Untileability

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- A region R is a finite union of cells in a regular lattice
- A *tile* τ is a fixed region.
- Let $\mathcal{T}$ be a set of tiles, we say that $\mathcal{T}$ *tiles* a region R when R can be can be written as a disjoint union of translates of the tiles in $\mathcal{T}$.

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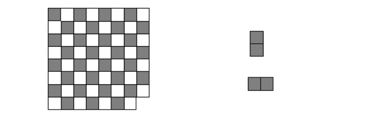

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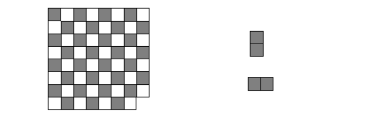

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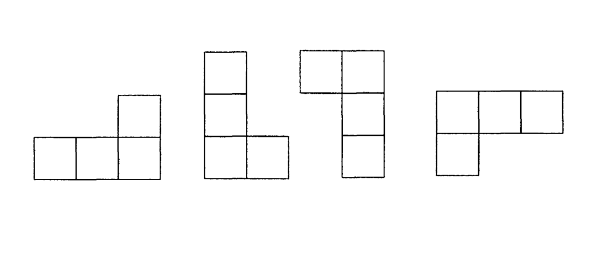
However another invariant can be found in terms of colouring maps and until recently these were the main tool used to prove the untileablility of a region! Therefore we consider colouring maps...

Example of a colouring argument

An example of a colouring argument showing the untileability of a 10 by 10 square in the square lattice by L-tetrominoes:



The 10×10 square and L-tetrominoes

Example of a colouring argument

The idea is that one numbers the cells of the square as shown below:

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5

Numbering of the square

we argue that every tile covers either three 5's and one 1, or three 1's and one 5. In either case we obtain a multiple of 8, but the sum over all the squares in the rectangle is 300.

Generalised colouring maps

Let $\mathbb{A}$ to be the free abelian group on the cells of a regular lattice. Given a tile placement of a tile in $\mathcal{T}$, we associate the element of $\mathbb{A}$ which is defined by being 1 on the squares which are covered and zero everywhere else.

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Let G be an abelian group, a homomorphism $f : \mathbb{A} \rightarrow G$ is called a *generalised colouring map* if

$$f(\tau) = e \; \forall \tau \in \mathcal{T}$$

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where e is the identity element of G.
If $f(R) \neq e$, then the region R is untileabile. This argument is called a *generalised colouring argument*.

Generalised colouring maps

Our previous argument can be seen as a generalised colouring argument, since the numbering can be seen as a map $f : \mathcal{R} \to \mathbb{Q}/\mathbb{Z}$ defined by:

$$f(x_{i,j}) = \begin{cases} \frac{1}{8} & \text{if } i \text{ is odd} \\ \frac{5}{8} & \text{if } i \text{ is even} \end{cases}$$

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Each tileable region maps to an integer and hence the identity in $\mathbb{Q}/\mathbb{Z}$, but $f(R) = \frac{1}{2}(mod\ \mathbb{Z})$.

Signed tilings

Consider a set of tiles $\mathcal{T}$ with weights of either $+1$ or -1 on each tile. Then we say that a region R has a *signed tiling* by $\mathcal{T}$ if there is a covering of R by these weighted tiles, so that the sum of the weights of the tiles that cover a cell inside R is 1 and outside R is zero.

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Remarks:

- Every tileable region is signed tileable.
- A generalised colouring argument will prove signed untileability (as $f(\tau^{-1}) = e$).
- The existence of a signed tiling $\implies$ there is no generalised colouring argument.

The tile homology group

The *tile homology group* of $\mathcal{T}$ is the quotient group $H(\mathcal{T}) = \mathbb{A}/B(\mathcal{T})$, where $B(\mathcal{T}) \subset \mathbb{A}$ is the subgroup generated by all the elements corresponding to possible placement of tiles in $\mathcal{T}$.

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Remarks:

- A region R has a signed tiling if and only if its element in $\mathbb{A}$ is a member of $B(\mathcal{T})$.
- A group homomorphism from $H(\mathcal{T})$ to any abelian group gives a generalised colouring argument.

Proposition (Michael Reid):

Let R be a region that does not have a signed tiling by the set of tiles $\mathcal{T}$. Then there is a numbering of all the cells with rational numbers such that

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- The map can be thought of as a map $\psi : \mathbb{A} \to \mathbb{Q}/\mathbb{Z}$ in which case it is a generalised colouring argument.
- If there is no signed tiling then there exists a generalised colouring argument.

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Proof:

- Let r $\in H(\mathcal{T})$ be the image of the region R. R has no signed tiling $\implies$ r is non-trivial.

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- Let $\langle r \rangle \subset H(\mathcal{T})$ be the cyclic group generated by r, then we can find a group homomorphism $\phi : \langle r \rangle \to \mathbb{Q}/\mathbb{Z}$ such that $\phi(r) \neq 0$.

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- Let $\langle r \rangle \subset H(\mathcal{T})$ be the cyclic group generated by r, then we can find a group homomorphism $\phi : \langle r \rangle \to \mathbb{Q}/\mathbb{Z}$ such that $\phi(r) \neq 0$.
- Then as $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, we can extend this homomorphism to the whole of $H(\mathcal{T})$, mapping the identity to the identity and hence signed tileable regions to the identity.

Signed tilings $\Leftrightarrow$ non-existence of GCAs

Since $\mathbb{A}$ is a free abelian group, there exist a homomorphism ψ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, two heads] & \mathbb{Q} \arrow[d, two heads] \\ H(T) \arrow[r, "\phi"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

The homomorphism diagram

and ψ is the homomorphism we wanted

$\square$

Remarks:

- Often, $H(\mathcal{T})$ is finitely generated, so $\phi(H(\mathcal{T})) \subset \mathbb{Q}/\mathbb{Z}$ is also finitely generated.

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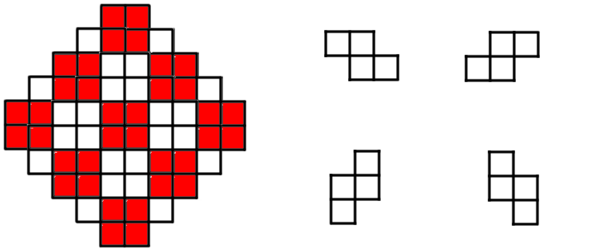
Multiplying by N we obtain a numbering of the cells, such that

1. any tile placement covers a total divisible by N, and
2. the region covers a total which is not divisible by N.

Colouring proof for the Aztec diamond

Recall the colouring argument of the Aztec diamond which proves that it cannot be tiled by skew tetrominoes for $n \equiv 1, 2 \pmod 4$:



Colouring of the Aztec diamond

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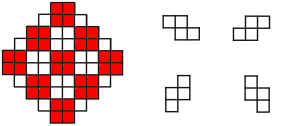

- Each tile covers an odd number of white and red cells.

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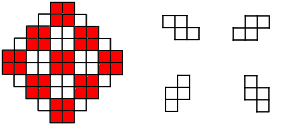

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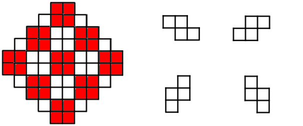

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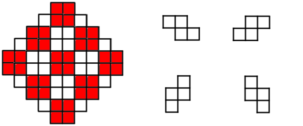

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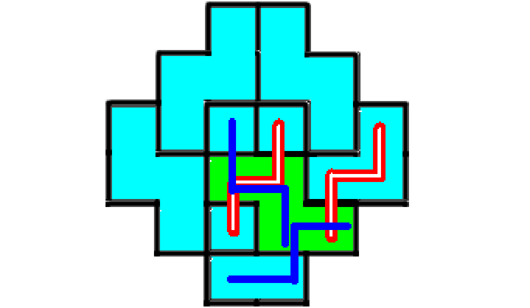
- The region contains an even number of black squares $\implies \frac{n(n+1)}{2}$ must be even.
- The Aztec square of order n is untileable for $n \equiv 1$ or 2 $(mod\ 4)$.

The existence of a signed tiling

This will not work for $n \equiv 3$ or 4 $(mod\ 4)$ as we can find signed tilings.



Signed tiling of the Aztec diamond

Hence we look for stronger invariants, boundary word invariants!

Observing the boundaries

As the above has shown, (generalised) colouring arguments approaches are not always successful.

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Idea:

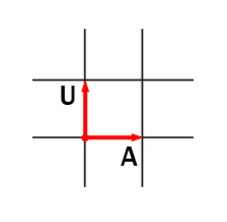
- Observe the boundaries traced by the tiles, as well as those of the region needing to be tiled.
- Develop boundary invariants, i.e. properties shared by all regions admitting a tiling.
- Try to conclude the untileability based on the above observations.

The square lattice as a free group

The oriented paths in the square lattice are described as words in the free group

$$\mathbb{F} =< A, U >$$

where A and U stand for "Across" and "Up". The directed paths corresponding to words of length 1, are the *edges* of the lattice.



The generators of a square lattice

- Such a directed path is called *closed* if it's starting point coincides with its ending point, and *simple* if it does not cross itself.

The topological boundary

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- We define the *topological boundary* of a region as follows:
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  - For a region, it is the union of the topological boundary of the cells it contains, discarding the edges that appear twice with opposite orientations.

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  - For a cell, it is the set of its bounding lattice edges, taken counterclockwise.
  - For a region, it is the union of the topological boundary of the cells it contains, discarding the edges that appear twice with opposite orientations.
- This presentation will treat only *simply connected regions*, i.e. regions whose complement in $\mathbb{R}^2$ is connected, and whose edges can be ordered into a simple directed path.

Determining the boundary

A boundary path of a simply connected region R can be uniquely determined from its first edge and it will be denoted by $\partial R(e)$.

$\partial R(e_1) = U^{-2}A^{-1}UA^{-1}UA^2$ $\qquad$ $\partial R(e_2) = UA^2U^{-2}A^{-1}UA^{-1}$

Two boundary words for the T-tromino

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$$[\partial R] = \left\{ W\partial R(e)W^{-1}, W \in \mathbb{F} \right\}$$

where e is an edge on the topological boundary of R. By the above, it is well-defined.

Combinatorics meets group theory

We will further use some standard terminology from group theory:

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- Lastly, $[G : G]$ will denote the *commutator subgroup* of G, i.e. the group generated by the commutators $W_1 W_2 W_1^{-1} W_2^{-1}$ for all $W_1, W_2 \in G$.

Defining the tile group

Back to the square lattice, the *cycle group* C is the subgroup of $\mathbb{F}$, consisting of all words associated to closed directed paths in the lattice. As we will later see, C is $[F : F]$, a normal subgroup, equal in fact to $N(\langle AUA^{-1}U^{-1}\rangle)$.

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To a set of tiles $\mathcal{T} = \left\{\tau_i, i \in \overline{1, m}\right\}$ we assign a subgroup of F, called the *tile group*, denoted by $T(\mathcal{T})$, which contains the combinatorial boundaries of all tiles in $\mathcal{T}$, that is

$$T(\mathcal{T}) = \langle W\partial\tau_i(e_i)W^{-1} : W \in \mathbb{F}, 1 \leq i \leq m\rangle$$

where $\partial\tau_i(e_i)$ is an oriented boundary of τ_i.

The tile homotopy group

In fact, $T(\mathcal{T})$ is also a normal subgroup of C, and we define the *tile homotopy group* to be the quotient

$$h(\mathcal{T}) = C/T(\mathcal{T})$$

The basic invariant we assign a region R which is to be tiled with a set of tiles $\mathcal{T}$, is the conjugacy class in $(\mathcal{T})$ corresponding to the combinatorial boundary $[\partial R]$.

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- As $T(\mathcal{T})$ is a normal group, it suffices to show that some oriented boundary $[\partial R(e)]$ is in $T(\mathcal{T})$. This follows by induction by the number of tiles needed to tile R.

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- The base case (only one tile is needed) follows immediately.

Topological considerations

Sketch of proof, continued:

- Simply connected regions can be described as topological disks with Jordan curve boundaries. (The particular case of a boundary touching itself at a vertex can be resolved if we considered a copy of the region "thickened" at that corner).

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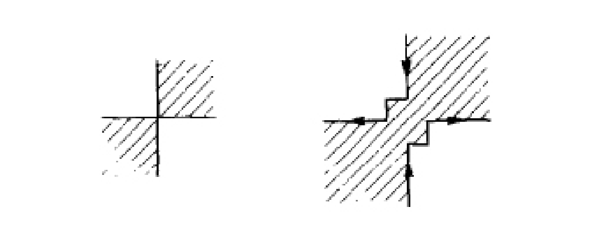
- Simply connected regions can be described as topological disks with Jordan curve boundaries. (The particular case of a boundary touching itself at a vertex can be resolved if we considered a copy of the region "thickened" at that corner).



Thickening of a corner

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- Now the following claim has an accessible (albeit technical) proof:

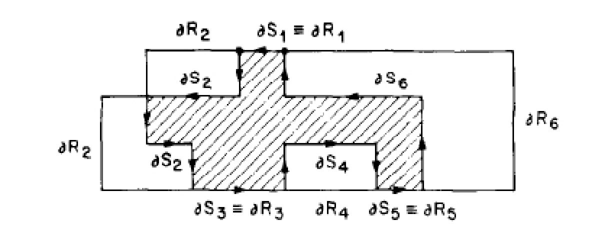
Claim: There exists a decomposition $R = R^* \cup R^{**}$ such that R^*, R^{**} are non-empty simply connected regions which can be tiled by $\mathcal{T}$, and there are directed edges e_1 of R^* and e_2 of R^{**} such that :

$$\partial R(e_1) = \partial R^{**}(e_2)\partial R^*(e_1)$$

Visualizing the decomposition

Sketch of proof, continued:

- The claim easily completes the induction step.



Visualizing the induction step

Introducing the Cayley graph

A construction allowing to easily observe the boundary invariants is the *Cayley graph*, denoted by $\mathcal{G}(G := F/K)$, where F is a free group and K is a normal subgroup of relations.

This graph (also known as *group diagram*), is drawn as follows:

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- For generators of order two there will be drawn undirected edges.

The tile group and Cayley graphs

The group diagram will be used in the context of tileability as follows:

- Let $H \supseteq T(\mathcal{T})$ be a normal subgroup and $\mathcal{G}_H$ the Cayley graph $\mathcal{G}_H$ corresponding to $\mathbb{G}/H$.

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- For the region R to be tiled, observe the directed path tracing the image in $\mathcal{G}_H$ of the topological boundary of the chosen region.
- If the above path does not close, a tiling is impossible.
- A closed path is necessary, but not sufficient. We will draw stronger conclusions based on *winding numbers*.

The winding number

Let s be a cell in our graph, $x_s \in Int(s)$, and P a closed directed path. The winding number $W(P;s)$ measures how many times does P enclose s in a counterclockwise direction, and is given by the formula

$$w(P;s) = \frac{1}{2\pi\imath}\oint_P \frac{1}{z - x_s}dz.$$

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Properties of the winding number:

- The formula above does not depend on the choice of x_s.
- It is additive: For two closed paths P_1, P_2 starting at the same point, one has:

$$w(P_1 P_2; s) = w(P_1; s) + w(P_2; s).$$

Applications to tileability

We will denote the set of cells (faces) of the Cayley graph $\mathcal{G}_H$ by $\mathcal{C}$. Upon fortunate choice of H,

$$\exists A \subset \mathcal{C} s.t. \sum_{s \in A} w(\partial\tau(e); s) = 0, \ \forall \tau \in \mathcal{T} \ (*)$$

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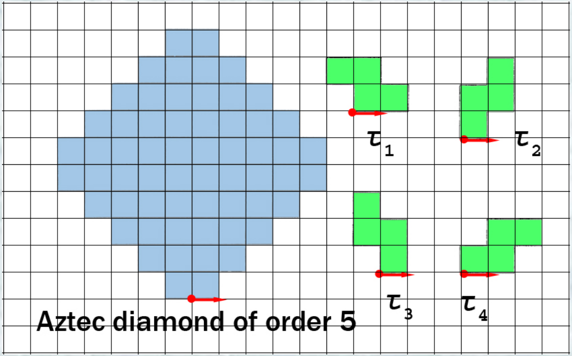
Remarks:

- More conditions like $(*)$ could be needed for a particular problem, or slightly modified versions.
- Finding such a group H, "In general, it trades one hard problem for another" (J. Conway).

Recalling the Aztec diamond

Consider the four tiles depicted below, and the order 5 Aztec diamond, with the marked leading edges:



The Aztec diamond and skew tetrominoes

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$$\partial\tau_4(e_4) = U^{-1}A^{-1}U^{-1}A^{-2}UAUA^2$$

whereas the Aztec diamond of order n can be assigned the following boundary word:

$$(AU^{-1})^n(U^{-1}A^{-1})^n(A^{-1}U)^n(UA)^n.$$

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Proposition:

The Aztec diamond can never be tiled by skew tetrominoes.

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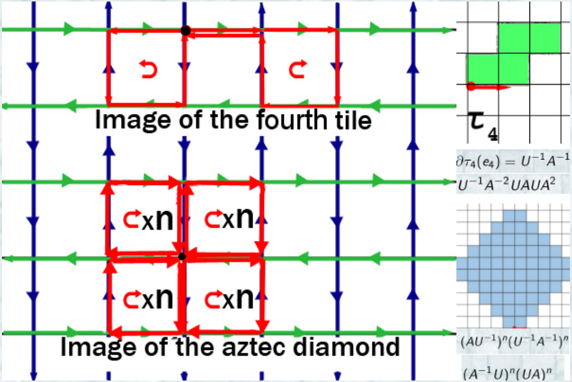
We have seen in the previous section the proof of the above for diamonds of size $n \equiv 1, 2\ (mod\ 4)$.

We have also seen that there exist signed tilings for $n \equiv 0, 3\ (mod\ 4)$, so in this case colouring arguments won't suffice. We will develop an approach using the notions introduced above.

Constructing the Cayley graph

Now take the subset H to be $N(\langle AUAU, AU^{-1}AU^{-1}\rangle)$, and construct the Cayley graph $\mathcal{G}_H$, pictured below.



The image of the region and the tiles in the Cayley graph

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Finalisation of the proof

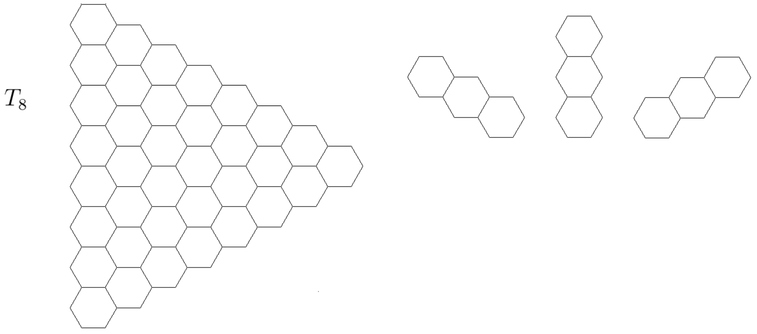
Note the following:

- Both the tiles and the Aztec diamond have their boundaries mapped to closed paths in $\mathcal{G}_H$.
- Tracing the contour of each τ_i, one easily notes that the sum of the winding numbers around all cells is 0.
- As the winding number is additive, every element in $T(\mathcal{T})$ maps to a directed closed path with 0 winding-number.
- However, the Aztec diamond has a winding number of either $4n$ or $-4n$ (depending on the starting point), hence untileability by skew tetrominoes can be concluded.

$\square$

Stating the tribone tiling problem

We will now try to apply some of these ideas on the following problem: Can we tile triangular regions T_n in the hexagonal lattice with the set of tiles $\mathcal{T}$ (called *tribones*) below?



Tiling by tribones?

The result we aim at

After playing around with the tiles you will find that a tiling is hard to find, so maybe it is untileable!

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It is impossible to tile the triangular region T_n in the hexagonal lattice by tribones and their 120° and 240° rotations.

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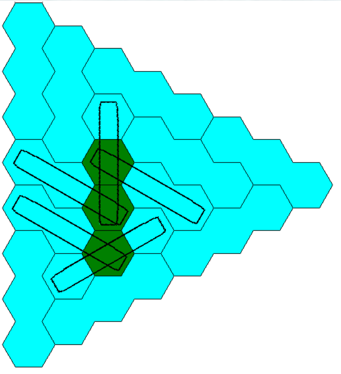
Clearly as the number of hexagons in T_n is $n(n+1)/2$, so either n or $n+1$ must be divisible by 3 (tile covers 3 hexagons). This rules out $n \equiv 1\ (mod\ 3)$.

Colouring arguments are not useful here

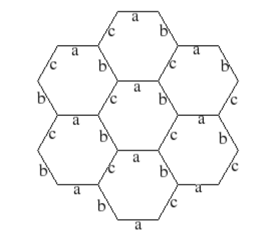
Notice that for $n = 8$ we can find a signed tiling of the triangle, so we are indicated once again that we need to argue otherwise than colouring arguments!



Signed tiling by tribones

Observations about the hexagonal lattice

We label the lines of the hexagonal lattice a, b, and c if they are at 0°, 120°, and 60° respectively, as undirected edges.



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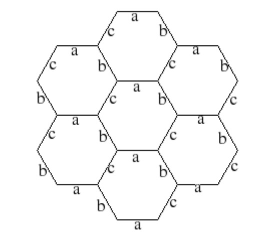


Hexagonal lattice as Cayley graph

We see that this is the Cayley graph of the group

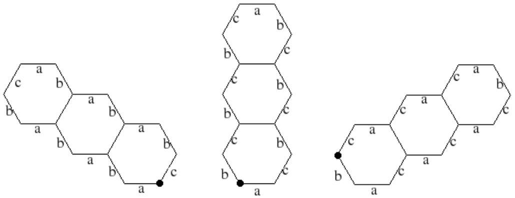
$$\mathbb{H} = \langle a, b, c | a^2 = b^2 = c^2 = (abc)^2 = 1 \rangle$$

Note that none of the edges are directed in this graph, as all the generators have order 2.

The tribone boundary words

We also note that the boundary words of the tiles are:

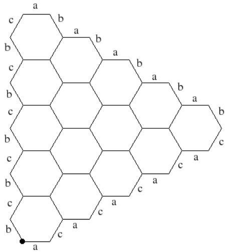
- $t_1 = (ab)^3 c (ab)^3 c$
- $t_2 = (bc)^3 a (bc)^3 a$
- $t_3 = (ca)^3 b (ca)^3 b$



The labeled tribones

The boundary of T_n

and that the boundary word of our region is $(bc)^n(ab)^n(ca)^n$.



The triangle in the hexagonal lattice

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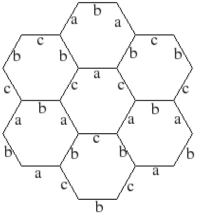
Both the combinatorial boundary $[\partial T_n]$ and the Tile group $T(\mathcal{T})$ are contained in $\mathbb{S}$

Proof of Claim:

It is enough to show that the generators of $T(\mathcal{T})$ and $[\partial T_n]$ are in $\mathbb{S}$. This can be done by looking at the Cayley graph (denoted $\mathcal{G}_S$ of $\mathbb{H}/\mathbb{S}$).

The relabeled hexagonal lattice

The Cayley graph of $\mathcal{G}_S := \mathbb{H}/N(\langle (bc)^3, (ca)^3, (ab)^3 \rangle)$ is



$\mathcal{G}_S$ of $\mathbb{H}/\mathbb{S}$

Observations regarding $\mathcal{G}_S$

- Surprisingly, it is again the hexagonal lattice, albeit differently labeled. This time, it is composed of infinitely many ab, bc and ca hexagons instead of abc hexagons.

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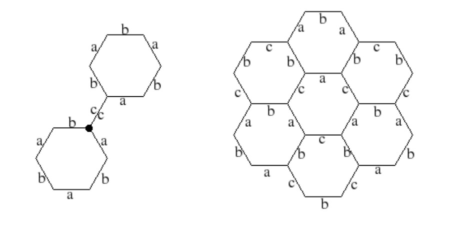
- What we first need to do is to show that shadow paths of the generators of $T(\mathcal{T})$ and $[\partial T_n]$ in $\mathcal{G}_S$ are closed.
- The case of generator $t_1 = (ab)^3c(ab)^3c$ is shown below and the results are similar for t_2 and t_3.



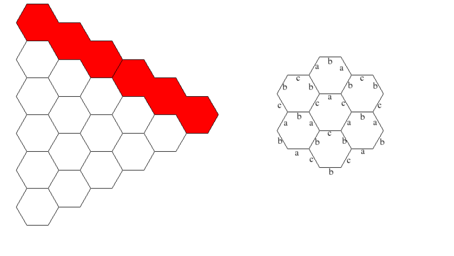
Shadow path of a generator in the Cayley graph

Reducing the analysis to $n = 3k$.

The combinatorial boundary of T_n is clearly in $\mathbb{S}$ when $n = 3k (k \in \mathbb{N})$, as the word is just $(bc)^{3k}(ab)^{3k}(ca)^{3k}$.

Reducing the analysis to $n = 3k$.

The combinatorial boundary of T_n is clearly in $\mathbb{S}$ when $n = 3k (k \in \mathbb{N})$, as the word is just $(bc)^{3k}(ab)^{3k}(ca)^{3k}$. When $n \equiv 2 \ (mod\ 3)$, we notice that if we add a line of tribones to one side of T_n we obtain T_{3k} for some k. In $\mathcal{G}_S$ this can be seen as adding in several closed loops to the loop of T_n to obtain a new loop, that of T_{n+1}.



Extending a triangle

Winding numbers in the $\mathcal{G}_S$

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$$w(P;S) = \sum_{s \in S} w(P;s)$$

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- Let S_1, S_2, and S_3 be the set of all the *ab*, *bc*, and *ca* hexagons respectively.
Taking any word u in $\mathbb{S}$, then define the group homomorphism $W : \mathbb{S} \to \mathbb{Z}^3$ by

$$W(u) = (w(u;S_1), w(u;S_2), w(u;S_3))$$

where u is considered to be the path of the word in $\mathcal{G}_S$.

Concluding untileability

With this definition we see that:

- $W(t_1) = W(t_2) = W(t_3) = (0, 0, 0)$, hence all conjugates are mapped to (0,0,0) and so $W(T(\mathcal{T})) = \{(0, 0, 0)\}$.

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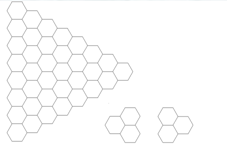
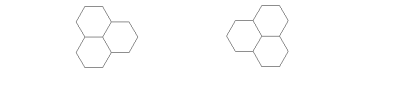
Therefore $[\partial T_n] \not\subset T(\mathcal{T})$, hence T_n is untileable by tribones, proving Conway's theorem.

$\square$

Tiling T_n by T_2

Another nice T_n tiling problem is the following: can you tile T_n by T_2 (and it's 180° rotation)?





Triangles of size 2 as tiles

Theorem (Conway, Lagarias):

The triangular region T_n in the hexagonal lattice can be tiled by T_2's if and only if $n \equiv 0, 2, 9,$ or 11 (*mod* 12).

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Proceeding with the proof:

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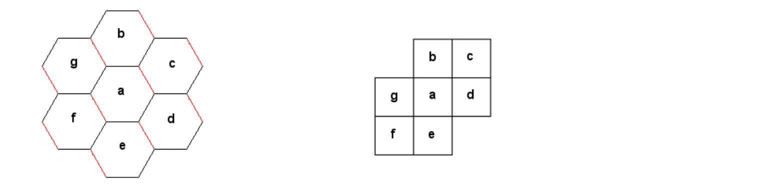
- We are left to show that T_n is untileable for $n \equiv 3, 5, 6, 8$ $(mod\ 12)$.

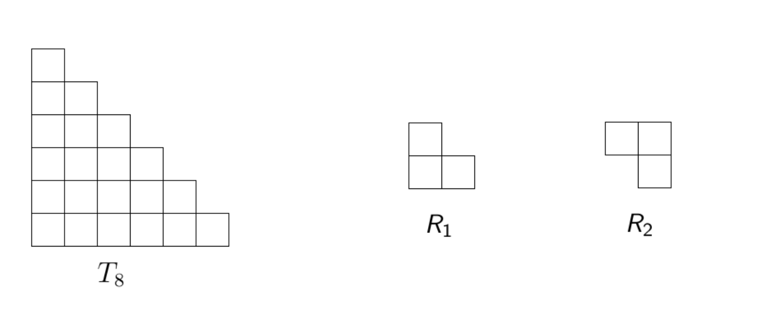
Changing to square lattice

The problem can be translated to an equivalent problem in the square lattice by 'contracting an edge and reshaping'.



Converting from the hexagonal to the square lattice

This converts the problem to:

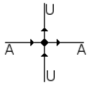


Restatement of the problem

The tile boundaries

Orient the edges in the square lattice and consider it as the Cayley graph of $\mathbb{F}$.

At each vertex we have:



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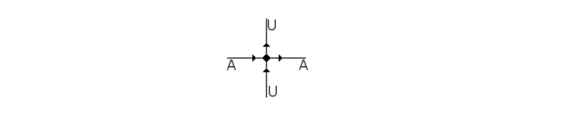

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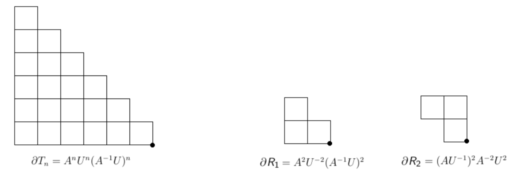
The boundary words of the tiles are:



Boundary words

Let $\mathbb{S} := N(\langle A^3, U^3, (U^{-1}A)^3\rangle) \subset \mathbb{F}$.
Once again $\mathbb{S}$ contains $[\partial T_n]$ and $T(\mathcal{T})$, as their paths are loops the Cayley graph $\mathbb{F}_S := \mathbb{F}/\mathbb{S}$:

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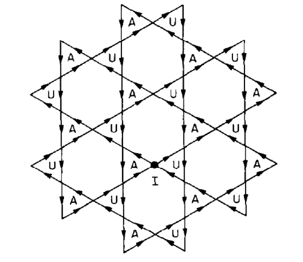

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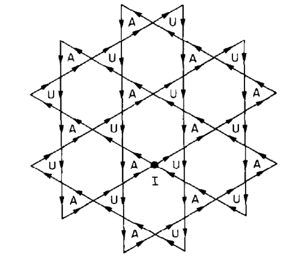

Shadow paths of the region and of the tiles

Note: due to the relations A^3, U^3 and $(A^{-1}U)^3$, we just need to consider $n \equiv 2$ or 3 $(mod\ 3)$ to show that ∂T_n is in $\mathbb{S}$.

Observing the winding numbers

Once again we use the winding number, consider the word as a path in $\mathbb{F}_S$ starting from a fixed vertex.
Let $S :=$ the set of hexagonal regions, then

- $w(\partial R_1; S) = 1$
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$$w(\partial T_n; S) = \sum_{i=1}^{m} w(W_i (\partial R_{k_i})^{\epsilon_i} W_i^{-1}; S)$$

$$= \sum_{i=1}^{m} \epsilon_i w(\partial R_{k_i}; S) \equiv \ (mod\ 2) \qquad (1.2)$$

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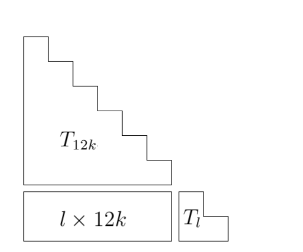
The triangular region T_n in the hexagonal lattice can be tiled by T_2's only if $n \equiv 0, 2, 9,$ or 11 $(mod\ 12)$.

In the other cases, a tiling is possible

- A 2×3 rectangle is tileable.
- A 5×6 rectangle is tileable.
- Hence $l \times 12k$ rectangles are tileable for l = 2, 9, 11 and, 12.

We can extend a tiling by observing a partition of T_{12k+l}:



Extending the tiling

Introducing the lozenges

We now consider the third type of regular plane lattice, the triangular one. A *lozenge* (rhombus) is defined to be a region consisting of two adjacent cells. Below is depicted the set $\mathcal{L}$ of the possible (non-equivalent by translation) placements of lozenges in the lattice.

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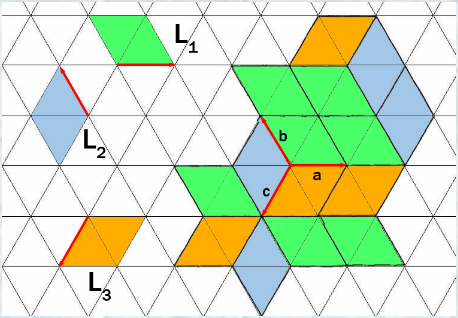

The lozenges in the triangular lattice

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- We establish a consistent labelling of the plane: one set of edges will be parallel to the horizontal axis, which set of edges will be named a. Analogously, the set of edges pointing at 120° will be labelled b, whereas the edges pointing at 240° will be labelled c. Let the free group generated by a,b, and c, be called $\mathbb{F}_\Delta$, and the corresponding cycle group be C_Δ.

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- The cycle group of the triangular lattice is $N(\langle abc, cba \rangle)$, the two words corresponding to anti-clockwise, and clockwise oriented boundaries of elementary triangles.

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By analyzing the tile homotopy group $h(\mathcal{L})$, one can note that the above boundary words translate in commutativity relations: $ab = ba$, $bc = cb$, $ca = ac$, which shows that that $h(\mathcal{L})$ is identified with a subgroup of $\mathbb{Z}^3$ and in fact $\mathbb{F}_\Delta/\mathcal{L}$ is isomorphic to $\mathbb{Z}^3$

Lifting lozenges in the 3D space

Consider the Cayley graph $\mathbb{F}_\Delta/\mathcal{L} \equiv \mathbb{Z}^3$. It can be naturally embedded into the Euclidean 3D space, as a cubical tesselation of the space, with the cubes are on their corners.

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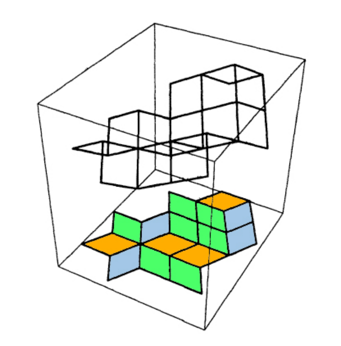

Lozenge tiling as the projection of a cubical tesselation

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Take a region in the plane that can be tiled by lozenges. It can be lifted, tile by tile, into the cubical skeleton. Each edge of a cube brings a change in height equal to ± 1. The edge of the plane covered by a lozenge, lifts to a diagonal of a cubical face, which brings a height change of 2.

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- The construction of the cubical tesselation indicates the necessity that this net rise in height is 0, in order to claim tileability.

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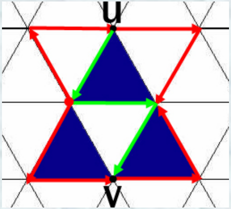
- Each lozenge covers a cell of each colour, so a clear condition for tileability, is that there is an equal number of white and blue triangles in a region.
- This is equivalent to saying that the boundary of R lifts to a closed path in $\mathbb{F}_\Delta/\mathcal{L}$.



Region with no lozenge tiling, but closed lifted boundary

Defining a minmal distance

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- The three vertices of a triangle take three diferent values modulo 3.

Finding a necessary and sufficient condition

Recalling the height function previously considered for the vertices in $\partial[R]$, for any $v,\ w$ on the boundary, with $h(w) \geq h(v)$ we have $h(w) - h(v) \geq d(v, w)$, as a necessary condition for a tiling to be possible.

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Motivation

We have seen in the beginning that the generalised colouring arguments could be interpreted as maps from the cells of the lattice to an abelian group, (for finitely generated $\mathcal{H}(\mathcal{T})$ there is a suitable choice of N, such that $\mathbb{Z}_N$ can be the chosen abelian group).

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The following theorem will explain several group theoretical notions regarding the cycle group C.

Explicite definition of C and important subgroups

Theorem (Conway, Lagarias)

1. The cycle group C consisting of all words W such that $P(W)$ is a closed, directed path in $\mathbb{Z}^2$, is $C = [\mathbb{F} : \mathbb{F}]$.

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3. The group $A_0 = C/[C : C]$ is a direct sum of a countable number of copies of $\mathbb{Z}$, which are in one-to-one correspondence with the cells c_{ij} of the lattice $\mathbb{Z}^2$. The projection map $\pi_{i,j} : C \to \mathbb{Z}$ onto the c_{ij}th summand of A_0 is given by the winding number $w(P(W); c_{ij})$.

Sketch of proof:

(1):

- "$[F:F] \subset C$": Note that a word defines a closed directed path iff $U = U^{-1}$ and $A = A^{-1}$. As this happens for all commutators $UVU^{-1}V^{-1}$ (where U and V are words in $\mathbb{F}$), conclusion follows.

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- "$C \subset [F : F]$" : If $c \in C$ has 2 or 4 edges, it is true. The proof is completed by induction, ordering the words in C lexicographically by (n, k, l), where n is the length of the word, k is the maximum value $i^2 + j^2$ of any vertex, and l is the number of vertices with value k. The basic idea is to "cut out" a cell from the furthermost corner of the cycle, decomposing it in terms of lexicographically lower cycles.

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- "$[C : C] \subseteq C_1$": Follows from the additivity of the winding numbers:

$$w(W_1 W_2 W_1^{-1} W_2^{-1}; c_{ij}) = w(W_1; c_{ij}) + w(W_2; c_{ij}) +$$

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- "$C_1 \subseteq [C : C]$": Follows by induction after the same lexicographical order as in (1).

Sketch of proof, continued:

(3)

Define the homomorphism $\pi = \otimes_{i,j}\pi_{i,j}$ from C to $\otimes_{(i,j)}\mathbb{Z}$, by $\pi_{i,j} = w(P(W); c_{ij})$.
By part (1), this map is well defined, by (2), its kernel is $[C : C]$.
Hence its image is isomorphic to $C/[C : C]$.

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- In the particular case when $A = \mathbb{Z}^k$, the above can model standard colouring arguments. In that case $\phi = \otimes_i \phi_i$, where $\phi_i(W)$, counts the sum of the winding numbers of $P(W)$ around cells coloured with the colour i.

Projecting the GCM to $h(\mathcal{T})$

A generalised colouring map checks whether

$$\phi([\partial R]) \in \phi(T(\mathcal{T}))$$

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No information is lost by performing this projection, therefore generalised colouring arguments can be considered as being specified by homomorphisms $\tilde{\phi}$ from the tile homotopy group $h(\mathcal{T})$ to abelian groups $\tilde{A}$.

(Re)introducing $B(\mathcal{T})$ and $H(\mathcal{T})$

The maximal information available about tilings in this context is given by the projection $\pi_s : h(\mathcal{T}) \to H(\mathcal{T})$, where $H(\mathcal{T})$ is the *tile homology group* introduced in the beginning, and in fact, the largest abelian quotient subgroup of $h(\mathcal{T})$.

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Looking at the natural projection $\tilde{pi} : C \rightarrow C/T(\mathcal{T})$, and taking the kernel of the map $\pi_s \circ \tilde{pi}$ to be $B(\mathcal{T})$, we obtain $H(\mathcal{T}) = C/B(\mathcal{T})$.

(Re)introducing $B(\mathcal{T})$ and $H(\mathcal{T})$

The maximal information available about tilings in this context is given by the projection $\pi_s : h(\mathcal{T}) \to H(\mathcal{T})$, where $H(\mathcal{T})$ is the *tile homology group* introduced in the beginning, and in fact, the largest abelian quotient subgroup of $h(\mathcal{T})$.

Looking at the natural projection $\tilde{pi} : C \to C/T(\mathcal{T})$, and taking the kernel of the map $\pi_s \circ \tilde{pi}$ to be $B(\mathcal{T})$, we obtain $H(\mathcal{T}) = C/B(\mathcal{T})$.

$B(\mathcal{T})$ is called the *tile boundary group* and it is the smallest normal subgroup of C containing $T(\mathcal{T})$ and $[C : C]$. In fact the following holds:

$$B(\mathcal{T}) = T(\mathcal{T})[C : C]$$

Restating the signed tiling theorem

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A region R has a signed tiling by $\mathcal{T}$ iff $[\partial R] \in B(\mathcal{T})$.

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Both directions are easy to show by direct expansion of the boundary word of the region in terms of the boundary of the tiles.

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(H2): $[\partial R]$ is in the tile group $T(\mathcal{T})$.

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(H2): $[\partial R]$ is in the tile group $T(\mathcal{T})$.

(H3): $[\partial R]$ is in the group $B(\mathcal{T})$.

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Then $(H1) \Rightarrow (H2) \Rightarrow (H3)$. In general, the implications are not reversible.

Proof of the above theorem

Proof:

The statement $(H1) \Rightarrow (H2)$ is exactly one of the theorems we have seen before. $(H2) \Rightarrow (H3)$ is immediate, as $T(\mathcal{T}) \subset B(\mathcal{T})$.

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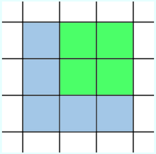
To see that $(H2)$ does not imply $(H1)$, consider the following example, where the set of tiles consists of a 3×3 square and a 2×2 square.



Region in the tile group, but untileable

Proof, continued:

Finally, all the examples of regions which had signed tilings (Aztec square, the triangle tiled by tribones) did belong to $B(\mathcal{T})$ (by the previous theorem), but the analysis of Cayley graphs showed that they don't belong to $T(\mathcal{T})$, therefore $(H3)$ does not imply $(H2)$.

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The End

Thank you for your attention!

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