2. Exercise sheet **Topics in Discrete Mathematics**

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Due for the exercise session: April 30, 2026

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(1) Let I be a family of n intervals on the real line. Show that either I contains $\lceil\sqrt{n}\rceil$ pairwise disjoint intervals or I contains $\lceil\sqrt{n}\rceil$ intervals sharing a common point.

(2) Let $U(x)$ and $D(x)$ be the *open* up-set/down-set of x in P that is

$$U(x) = \{y > x \mid y \in P\}, \qquad D(x) = \{y < x \mid y \in P\}.$$

Prove that for every interval order P, we have

$$|\{D(x) \mid x \in P\}| = |\{U(x) \mid x \in P\}|.$$

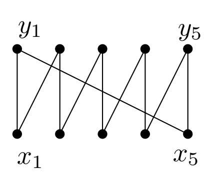
(3) Let P and Q be orders that both have a global minimum $\mathbf{0}$ and a global maximum $\mathbf{1}$ (and $\mathbf{0} \neq \mathbf{1}$ in both orders). Show that

$$\dim(P \times Q) = \dim(P) + \dim(Q).$$

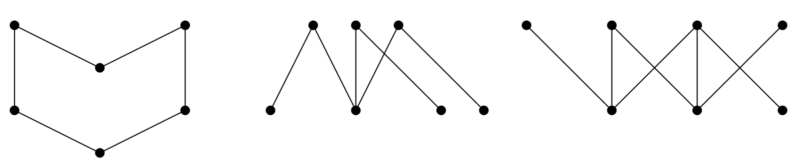
(4) A poset is 3-*irreducible* if it has dimension 3 and after removing any element the dimension drops to 2. There is a complete list of 3-irreducible posets (it includes some infinite families).

Prove that the dimension of the posets below is at least 3.

a. The *crown* C_n of order n is a poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $x_i < y_i$ and $x_i < y_{i+1}$ for $i \in \{1, \ldots, n\}$ (cyclically) and no other strict comparabilities.



b. Three sporadic examples: the chevron, the spider, and one more.



(5) Let G be a plane 3-connected graph. Characterize the critical pairs of the incidence poset $P_{V,E,F}(G)$ of vertices, edges and faces of G

(6) Let $\mathcal{B}_n(k_1, k_2, ..., k_r)$ be the containment order of all subsets of n with cardinalities $k_1, \ldots, k_r$. Characterize the critical pairs of $\mathcal{B}_n(k_1, k_2, ..., k_r)$.

(7) A set R of critical pairs of a poset P is called *reversible* if all critical pairs of R can be reversed by a single linear extension of P. Characterize the minimal non-reversable sets of critical pairs.

(8) Let $\mathcal{H}(P) = (C, \mathcal{E})$ be the hypergraph where C is the set of critical pairs of P and $\mathcal{E}$ is the set of minimal non-reversable sets of critical pairs. Show that $\dim(P)$ can be defined purely in terms of $\mathcal{H}$.

(9) Given a Schnyder Wood S of a triangulation $T = (V, E)$ we define a coloring of the angles of T with colors r, g, b: the two angles at the head of an i-colored edge of S are colored i.

   a. Give a definition of these angle colorings in terms of local properties which do not refer to Schnyder Woods.

   b. Show that there is a bijection between your angle colorings and Schnyder Woods.