5. Exercise sheet **Topics in Discrete Mathematics**

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(1) Let $e_k(P)$ be the number of k-edges of a set P of points in general position in the plane and let $|P| = n \geq 2k+3$.

  a. Show that $e_k \geq 2k+3$.

  b. Show that the bound of **a.** is best possible for every $n \geq 2k+3$. Hint: find a solution with $2k+3$ points and then add further points.

(2) Let R_1, R_2, R_3 be three rays starting in x with an angle of $120°$ between any two of them. For $i = 1, 2, 3$, $n \geq k$ and $j = 1, \ldots, n$ let $D_{i,j}$ be a disc of radius $\varepsilon \leq 1/10$ whose center is on R_i at distance j from x. Let P be obtained by choosing one point from each $D_{i,j}$ such that the set of $3n$ points is in general position. Determine the number e_k of k-edges and of $E_\ell = \sum_{k=0}^{\ell} e_k$ for this point set.

(3) Let P be a set of points such that all points belong to one of two circles C, C'. Find bounds for the number of k-edges of P.

(4) Prove that a set of n points in general position in the plane admits a plane Hamilton path, i.e., a crossing-free spanning path. (There are many proofs, we recommend that you try finding more than one.)

(5) It is conjectured that the crossing number of the complete bipartite graph $K_{2m,2n}$ equals $4\binom{m}{2}\binom{n}{2}$. Here we describe a drawing of the graph on the sphere: Let M' and N' be sets of m and n points in general position on the sphere, respectively. Let M and N be obtained from M' and N' by adding their antipodes, i.e., $|M| = 2m$ and $|N| = 2n$. Now consider the drawing of $K_{2m,2n}$ where edges are represented by geodesic arcs between a point of M and a point of N. Estimate the number of crossings by considering intersections of great circles.

(6) Show that in a simple drawing of K_n every vertex is incident to an empty 4-cycle.