt{n}$ chains and a family $\mathcal{A}$ of at most $\sqrt{n}$ antichains such that the chains and antichains in $\mathcal{C} \cup \mathcal{A}$ cover all elements of P.*

Hence, the right edges of a vertex v can be covered with at most $\sqrt{r_v}$ chains and $\sqrt{r_v}$ antichains. Let $C^r(v)$ be the set of edges covered by chains and $A^r(v)$ be the set of edges covered by antichains.

One of $\bigcup_v C^r(v)$ and $\bigcup_v A^r(v)$ contains at least one half of all edges of G. Let G' be the graph restricted to the edges of the larger class, clearly $e(G') > e(G)/2$. The *right block* of an edge of G' is the set of edges of the chain (or antichain) it belongs to. The order of edges of a right block is the order by decreasing slope, in this order the edges of a right block form an x-increasing (chain) or x-decreasing (antichain) sequence.

Let l'_v be the left degree of vertex v in G', clearly $l'_v \le l_v$. We now uniformize the left edges of every vertex v in G'. Let $L_s(v)$ be the order by increasing slope and $L_x(v)$ be the order by increasing x-coordinate. As before, the intersection order $L_s(v) \cap L_x(v)$ can be covered by at most $\sqrt{l'_v}$ chains and $\sqrt{l'_v}$ antichains. Let $C^l(v)$ be the set of edges covered by chains and $A^l(v)$ be the set of edges covered by antichains. Let G'' be the graph restricted to the edges of the larger class of $\bigcup_v C^l(v)$ and $\bigcup_v A^l(v)$, clearly $e(G'') > e(G)/4$. The *left block* of an edge of G'' is the set of edges of the chain (or antichain) it belongs to. The order of edges of a left block is the order by increasing slope, in this order the edges of a left block form an x-increasing (chain) or x-decreasing (antichain) sequence.

The restriction of a right block of G' to the edges of G'' is a right block of G''. For two edges with a common endpoint, $e = vu$ and $e' = vw$, we say that (e, e') is a *right-zag*, if e' follows immediately after e in the same right block at vertex v. Analogously, for two edges, $e = uv$ and $e' = wv$, we say that (e, e') is a *left-zag*, if e' follows immediately after e in the same left block at v.

A path $e_1, e_2, \ldots, e_m$ of G'' is a *zig-zag* if in every pair e_i, e_{i+1} of consecutive edges is either a right-zag or a left-zag with left and right alternating along the path.

Claim 1. Every zig-zag of G'' contains at most $2k$ edges.

There are four cases, the right/left blocks can be increasing/decreasing, these cases correspond to the four order relations $\prec_i$. We detail one case, the other cases can be handled analogously.

Assume that all right blocks of G' (and hence of G'') are x-increasing and all left blocks of G'' are x-decreasing. Let $e_1, e_2, \ldots, e_{2k+1}$ be a zig-zag of length $2k+1$. We claim that $e_{i+2} \prec_3 e_i$ for all $1 \le i \le 2k-1$. Consider e_i, e_{i+1}, e_{i+2} and let a, b, c, d be the sequence of vertices of this zig-zag. The pair (e_i, e_{i+1}) is a right-zag or a left-zag at vertex b. We distinguish these two cases, see Figure 1.8.

(1) If (e_i, e_{i+1}) is a right-zag then $x(a) < x(c)$. Since (e_{i+1}, e_{i+2}) is a left-zag at c it follows that $x(b) > x(d)$. Clearly, e_{i+2} is below e_i, so $e_{i+2} \prec_3 e_i$.

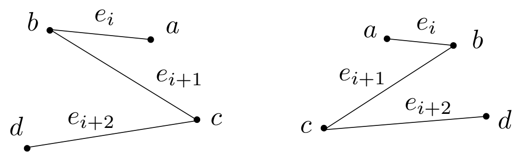

Figure 1.8 Edges e_i, e_{i+1} form a right-zag or a left-zag at b.

(2) If (e_i, e_{i+1}) is a left-zag then $x(a) > x(c)$. Since (e_{i+1}, e_{i+2}) is a right-zag at c it follows that $x(b) < x(d)$. Again, e_{i+2} is below e_i and $e_{i+2} \prec_3 e_i$.

Consequently, a zig-zag of length $2k+1$ contains a chain $e_{2k+1} \prec_3 e_{2k-1} \prec_3 \dots \prec_3 e_1$ of length $k+1$. This is a contradiction, since the edges of a chain are pairwise disjoint. △

Claim 2. If Z is the number of maximal zig-zags in G'', then $Z \leq 2\sqrt{e(G)n}$.

Let $e_1, e_2, \dots, e_m$ be a maximal zig-zag if v is the vertex of e_1 which is not incident to e_2 then e_1 is the first element of its block at v. Therefore, the number of maximal zig-zags starting at a vertex v is at most the number of blocks of edges at v. By construction v has at most $\sqrt{r_v}$ right blocks and at most $\sqrt{l'_v} \leq \sqrt{l_v}$ left blocks. This can be turned into an estimate for Z. Apply a variant of the Cauchy-Schwarz inequality, namely, $(\sum_{i=1}^n a_i)^2 \leq n(\sum_{i=1}^n a_i^2)$ twice and use the obvious equation $\sum_{i=1}^n (r_i + l_i) = 2e(G)$, to obtain:

$$Z \leq \sum_{i=1}^n (\sqrt{r_i} + \sqrt{l_i}) \leq \sqrt{n \sum_{i=1}^n (\sqrt{r_i} + \sqrt{l_i})^2} \leq \sqrt{n \sum_{i=1}^n 2(r_i + l_i)} = \sqrt{4ne(G)}.$$

△

Each edge of G'' is covered by at least one maximal zig-zag and each zig-zag contains at most $2k$ edges, hence $e(G'') \leq 2kZ$. By construction $e(G'') \geq e(G)/4$, together with Claim 2 this yields $e(G) \leq 16\,k\sqrt{e(G)n}$. Squaring the inequality and dividing by $e(G)$ we obtain $e(G) \leq 256\,k^2 n$. □

1.6 Geometric Graphs without Parallel Edges

Assuming general position for the endpoints of two disjoint segments in the plane we can distinguish two cases. Either the convex hull of their four endpoints is a triangle or a quadrangle, in the second case we say that the two *edges are parallel*, see Figure 1.9.

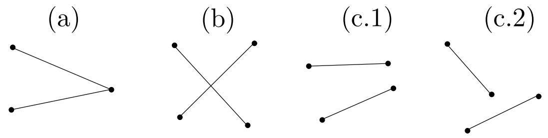

Figure 1.9 The four possible positions of two edges in a geometric graph, (a) adjacent, (b) crossing, (c.1) parallel, (c.2) stabbing.

A geometric graph on n points with no $k+1$ pairwise parallel edges can have more edges than a geometric graph with no $k+1$ pairwise disjoint edges. In the case of parallel edges, already the solution to the extremal problem for $k = 1$ requires the use of interesting techniques.

Kupitz gave examples showing that in the absence of parallel edges a geometric graph can have as much as $2n - 2$ edges, see Figure 1.10, he conjectured that this is the maximum.

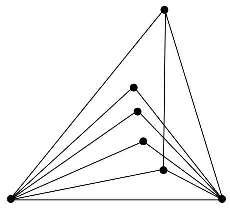

Figure 1.10 The construction of Kupitz for $n = 7$.

Theorem 1.13 *A geometric graph on* n *vertices with no two parallel edges can have at most* $2n - 2$ *edges.*

Let G be a geometric graph on n vertices and with $e(G)$ edges. For an edge e of G let l_e be the line supporting e. Choose a circle C such that all vertices $v_1, v_2, \ldots, v_n$ of G are in the interior of C. For every edge $e = v_i v_j$ consider the two points of intersection of line l_e with circle C and label them with the index of the closer vertex, i.e., traversing l_e we find point i, vertex v_i, vertex v_j and point j in this order. Let $S(G)$ be the circular sequence obtained from reading the labels of the intersection points along C in clockwise order. The length $|S(G)|$ of $S(G)$ is $2e(G)$.

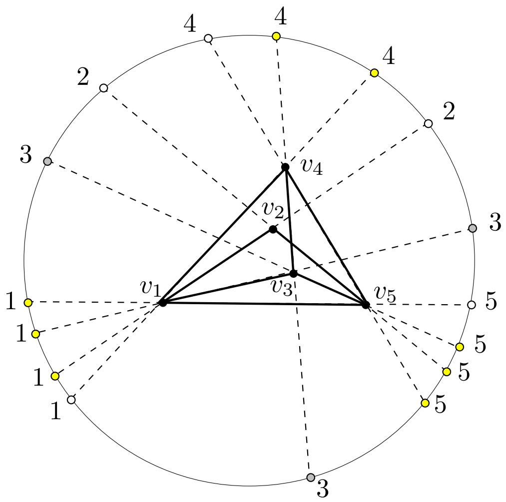

Figure 1.11 A graph G with circular sequence $S(G) = (4, 4, 4, 2, 3, 5, 5, 5, 5, 3, 1, 1, 1, 1, 3, 2($.

From $S(G)$ we construct a reduced circular sequence $RS(G)$ by erasing all the labels which are equal to their predecessor labels. The reduced sequence for the graph in Figure 1.11 is $RS(G) = (4, 2, 3, 5, 3, 1, 3, 2, ($.

Lemma 1.14 *If G is a geometric graph without parallel edges, then $|RS(G)| \geq e(G)$.*

Proof. Consider an edge $e = v_i v_j$ and the labels $i = i_e$ and $j = j_e$ in $S(G)$ that come from e. Assuming that the successor of i_e in $S(G)$ is i and the successor of j_e is j we find parallel edges e_i and e_j as indicated in Figure 1.12. This shows that if G has no

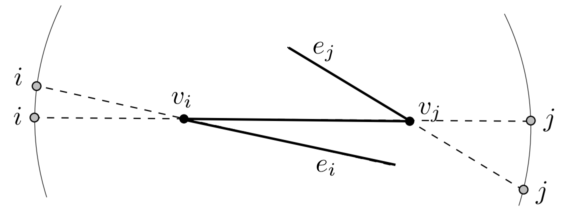

Figure 1.12 An edge with two erased successor labels implies a pair of parallel edges.

parallel edges, then every edge has at least one remaining label in $RS(G)$. In other words $|RS(G)| \geq e(G)$. □

Lemma 1.15 *If G is a geometric graph without parallel edges, then there is no circular subsequence i, j, i, j in $|RS(G)|$.*

Proof. Suppose that there is a subsequence i, j, i, j in $RS(G)$. Let ℓ be the line through v_i and v_j. A rotation makes ℓ vertical, such that v_i is above v_j. One of the following two events is unavoidable: Reading clockwise there is a ji subsequence right of ℓ or a ij subsequence left of ℓ. In the second case exchange the labels i and j and rotate. Now we have the situation illustrated in Figure 1.13. The crossing right of ℓ of the supporting lines

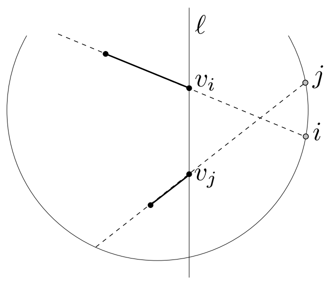

Figure 1.13 A i, j, i, j subsequence implies a pair of parallel edges.

of two edges situated left of ℓ shows that these two edges are parallel, a contradiction. □

The next lemma implies that $|RS(G)| \leq 2n - 2$. Together with Lemma 1.14 this completes the proof of Theorem 1.13.

Lemma 1.16 *If R is a circular sequence with entries from a set of $n > 1$ symbols such that no two adjacent entries are identical and R contains no circular subsequence of type $abab$, then the length of R is at most $2n - 2$.*

Proof. Let R be a circular sequence on n symbols such that no two adjacent entries are identical and R contains no circular subsequence of type *abab*. In the literature such a sequence is called a circular Davenport-Schinzel sequence of order 2. Let a be a symbol with $k \geq 2$ occurrences in R and decompose the sequence using this symbol, $R = aS_1aS_2a \dots aS_k$. The forbidden pattern enforces that for $i \neq j$ the subsequences S_i and S_j have no symbol in common. Let λ_i be the number of symbols of S_i. If we consider S_i as a circular sequence it contains no *abab* but the first and the last element may be equal. After removal of such a duplication the length of the sequence is at most $2\lambda_i - 2$, by induction. Therefore, $|S_i| \leq 2\lambda_i - 1$ and

$$|R| \leq \sum_{i=1}^{k}(2\lambda_i - 1) + k = 2\sum_{i=1}^{k}\lambda_i = 2(n-1).$$

□

1.7 Notes and References

A good introduction to many aspects of graph theory is the book of West [34], with its more then 500 references this book can serve as a starting point for deeper studies. Biggs et al. [4] give a nice account to the early history of graph theory. Extremal graph theory is the topic of a survey by Bollobás [5]. Four different proofs of Turán's theorem can be found in The Book by Aigner and Ziegler [2]. The existence of a straight line drawing for every planar graph is a theorem independently obtained by Wagner [32], Fáry [12] and Stein [26]. As part of his "Geometry Junkyard" Eppstein has collected fifteen proofs of Euler's Formula [10]. A chapter in The Book [2] is devoted to applications of the formula. The classification of regular polytopes is another application, see e.g. [34]. Extensions of Euler's formula in the theory of polytopes, in particular the Euler-Poincaré formula, are presented in the polytope book of Ziegler [36]. West [34] reproduces ideas of Thomassen to give a joint proof for the theorems of Kuratowski and Tutte. The theorem of Tutte is one of the main results in the next chapter of this book. The hard direction of Steinitz's theorem, to produce a polytope with a prescribed skeleton, can be proven in two ways. An inductive approach based on ΔY transformations is detailed by Ziegler [36]. Richter-Gebert [25] extends the approach used by Tutte to prove his theorem, this leads to a special plane drawing of the graph which has the property that the vertices can be lifted so that they form the desired polytope. More detailed comments about this approach can be found in the notes section of Chapter **??**.

The book [22] of Mohar and Thomassen contains a rich chapter on planar graphs with many references. In particular they are careful about the use of the Jordan Curve Theorem in this theory.

Theorem 1.8 is a byproduct of Whitney's characterization of all plane embeddings of a 2-connected planar graph. He shows [35] that all these embeddings can be obtained from a given one by a series of *switchings* of two connected components. The construction of the dual graph introduced here is geometric. A multi-graph G^* is an *algebraic dual* of G if there is a bijection $E(G^*) \leftrightarrow E(G)$ between the edge-sets which maps simple cycles of G to minimal cuts of G^* and vice versa. Whitney characterized planar graphs as those graphs admitting an algebraic dual. The study of cuts and duals leads to the definition of a *matroid* and the rich theory thereof, West [34] gives an introduction and further references.

Outerplanar graphs and convex geometric graphs.

According to Pach [23] Theorem 1.9 was first obtained by Kupitz 1982. To show that the bound of the theorem is best possible the family $G_k(n)$ is described in [23]. The "dual" extremal problem was solved by Capoyleas and Pach [9]: The maximum number of edges that a convex geometric graph can have without containing $k+1$ pairwise crossing edges is $k(2n-2k-1)$ for all $n \geq 2k+1$.

Kupitz and Perles [20] study the maximum number of edges that a convex geometric graph can have without containing $k+1$ disjoint edges in convex position, that is all the edges are edges of the convex hull of their vertices. If $n \geq 2k+1$ then this number is $t_k(n) + n - k$, where $t_k(n)$ is the Turán number of Theorem 1.2.

Geometric graphs with no $k+1$ pairwise disjoint edges.

Besides the study of planar graphs, the first investigations on geometric graphs were in the context of repeated distances. For example the problem of determining the maximum number of diametral pairs of a set of n points in the plane is closely related to the maximum number of edges of a geometric graph with no disjoint edges. A related problem was posed by Hopf and Pannwitz in 1934. They ask for a proof that only for odd n it is possible to choose n points such that all the pairs x_i, x_{i+1}, $i = 1, .., n$ are diametral. Sutherland [27] and Fenchel [13] proposed solutions. Erdős [11] proved the bound od Theorem 1.10 in the context of distance problems.

The question about the maximum number $d_k(n)$ of edges of a geometric graph with no $k+1$ pairwise disjoint edges was raised by Kupitz and Perles [18] and again stated by Akiyama and Alon [3]. Alon and Erdős proved, $d_2(n) \leq 6n$. Goddard et al. [15] improved to $d_2(n) \leq 3n+1$, they also proved $d_3(n) \leq 10n+1$ and gave the first general upper bound $d_k(n) \in O(m(\log n)^{k-3})$. Pach and Törőcsik [24] introduced the order relations $\prec_i$, $i = 1, 2, 3, 4$, on disjoint edges. As an application of Dilworth's Theorem about chain decompositions of orders they could show that $d_k(n)$ is linear in n, more precisely $d_k(n) \leq k^4 n$. Tóth and Valtr [29] added the concept of a zig-zag and improved to $d_k(n) \leq k^3(n+1)$. They also construct examples to show the lower bound $d_k(n) > 1.5(k-1)n - 2k^2$. Tóth [28] obtained the bound on $d_k(n)$ of order $ck^2 n$ (Tóth says $c = 2^9$ but his argument only yields $c = 2^{13}$). Tóth is using a greedy approach based on the Erdős-Szekeres Theorem to define the blocks of edges at each vertex. Replacing this by an application of the more sophisticated Theorem 1.12 made the slightly improved constant in Theorem 1.11 possible.

Theorem 1.12 is an offspring of the Greene-Kleitman theory about chain and antichain families in orders. A nice proof based on min-cost flows is given by Frank [14]. The Greene-Kleitman Theory was surveyed by West [33] and more recently by Britz and Fomin [7].

It is widely believed that $d_k(n) \sim ckn$ for some very moderate c. It would already be very nice to know this for the restricted class of those geometric graphs admitting a line which intersects all the edges. Let $dl_k(n)$ be the maximum number of edges such a geometric graph can have if it has no $k+1$ pairwise disjoint edges. If $dl_k(n) \leq ckn$ the bound on $d_k(n)$ would drop to $d_k(n) \leq c'(k \log k)n$.

Conway defines a *thrackle* as a graph drawn in the plane such that any two distinct edges either have a common vertex or meet at exactly one point where they cross, i.e., a thrackle is a topological graph without disjoint edges. Every cycle is a thrackle (exercise). Conway conjectured that a thrackle with n vertices has at most n edges, i.e., that the bound of Theorem 1.10 remains valid in the more general context of topological graphs.

For about 40 years the best known upper bound on the number of edges of a thrackle

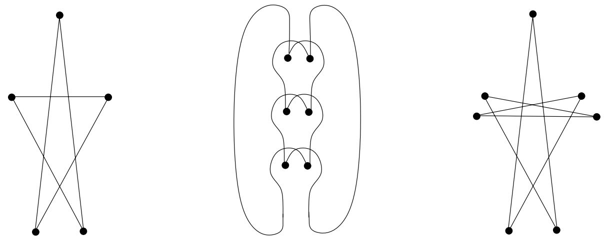

Figure 1.14 Thrackle-representations of the 5-, 6- and 7-cycle.

was $O(n^{3/2})$. This follows from the fact that a subgraph of a thrackle is a thrackle and the four-cycle is not a thrackle. In a very nice paper Lovász, Pach and Szegedy [21] show that the number of edges of a thrackle on n vertices cannot exceed $2n - 3$. Cairns and Nikolayevsky [8] improved the bound from $2n - 3$ to $(3/2)(n - 1)$.

Graphs with no $k + 1$ pairwise parallel edges.

Kupitz [19] was the first who considered geometric graphs with no pair of parallel edges. He constructed examples of such graphs with n vertices and $2n - 2$ edges and conjectured that this is the maximum. Katchalski and Last [17] proved an upper bound of $2n - 1$. The argument was sharpened by Valtr [30] to yield the conjectured bound. The proof given here is a simplification of the original argument, it first appeared in Valtr [31]. Valtr also considers geometric graphs with no $k + 1$ pairwise parallel edges, using Dilworth's theorem and generalized Davenport-Schinzel sequences he proves that such graphs have $O(n)$ edges.

Graphs with no $k + 1$ pairwise crossing edges.

For $k = 1$ these graphs are just the planar graphs. Graphs with no three pairwise crossing edges have been named quasi-planar. Agarwal et al. [1] have shown that quasi-planar graphs have a linear number of edges. A nice question stated in that paper is: Can the edges of a quasi-planar graph G be colored with a constant number of colors, such that the edges in each color class form a planar subgraph of G? For general k the currently best bound for the number of edges of a graph with no $k + 1$ pairwise crossing edges is $O(n \log n)$, due to Valtr [30]. He makes use of the bound for the extremal problem for pairwise parallel edges. The key idea is to use the mapping $T : (x, y) \to (1/x, y/x)$, this sends a pair of crossing edges, both with one endpoint in the halfplane $x < 0$ and one in the halfplane $x > 0$, to a pair of parallel edges.

In the case of convex geometric graphs the problem has a complete solution. Capoyleas and Pach [9] have shown that for $n \geq 2k + 1$ the maximum number of edges of a convex geometric graph without $k + 1$ pairwise crossing edges is exactly $(2n - 2k - 1)k$.

Several authors have considered more complex extremal problems for convex geometric graphs. A convex geometric graph H with vertices $w_1, \ldots, w_r$ in cyclic order is a *convex subgraph* of a convex geometric graph G if there is a circular subsequence of vertices $v_1, \ldots, v_r$ of G, such that $w_i w_j \in E(H)$ implies $v_i v_j \in E(G)$. Kupitz and Perles [20] solve the extremal problem for the graph H on $2k$ vertices with k edges in convex position, i.e., $E(H) = \{w_1 w_2, w_3 w_4, \ldots, w_{2k-1} w_{2k}\}$.

Gritzmann et al. [16] study the following situation: H is a 2-connected outerplanar graph on $k + 1$ vertices represented as a convex $(k + 1)$-gon with some non-crossing chords. They show that a convex geometric graph not containing H as convex subgraph can have at most $t_k(n)$ edges, where $t_k(n)$ is the Turán number. In a recent paper [6]

Braß, Károlyi and Valtr review the known results and work towards an extremal theory for convex geometric graphs.

Bibliography

[1] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir, *Quasi-planar graphs have a linear number of edges*, Combinatorica, 17 (1997), pp. 1–9.

[2] M. Aigner and G. M. Ziegler, *Proofs from* The Book, Springer-Verlag, 1998.

[3] J. Akiyama and N. Alon, *Disjoint simplices and geometric hypergraphs*, in Combinatorial Mathematics;, G. S. Blum et al., ed., vol. 555, Annals of the New York Academy of Sciences, 1989, pp. 1–3.

[4] N. L. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory 1736–1936*, Claredon Press, 1976.

[5] B. Bollobás, *Extremal graph theory*, Handbook of Combinatorics, Vol II, Graham, Grötschel, Lovász (eds) (1995), pp. 1231–1292.

[6] P. Brass, G. Károlyi, and P. Valtr, *On a Turán-type extremal theory for convex geometric graphs*, in Discrete and Computational Geometry, The Goodman and Pollack Festschrift, vol. 25 of Algorithms and Combinatorics, Springer Verlag, 2003, pp. 275–300.

[7] T. Britz and S. Fomin, *Finite posets and Ferrer shapes*, Adv. Math., 158 (2001), pp. 86–127.

[8] G. Cairns and Y. Nikolayevsky, *Bounds for generalized thrackles*, Discrete Comput. Geom., 23 (2000), pp. 191–206.

[9] V. Capoyleas and J. Pach, *A Turán-type theorem on chords of a convex polygon*, J. Comb. Theory (B), 56 (1992), pp. 9–15.

[10] D. Eppstein, *Fifteen proofs of Euler's formula: $V - E + F = 2$* . http://www.ics.uci.edu/~eppstein/junkyard/euler.

[11] P. Erdős, *On a set of distances of n points*, Amer. Math. Monthly, 53 (1946), pp. 248–250.

[12] I. Fary, *On straight lines representation of planar graphs*, Acta Sci. Math. Szeged, 11 (1948), pp. 229–233.

[13] W. Fenchel, *Lösung Aufgabe 167 von Hopf und Pannwitz*, Jahresberichte der Deutschen Mathem. Vereinig., 45 (1935), pp. 34–35.

[14] A. Frank, *On chain and antichain families of partially ordered sets*, J. Comb. Th. (B), 29 (1980), pp. 176–184.

[15] W. Goddard, M. Katchalski, and D. J. Kleitman, *Forcing disjoint segments in the plane*, Europ. J. Comb., 17 (1997), pp. 391–395.

[16] P. Gritzmann, B. Mohar, J. Pach, and R. Pollack, *Embedding a planar triangulation with vertices at specified points*, Amer. Math. Monthly, 98 (1991), pp. 165–166.

[17] M. Katchalski and H. Last, *On geometric graphs with no two edges in convex position*, Discrete Comput. Geom., 19 (1998), pp. 399–404.

[18] Y. S. Kupitz, *Extremal problems in combinatorial geometry*, Aarhus University Lecture Notes Series 53, Aarhus University, 1979.

[19] ——, *On pairs of disjoint segments in convex position in the plane*, Annals of Discr. Math. 20, Rosenfeld, Zaks (eds) (1984), pp. 203–208.

[20] Y. S. Kupitz and M. Perles, *Extremal theory for convex matchings in convex geometric graphs*, Discrete Comput. Geom., 15 (1996), pp. 195–220.

[21] L. Lovász, J. Pach, and M. Szegedy, *On Conway's thrackle conjecture*, Discrete Comput. Geom., 18 (1997), pp. 369–376.

[22] B. Mohar and C. Thomassen, *Graphs on Surfaces*, John Hopkins University Press, 2001.

[23] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, John Wiley & Sons, 1995.

[24] J. Pach and J. Törőcsik, *Some geometric applications of Dilworth's theorem*, Discrete Comput. Geom., 12 (1993), pp. 1–7.

[25] J. Richter-Gebert, *Realization spaces of polytopes*, vol. 1643 of Lecture Notes in Mathematics, Springer-Verlag, 1997.

[26] S. K. Stein, *Convex maps*, Proc. Amer. Math. Soc., 2 (1951), pp. 464–466.

[27] J. W. Sutherland, *Lösung Aufgabe 167 von Hopf und Pannwitz*, Jahresberichte der Deutschen Mathem. Vereinig., 45 (1935), pp. 33–34.

[28] G. Tóth, *Note on geometric graphs*, J. Comb. Theory (A), 89 (2000), pp. 126–132.

[29] G. Tóth and P. Valtr, *Geometric graphs with few disjoint edges*, in Annual ACM Symposium on Computational Geometry, 1998, pp. 184–191.

[30] P. Valtr, *On geometric graphs with no k pairwise parallel edges*, Discrete Comput. Geom., 19 (1998), pp. 461–470.

[31] ——, *Generalizations of Davenport-Schinzel sequences*, in Contemporary trends in discrete mathematics, R. Graham et al., ed., vol. 49 of DIMACS, Ser. Discrete Math. Theor. Comput. Sci., DIMACS, 1999, pp. 349–389.

[32] K. Wagner, *Bemerkungen zum Vierfarbenproblem*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 46 (1936), pp. 26–32.

[33] D. B. West, *Parameters of partial orders and graphs: Packing, covering and representation*, in Graphs and Orders, I. Rival, ed., D. Reidel, 1985, pp. 267–350.

[34] ——, *An Introduction to Graph Theory*, Prentice Hall, 1995.

[35] H. Whitney, *2-isomorphic graphs*, Amer. J. Math., 55 (1933), pp. 245–254.

[36] G. M. Ziegler, *Lectures on Polytopes*, vol. 152 of Graduate Texts in Mathematics, Springer-Verlag, 1994.