**3. Exercise sheet** **Topics in Discrete Mathematics**
**Stefan Felsner** **Summer 2026**
Due for the exercise session: May 7, 2026

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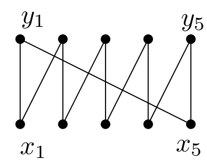
(1) A poset is 3-*irreducible* if it has dimension 3 and after removing any element the dimension drops to 2. There is a complete list of 3-irreducible posets (it includes some infinite families).

The *crown* $C_n$ of order $n \geq 3$ is the poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $x_i < y_i$ and $x_i < y_{i+1}$ for $i \in \{1, \ldots, n\}$ (cyclically) and no other strict comparabilities. Prove that the dimension of crowns is at least 3.




(2) Let $G$ be a plane 3-connected graph. Characterize the critical pairs of the incidence poset $P_{V,E,F}(G)$ of vertices, edges and faces of $G$

(3) Let $\mathcal{B}_n(k_1, k_2, ..., k_r)$ be the containment order of all subsets of $n$ with cardinalities $k_1, \ldots, k_r$. Characterize the critical pairs of $\mathcal{B}_n(k_1, k_2, ..., k_r)$.

(4) A set $R$ of critical pairs of a poset $P$ is called *reversible* if all critical pairs of $R$ can be reversed by a single linear extension of $P$. Characterize the minimal non-reversable sets of critical pairs.

(5) Let $\mathcal{H}(P) = (C, \mathcal{E})$ be the hypergraph where $C$ is the set of critical pairs of $P$ and $\mathcal{E}$ is the set of minimal non-reversable sets of critical pairs. Show that $\dim(P)$ can be defined purely in terms of $\mathcal{H}$.

(6) For a subset $A$ of a poset $P$ we define $C[A] = Pr[Su[A]]$, where

$$Pr[A] = \{y \mid y \leq a \text{ for all } a \in A\}, \text{ and } \quad Su[A] = \{y \mid a \leq y \text{ for all } a \in A\}.$$

**a.** $A \subseteq C[A]$, and $A \subseteq B \implies C[A] \subseteq C[A]$, and $C[C[A]] = C[A]$, i.e., $A \to C[A]]$ is a closure operator.

**b.** If $A$ and $B$ are closed sets in the sense of **a**, then so is $A \cup B$.

**Remark.** The containment order of closed sets is known as the *Dedekind-MacNeille completion* of $P$, some authors call it the *completion by cuts* of $P$.

(7) Given a Schnyder Wood $S$ of a triangulation $T = (V, E)$ we define a coloring of the angles of $T$ with colors $r, g, b$: the two angles at the head of an $i$-colored edge of $S$ are colored $i$.

**a.** Give a definition of these angle colorings in terms of local properties which do not refer to Schnyder Woods.

**b.** Show that there is a bijection between your angle colorings and Schnyder Woods.

(8) Find a characterization of the plane triangulations which have a unique Schnyder Wood.