The bijection between arrangements and zonotopal tilings

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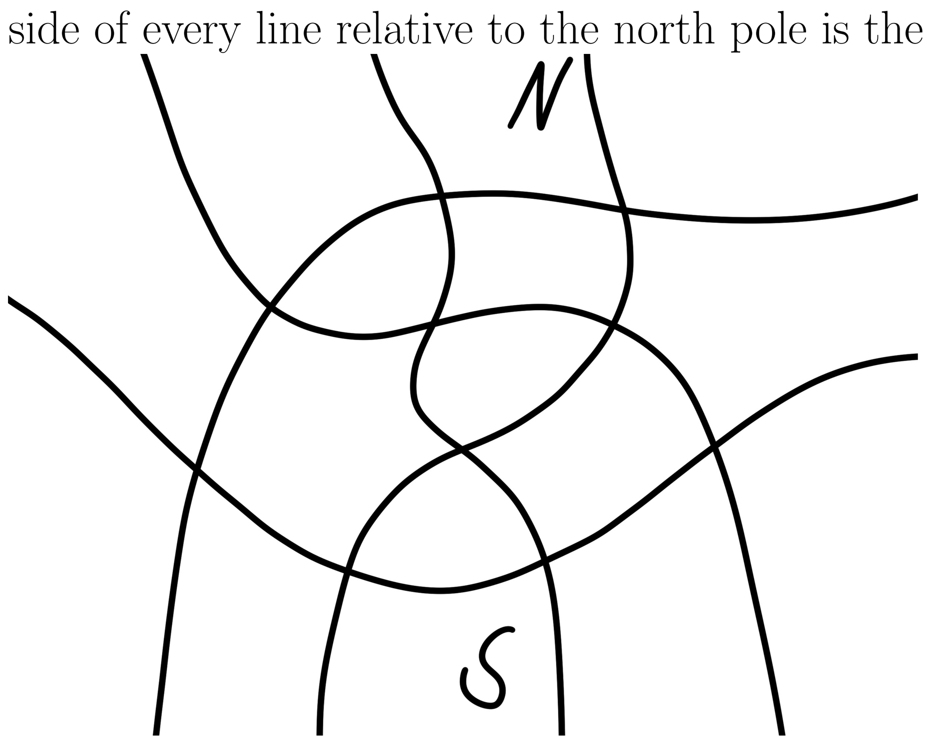
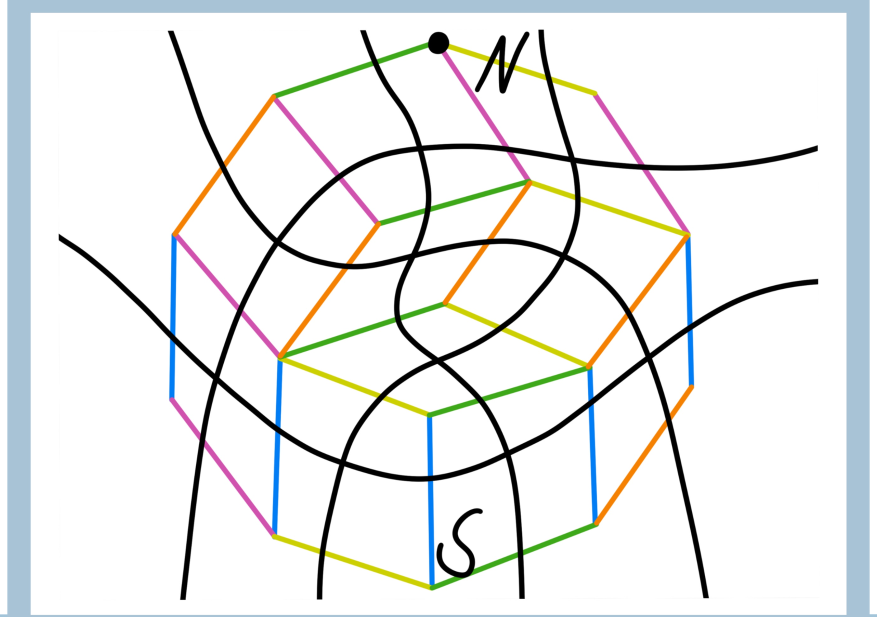
Definition: Pseudoline, Arrangement

A *pseudoline* is a curve $c\colon \mathbb{R} \to \mathbb{R}^2$ that does not intersect itself and that goes to infinity on both sides.
An *arrangement* of $n \in \mathbb{N}$ pseudolines is a collection of n pseudolines such that each pair crosses exactly once and has no further intersections. Also every point is contained in at most two lines. With one of the unbounded cells marked, the arrangement becomes a *marked* arrangment. We call the marked cell the *north pole*. The cell that is on the other side of every line relative to the north pole is the *south pole*.



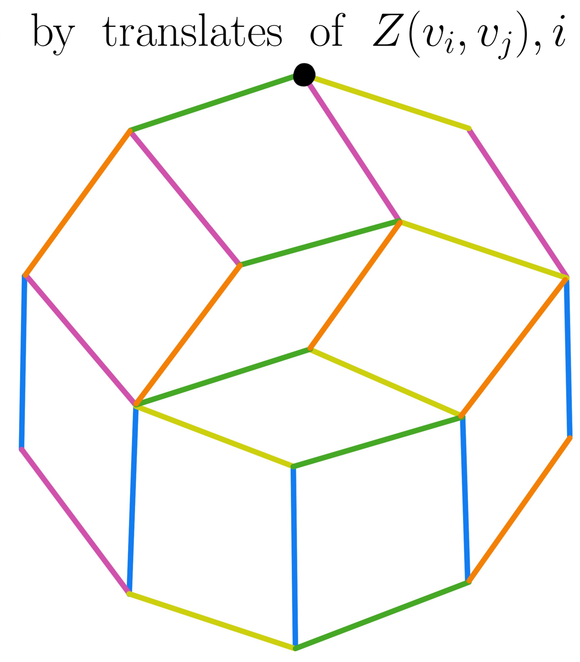


Definition: Marked zonotope, Zonotopal tiling

A *n-zonotope* is a centrally symmetric $2n$-gon in the plane. Alternatively, it is the Minkowski sum of n line segments in the plane. To $0 \neq v_i \in \mathbb{R}^2$ we associate $[-v_i, v_i]$ for $i \in [n]$. Then

$$Z(V) = \left\{ \sum_{i=1}^{n} c_i v_i : c_i \in [-1,1] \right\}$$

is the zonotope associated to $V = v_1, \dots, v_n$. With one vertex marked we obtain a marked zonotope.
A *zonotopal tiling* of $Z(V)$ is a tiling of $Z(V)$ by translates of $Z(v_i, v_j), i \neq j$.



Bijection: Arrangement $\rightarrow$ Tiling

The Arrangement can be seen as a graph with some edges connecting to infinity. If we draw the dual graph to the arrangement above, we obtain the zonotopal tiling above to the right.

Proof:

We now want to give an overview over the proof that this actually gives a bijection.

Bijection: Tiling $\rightarrow$ Arrangement

The tiling has n parallel classes, marked with different colors above. If we draw pseudolines through these classes, we obtain the arrangement above to the left.

Definition: Sweep of an arrangement

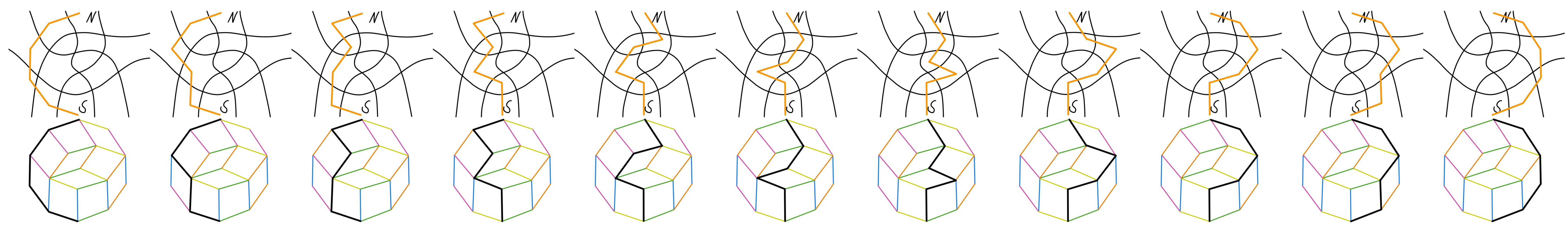
A sweep of an arrangement is a set of pseudolines $p_0, \dots, p_m$, $m = \binom{n}{2}$ with the following properties:

- For every $i \in [m]$, p_i could be added to the arrangement to obtain an arrangement of $n+1$ lines, i.e. p_i crosses each lines of the arrangement exactly once, has no further intersections with any line and does not pass through an intersection point of the arrangement.
- All pseudolines of the sweep start in the north pole and end in the south pole.
- p_0 has all intersection points of the arrangement to its right, p_m has them all on its left.
- p_i has exactly one intersection point extra to its left when compared to p_{i-1}.

Definition: Sweep of a zonotopal tiling

A sweep of a zonotopal tiling is a set of curves $p_0, \dots, p_m$, $m = \binom{n}{2}$ along the edges of the tiling with the following properties:

- For every $i \in [m]$, p_i uses exactly one edge out of each of the parallel classes.
- All curves of the sweep start at the marked vertex and end at the opposite vertex.
- p_0 walks along the left boundary of the zonotope, p_m along the right boundary.
- p_i differs from p_{i-1} exactly in the path taken across one rhombus.



Association to allowable sequences

Considering the sweeps above, in both cases we can interpret them as a sequence of permutations.
In the case of the arrangement, we number the lines from 1 to n starting at the north pole, going counterclockwise. Then the permutation associated to p_i is the order that p_i crosses the lines of the arrangement.
In the case of the tiling, we number the parallel classes from 1 to n starting at the marked vertex, going counterclockwise. Then the permutation associated to p_i is the order that p_i uses the parallel classes.
In both cases we get the same sequence of permutations. It has the following properties:

- p_0 is the identity and p_m is the reverse identity.
- p_i is obtained from p_{i-1} by applying exactly one adjacent transposition (adjacent in p_i).
- Every pair $x, y \in [n]$ is switched exactly once over the course of the sequence.

Such a sequence is called an *allowable sequence of permutations*.
In the case above we obtain the sequence

12345, 12435, 14235, 14253, 41253, 42153,
42513, 45213, 54213, 54231, 54321.

The other direction: Allowable sequence $\rightarrow$ Tiling with sweep

If we are given an allowable sequence $p_0, \dots, p_m$ of permutations, we can reconstruct the arrangement and the tiling, each with a sweep, that it came from. Here we only hint at how to do this in the case of the tiling:
Draw the left boundary of the zonotope, this is p_0. We place one rhombus at a time. Assume we have placed rhobi such that the right boundary of the shape is p_i (i.e. the order of the parallel classes is in the order of p_i). Then the adjacent transposition that we need to apply to obtain p_{i+1} is the transposition of two parallel classes that occur after one another on the right boundary of the shape we already have. Hence, exactly two sides of the corresponding rhombus are already used. We can place the rhombus now to obtain p_{i+1} as right boundary.