

2. A colouring argument

We colour the Aztec diamond in red and white, as in the picture on the right.

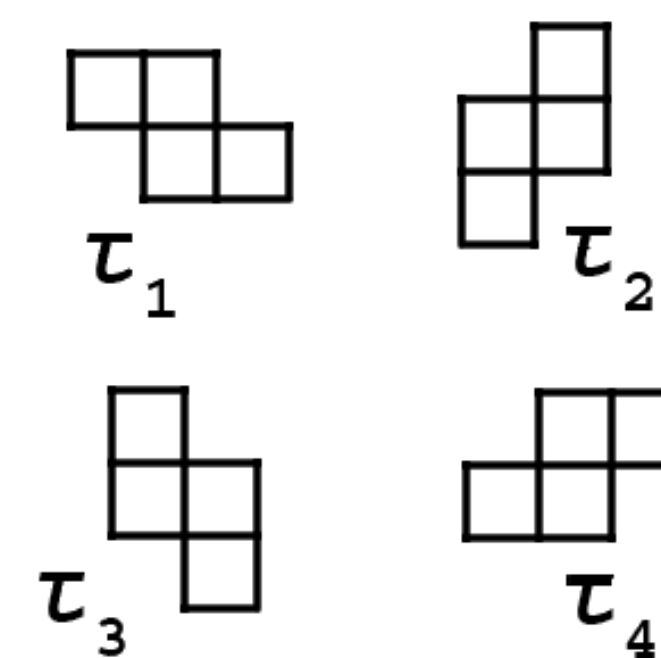
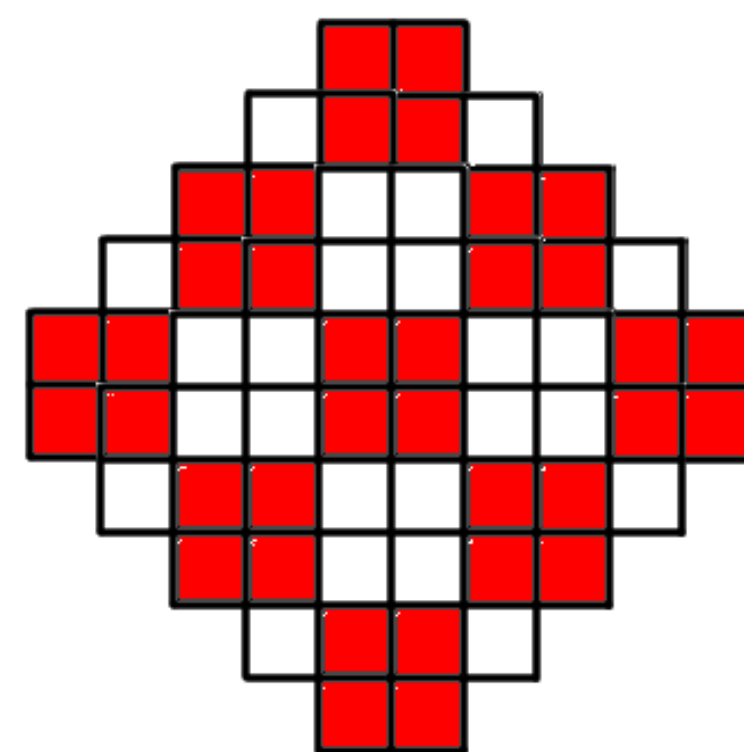
- Each tile covers an odd number of red cells, therefore an odd number of tiles covers an odd number of red cells.
- There are $2n(n+1)$ squares in the Aztec diamond, therefore $\frac{n(n+1)}{2}$ tiles are needed.
- The region contains an even number of red cells, therefore $\frac{n(n+1)}{2}$ must be even.
- We conclude that the Aztec diamond of order n is untileable for $n \equiv 1, 2 \pmod{4}$, but this method fails for other values of n .

Several other tiling problems can be solved by suitably numbering (colouring) the cells and observing invariant properties of tileable regions.

1. Can the Aztec diamond be tiled?

A challenging question is whether a region of a regular lattice can be tiled (perfectly covered with no overlaps) by a given set of tiles.

Example: The Aztec diamond of order n is the union of all cells in the square lattice, which are less than n steps away from a fixed origin. Can it be tiled by skew tetrominoes?



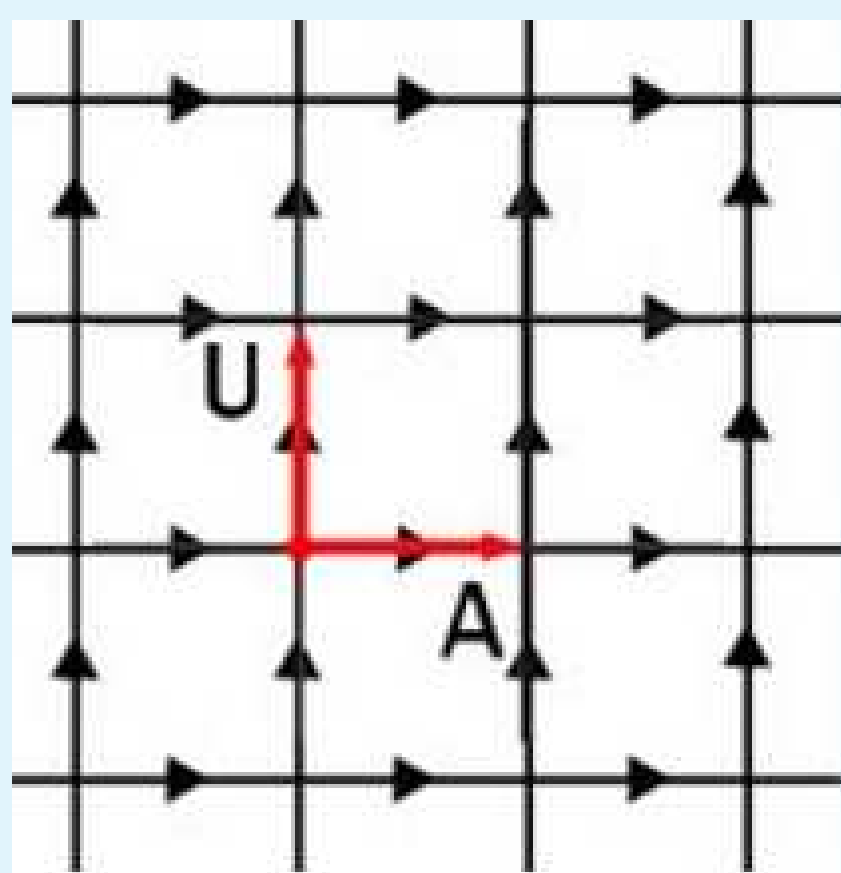
5. Conclusion

- Each path traced by Bernd is a closed loop.
- The path of each tetromino encloses one cell of the grid in clockwise direction, and a cell in counterclockwise direction.
- It can be shown that the path around every tileable region encloses an equal number of cells in each orientation.
- However, all cells enclosed by the boundary path of the Aztec diamond have the same orientation.
- From the above, we can conclude untileability for all n .

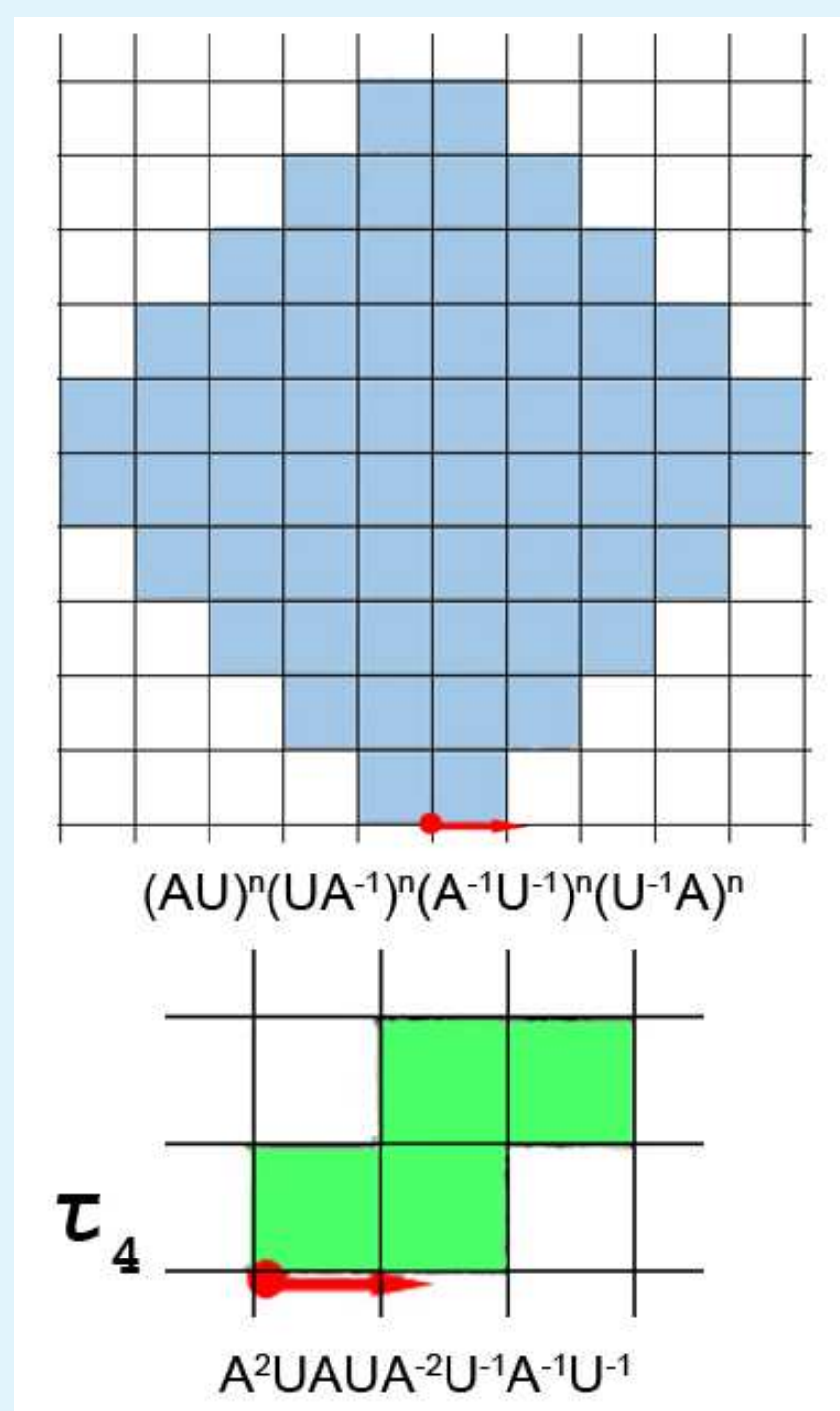
In general, one can construct a group containing the boundary words of all tileable regions. Its quotient can be drawn as a Cayley graph, which can be used to show that the boundary of a region is not in the original group.

3. A walk in Moscow

The streets of Moscow form a square grid, and on each of them the traffic is directed as in the picture below.

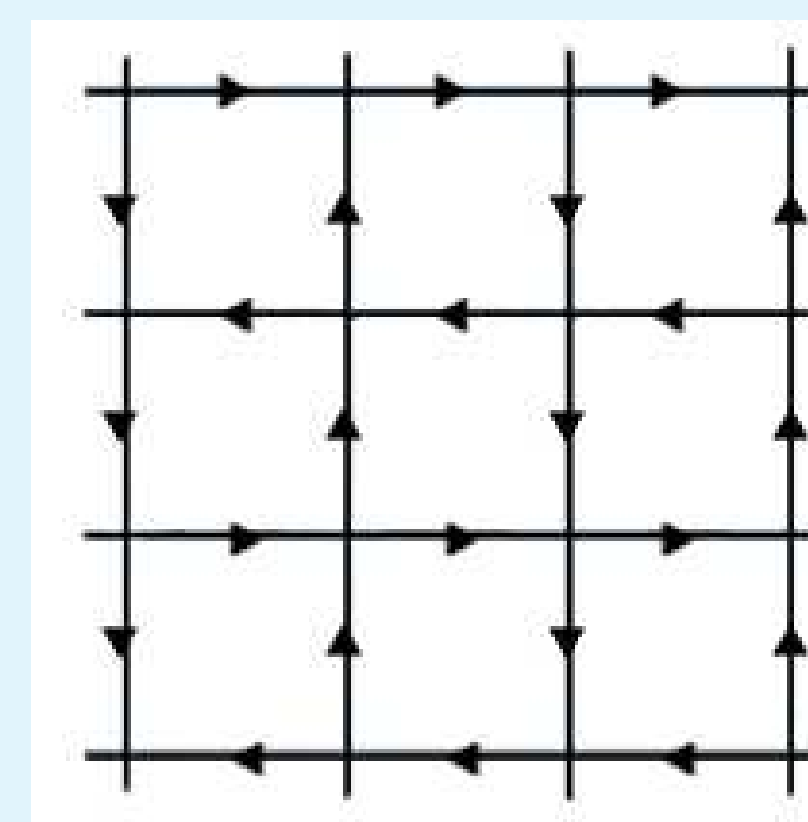


Mikhail starts at a crossroad in Moscow and traces the boundary of the Aztec diamond, writing down the direction he walks in (U if vertical, A if horizontal, adding the power -1 whenever he goes against traffic). He does the same for the skew tetrominoes.



4. A walk in Berlin

Berlin has a different traffic system than Moscow, as depicted below:



Bernd received the notes that his friend Mikhail in Moscow took during his walk. He follows the directions written down and walks the trajectories shown in the pictures on the right.

