1. Exercise sheet **Topics in Discrete Mathematics**
Stefan Felsner **Summer 2026**
Due for the exercise session: April 23, 2026

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(1) **a.** How many edges has the diagram of the Boolean lattice $\mathcal{B}_n$?

b. How many edges has the comparability graph of $\mathcal{B}_n$?

c. How many 3-chains does the Boolean lattice $\mathcal{B}_n$ have?

d. How many 2-antichains does the Boolean lattice $\mathcal{B}_n$ have?

(2) Given a collection of orders $(X, \leq_i)$ on a common ground-set X. Show that $(X, \bigcap_i \leq_i)$ is again a partial order.

(3) **a.** Let G be a comparability graph, show that G contains no odd cycle of length ≥ 5 as induced subgraph.

b. Let G be a incomparability graph, show that G contains no odd cycle of length ≥ 5 as induced subgraph.

(4) Let $\dim^*(P)$ be the least integer d such that the elements of G can be embedded into $\mathbb{R}^n$ in such a way that for every x, y in P we have $x \leq y$ in P if and only if the point of x is less or equal the point of y in the product order on $\mathbb{R}^n$. Prove that $\dim(P) = \dim^*(P)$.

(5) Let G be a graph with a girth $\mathsf{girth}(G)$. Show that if $\mathsf{girth}(G) > \chi(G)$, then G is a cover graph.

(6) Let P be an order. Characterize the pairs (x, y) with the property that

- $P + (x < y)$ is an order,
- $P - (x < y)$ is an order.

(7) Prove that the following conditions are equivalent:

a. G is a comparability graph of a poset of dimension at most 2;

b. G is a containment graph of intervals on a line;

c. G is a permutation graph;

(8) Let I be a family of n intervals on the real line. Show that either I contains $\lceil\sqrt{n}\rceil$ pairwise disjoint intervals or I contains $\lceil\sqrt{n}\rceil$ intervals sharing a common point.

(9) Let $U(x)$ and $D(x)$ be the *open* up-set/down-set of x in P that is

$$U(x) = \{y > x \mid y \in P\}, \qquad D(x) = \{y < x \mid y \in P\}.$$

Prove that for every interval order P, we have

$$|\{D(x) \mid x \in P\}| = |\{U(x) \mid x \in P\}|.$$

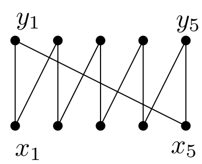
(10) Let P and Q be orders that both have a global minimum $\mathbf{0}$ and a global maximum $\mathbf{1}$ (and $\mathbf{0} \neq \mathbf{1}$ in both orders). Show that

$$\dim(P \times Q) = \dim(P) + \dim(Q).$$

(11) A poset is 3-*irreducible* if it has dimension 3 and after removing any element the dimension drops to 2. There is a complete list of 3-irreducible posets (it includes some infinite families).

Prove that the dimension of the posets below is at least 3.

a. The *crown* C_n of order n is a poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $x_i < y_i$ and $x_i < y_{i+1}$ for $i \in \{1, \ldots, n\}$ (cyclically) and no other strict comparabilities.



b. Three sporadic examples: the chevron, the spider, and one more.