Enumeration of Tilings

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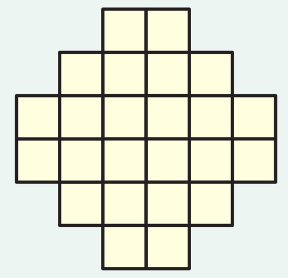
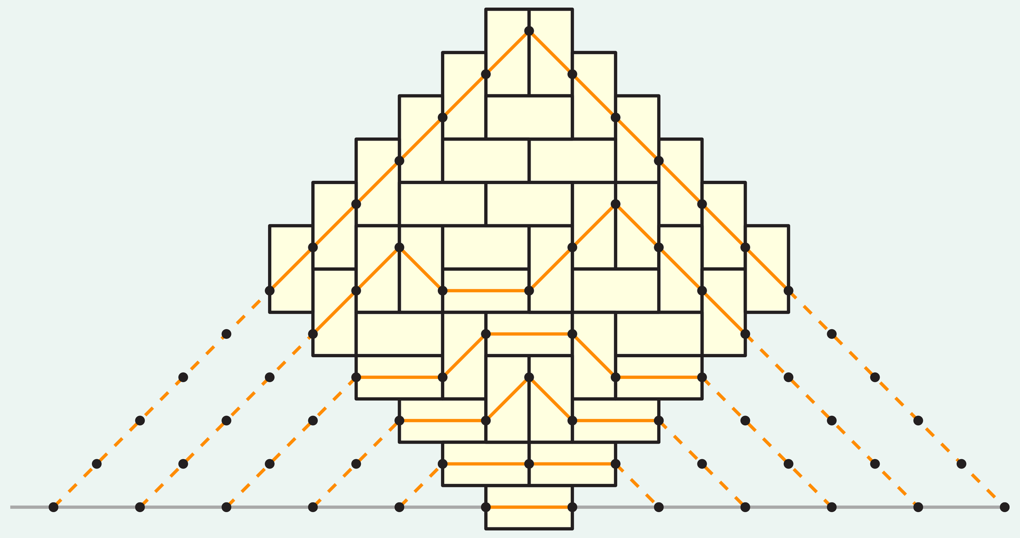
Domino Tilings of the Aztec Diamond

The Aztec Diamond of order n, called AD_n, consists of $2n(n+1)$ unit squares. We are interested in the number of tilings $AD(n)$ of the Aztec Diamond of order n with dominos. At right we see the AD_3.



Aztec Diamond Theorem: $AD(n) = 2^{\frac{n(n+1)}{2}}$

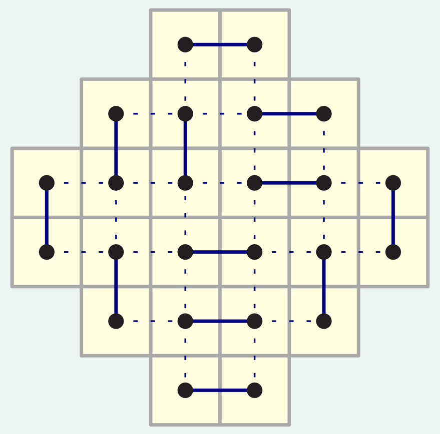
First sketch of a proof using Schröder Paths



We can identify every domino tiling with a set of **non-intersecting Schröder Paths** and the other way around. Schröder Paths have the property that they only make horizontal, up and down steps, and that they never go below the x-axis. With the **Lemma of Gessel-Viennot** we are able to count the number of these sets of paths and thus get $AD(n)$.

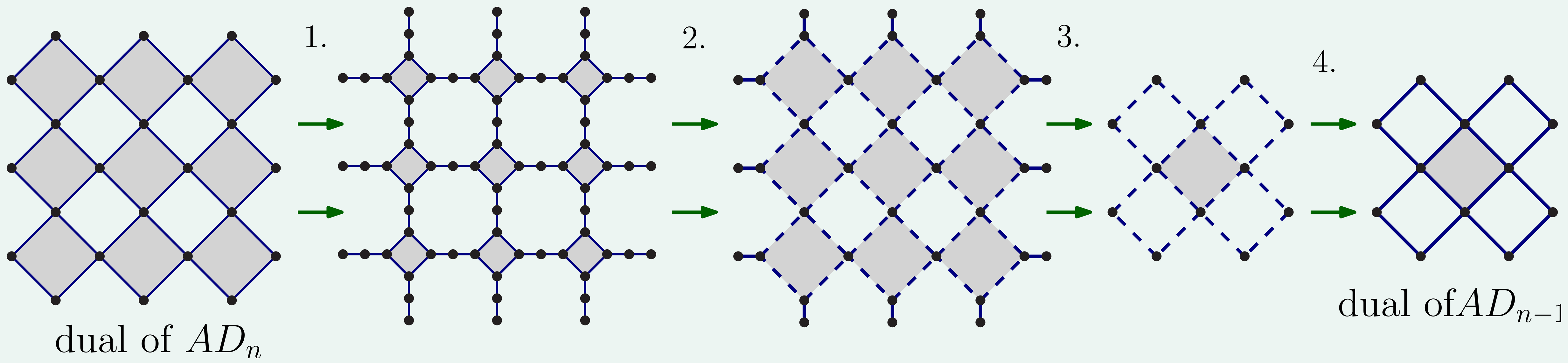
Second sketch of a proof using perfect matchings

Every domino tiling corresponds to a perfect matching of the dual graph of AD_n. If we give every edge e a weight $w(e)$ and define the weight of a matching M with $w(M) := \prod_{e \in M} w(e)$, then the sum over all these weights of matchings equals the number of domino tilings, in case that every edge has weight 1:



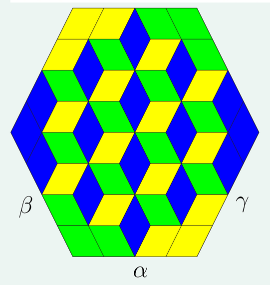
$$S(n) := \sum_{M \text{ perf. matching}} w(M) = AD(n)$$

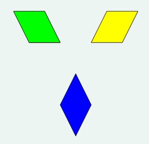
To count these weights of matchings, we can do some nontrivial transformations on the dual graph as shown below and focus on the sum of weights of the matchings. Straight lines have weight 1 and dotted lines weight $\frac{1}{2}$.



In step 1 and 3 there are bijections between the perfect matchings of the graphs. In step 2 we use a lemma to replace small cells by big cells and just like in step 4 the sum over the weights of the matchings differs by a factor here. In the end we have $S(n) = 2^n S(n-1)$.
Hence also the recursion $AD(n) = 2^n AD(n-1)$ holds. With the start value $AD(1) = 2$ and the induction principle it is easy to show the theorem now.

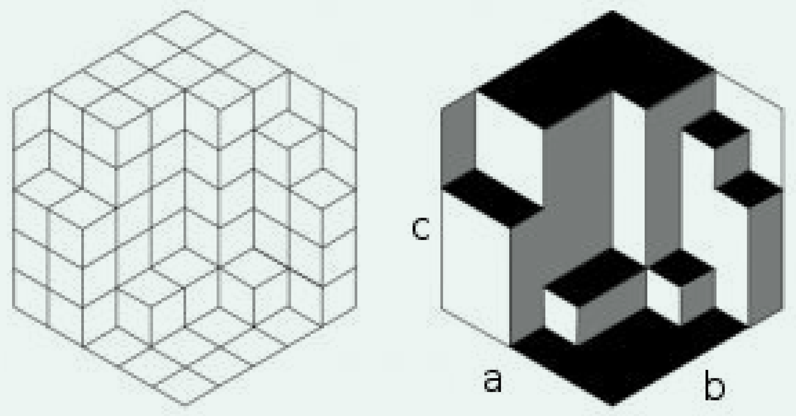
How many Calissons of each taste are there in a hexagonal box?



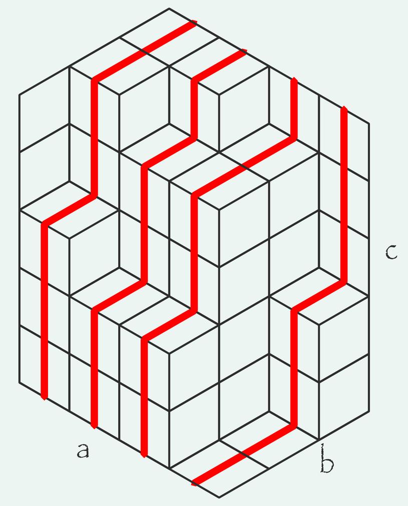


Calissons are some famous french sweet shaped like a regular rhombus. They come in a hexagonal box. If oriented by taste/colour the question is, how many of each we get within an arbitrary filled box (rhombic tiling)?
Answer: We get $\alpha \times \beta$ green ones, $\alpha \times \gamma$ yellow ones and $\beta \times \gamma$ blue ones. This can easily be checked by imagining certain angles of view, i.e. bottom-left for the green ones.
In case of a regular $n \times n \times n$ box, there are exactly n^2 Calissons of each taste, no matter the tiling.

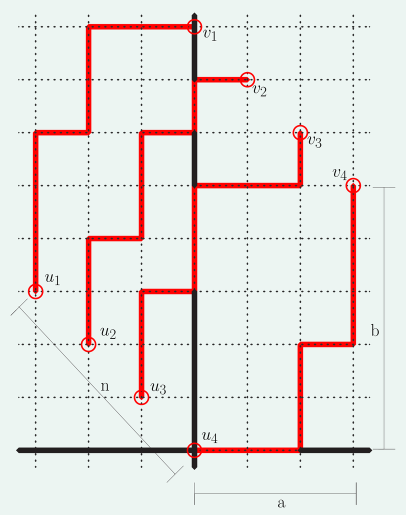
How many Rhombic Tilings of a Hexagon are there?



Plane Partition: *A well known bijection gives us exactly one plane partition bounded by an $(a \times b \times c)$ - box for each rhombic tiling of an (a, b, c, a, b, c) - hexagon.*



Paths upon PlanePartitions: *Starting on one of the base sides of the plane partition, we can traverse the rhombus tiling crossing only edges parallel to the one we started with. We arrive at characteristic paths upon the plane partition. By rotation and translation we can arrange those paths still non-intersecting in the grid $\mathbb{Z}^2$*



Non-intersecting Lattice Paths: *Using the Lemma of Gessel-Viennot to count the number of non-intersecting lattice paths N(n,a,b), we arrive at*

$$N(n,a,b) = det_{1 \le i,j \le n}\left(\binom{a+b}{a-i+j}\right)$$

(Evaluation of the determinant by C.Krattenthaler using the condensation formula of Desnanot)

MacMahon Formula (1915)

The total number N(a,b,c) of Rhombic Tilings of a $(a \times b \times c)$-hexagon is then given by MacMahon's formula as

$$N(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$