7. Exercise sheet **Topics in Discrete Mathematics**

Stefan Felsner **Summer 2026**

Due for the exercise session: June 18, 2026 —
The Dies Mathematicus starts at 15:30 the exercise session will be shorter than usual.

---

(1) We look at tilings of a triangular grid with rhombii, i.e., unions of two triangles of the grid sharing an edge. The region is triangular grid with sidelength n, i.e., with $n+1$ vertices on each side, and n holes. A hole is a triangle in the grid which has the same orientation as the full region, i.e., no two holes share an edge. Holes are not covered by rhombi.

a. Show that a tiling can only exist if each subtriangle of sidelength k contains at most k holes.

b. Show that the condition of **a.** is also sufficient for the existence of a tiling.

(2) Consider a $n \times n$ board with some red cells, we call them *infected*. A previously uninfected cell will become infected if at least two of its (up to four) neighbors are infected. Show that an initial infection must consist of at least n infected cells to spread such that eventually all cells of the board are infected.

(3) Consider a plane bipartite graph G with n white and n black vertices. A perfect matching of G can be considered as a permutation of n and permutations have signs. Find conditions which ensure that any two perfect matchings of G have the same sign.