Structure of Domino Tilings

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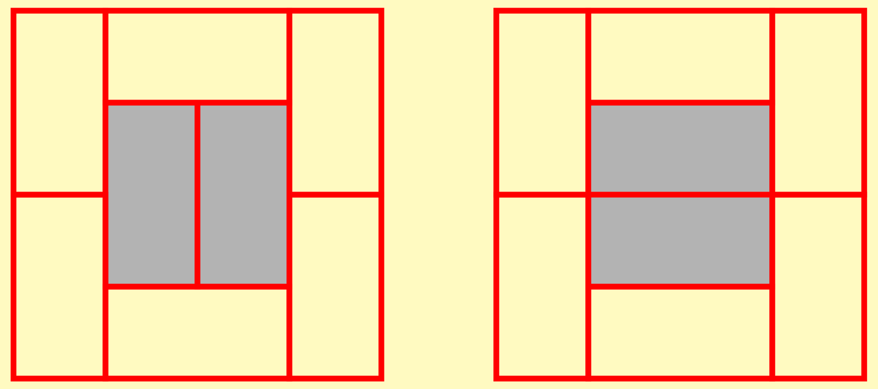
Objective

Imagine a checkerboard covered by dominoes in a way that each domino covers to adjacent fields. Pairs of parallel dominos in a tiling can be rotated by 90° to get a different tiling.

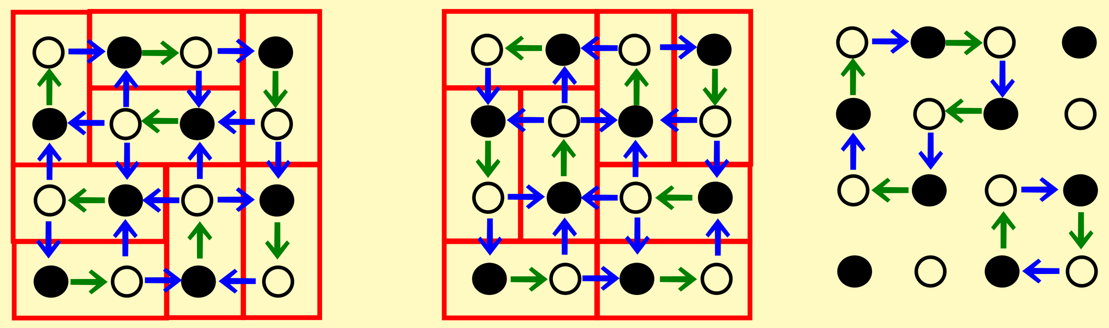


We will examine the implication of this simple transformation with regard to order and structure of the set of all domino tilings.

Although we will not always note that, any results we maintain will hold also for lozenge tilings.

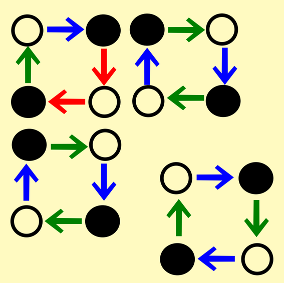
Transforming orientations

Reverting a directed Cycle in an α-orientation will provide another α-orientation as in every vertex one inedge and one outedge is switched.



In fact, each α-orientation can be transformed into any other by subsequently reversing directed cycles. That is because in each vertex for every new inedge there has to be a new outedge.

If in the interior of a directed cycle there is a directed path, one can divide the cycle in two smaller cycles which can be reversed consecutively.



We will repeat that process until we end up with small cycles that cannot be dividet further. We will call those *essential cycles*.

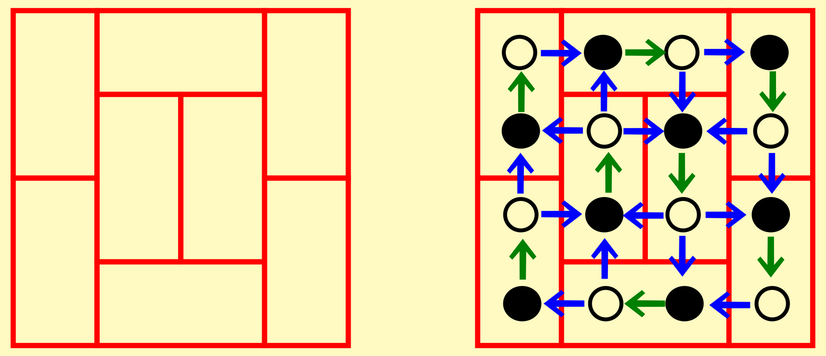
Model

Domino and lozenge tilings correspond to perfect matchings on the underlying graph which is bipartite and planar.
Based on that observation we will orient the graph so that for bipartition (A,B):

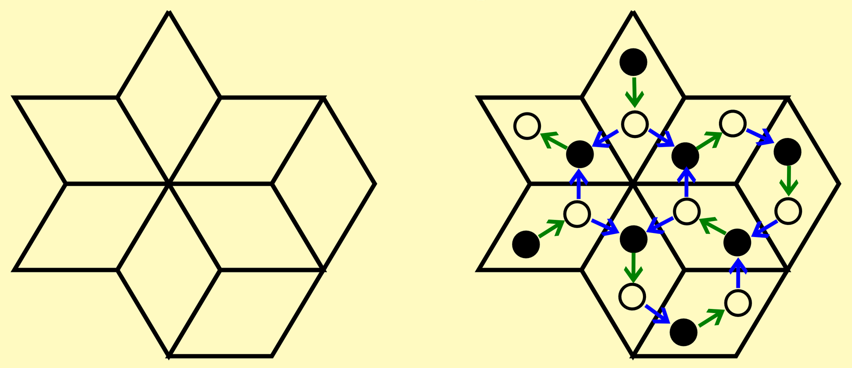
1. Each vertex in A (black) will have one outedge.
2. Each vertex in B (white) will habe one inedge.

We will call an orientation meeting those requirements an α-orientation.
An α-orientation will provide a perfect matching and hence a tiling by taking all the single outedges of vertices in A.

domino tilings ⇔ orientations of rectangular grid



lozenge tilings ⇔ orientations of triangular grid



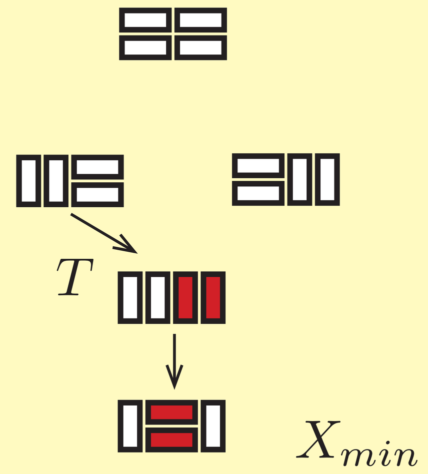
Flips and Flops

Although for arbitrary graphs, essential cycles may not have an empty interior or be even be contain in each other, in our case all directed cycles will decompose into the elemental 4-cycles.
Hence, just by reverting those 4-cycles we can transform an orienation into any other.
We will to refer to reverting a counter-clockwise 4-cycle as a *flip* and reverting a 4-cycle as a *flop*.
Both correspond to the transformation mentioned in the introdoction. However, there is no simple rule whether a given rotation of two dominoes is a flip or a flop.

Main result

The Poset $P = (M, \leq)$ where M is the set of all tilings together with the flip realtion $\leq$ [a] is a distributive lattice. This is a direct result from Stefan Felsner's paper[b]. It follows a sketch of the proof.

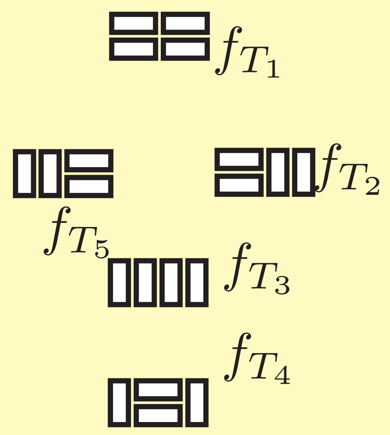
We show that every max. flip sequence on a tiling T leads to X_{min}



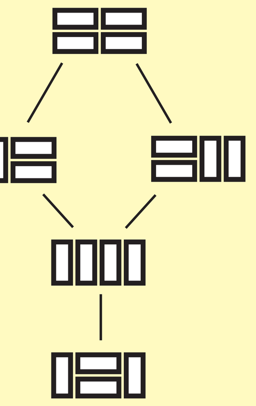
Let T be a tiling. We count the number of flips from an arbitary flipsequence from X_{min} to T. Let f_T be the function with the property $f_T(\text{Flip i}) =$ #(Flip i) in flipsequence
It follows that:
$f_T(1) = 0, f_T(2) = 1$
$f_T(3) = 1$ if we declare the flips as

1	2	3

There is a order-stable bijection between the set of all tilings and the set of their induced count functions.



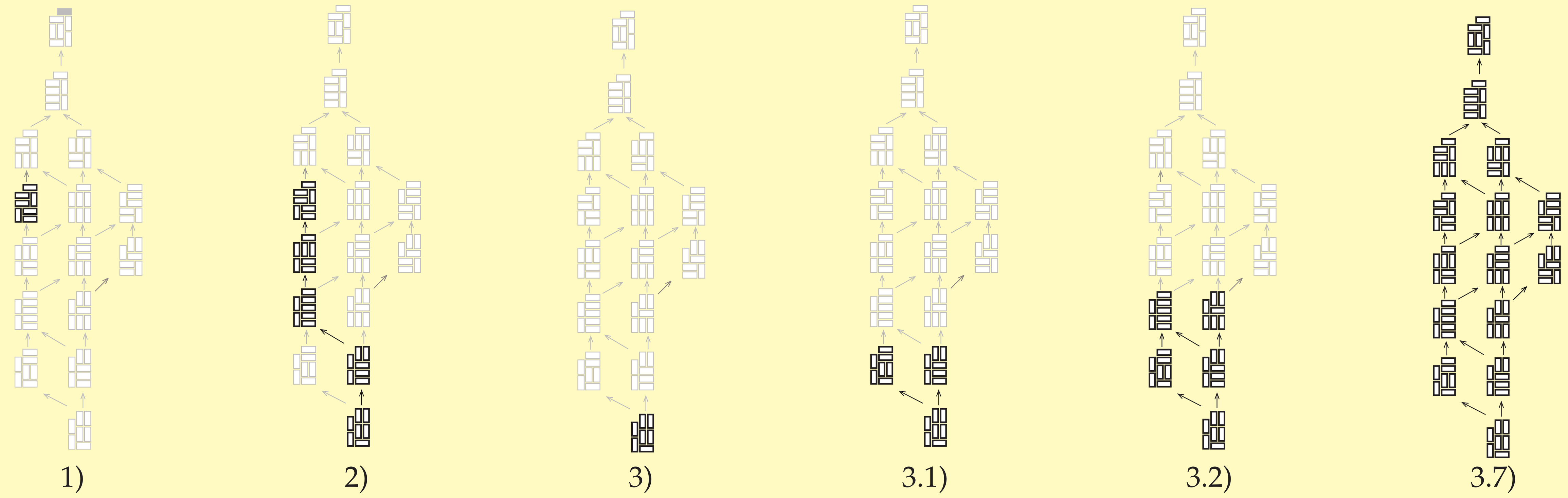
The functions induce a distrubitve Lattice. The cover relation are flips between the corresponding Tilings.



[a] i.e. for $A_1, A_2 \in M$ $A_1 \leq A_2$ iff there is a flip sequence from A_1 to A_2
[b] Lattice Structre from Planar Graphs

Determine the distributive lattice

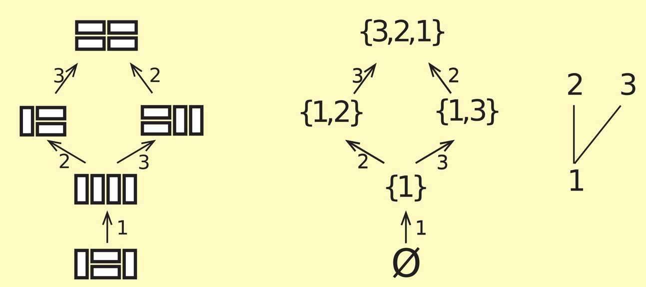
The following three steps give an example how to determine all tilings, in particular we will construct the distributive lattice that is implied by the flip-order.
1) Find a arbitrary tiling 2)Keep flipping unitl you reach X_{min} 3) Flop as long as possible from X_{min}



Tilings and downsets

From the Lattic-Therorem it follows that the tiling lattic is isomorphe to a downset lattic that is induced by a poset P.
The elements of the poset P correspond to the tiling flips from X_{min} to the actual tiling. So for instance the downset $\{3, 1\}$ corresponds to the tiling that you can reach, if you flip cycle 3 and 1 from X_{min}[a]. In this example, the elements of P coresond to the flips in the follwing way:

3	1	2



[a] Note that this implies that there are two orientations A_1, A_2, which need to flip every flip at least once in a transformation from A_1 to A_2