4. Exercise sheet **Topics in Discrete Mathematics**

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(1) Given a Schnyder Wood S of a triangulation $T = (V, E)$ we define a coloring of the angles of T with colors r, g, b: the two angles at the head of an i-colored edge of S are colored i.

  a. Give a definition of these angle colorings in terms of local properties which do not refer to Schnyder Woods.

  b. Show that there is a bijection between your angle colorings and Schnyder Woods.

(2) Find a characterization of the plane triangulations which have a unique Schnyder Wood.

Let $G = (V, E)$ be a connected graph. A function $h : V \to \mathbb{R}$ is *harmonic* at a vertex v if

$$\sum_{u \in N(v)} (h(v) - h(u)) = 0.$$

Vertices where h is not harmonic are called *poles*.

(3) Show that a nonconstant function on V has at least two poles.

(4) Show that if $S \subset V$ is nonempty and $h_0 : S \to \mathbb{R}$ a function, then there is at most one h which equals h_0 on S and is harmonic at each vertex of $V \setminus S$.

(5) Show that the Laplacian of G is singular and its pricipal minor is non-singular, without using the matrix-tree theorem.

Let Q be a plane quadrangulation with a bipartition V_B, V_W (i.e., a 2-coloring with white and black). Let $R = \{r_0, r_1\}$ be the set of white vertices of the outer face. A *separating decomposition* of Q is an orientation and coloring of the edges of Q with colors red and blue such that

  a. Every vertex v in $V \setminus R$ has two outgoing edges, one of each color, and these outgoing edges separate incoming edges of different color at v. If v is white the clockwise angle between the red and the blue outgoing edges contains red incoming edges, if v is black this same angle contains blue incoming edges.

  a. Vertices of R only have incoming edges, all edges at r_0 are red and all edges at r_1 are blue.

(6) Show that every plane quadrangulation admits a separating decomposition.

(7) Show that in a plane with a separating decomposition a closed curve can be drawn which has all red edges on one side and all blue edges on the other.