Combinatorial generation via permutation languages

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Overview

Presented here are many applications for Algorithm J, a greedy algorithm that exhaustively generates so-called zigzag languages and by extension many interesting families of combinatorial objects. The algorithm moves an entry in some permutation by k steps to either direction, an operation called a *jump* of length k.

Algorithm J

1. Start with π_0
2. Choose length-wise minimal jump of largest possible entry to get a new permutation. If no such jump exists or if both directions work, terminate.
   Repeat Step 2.

Pattern-avoiding Languages

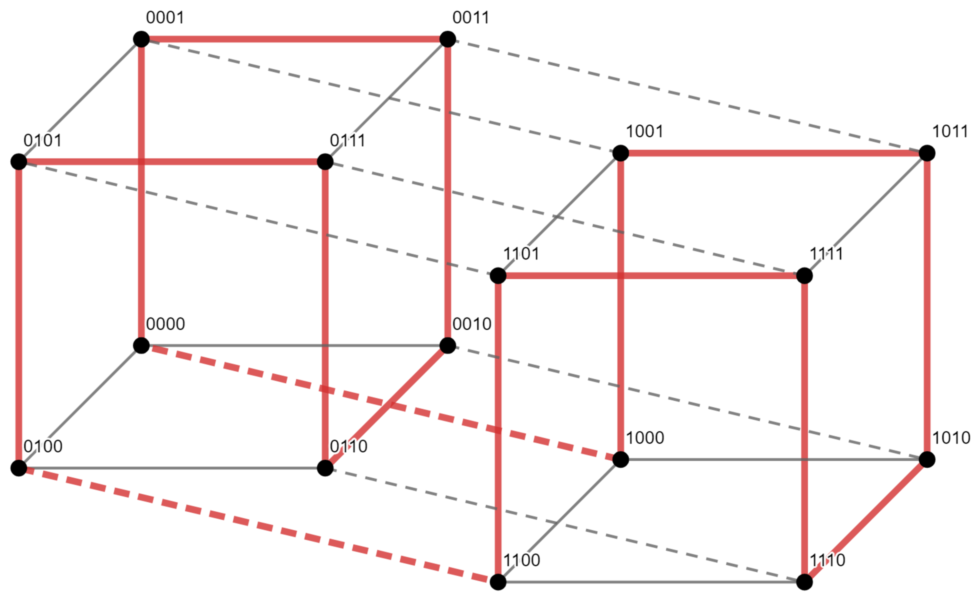
Some of the most noteworthy zigzag languages are the subsets of the symmetric group that avoid patterns which have their largest entry in either the first or last position. We call such patterns *tame*. For a more rigorous definition of zigzag languages and tameness conditions for other types of patterns, consult [1].

Noteworthy Examples

There is a bijection between $S_n(132 \wedge 231)$, the language with no *peaks*, and the set of all binary strings of length $n-1$: construct a permutation by parsing a bit-string and adding the next entry to the right, if the current bit is a zero and to the left, if it is a one:

Example:

$$f(\varepsilon) = 1$$
$$f(0) = 12$$
$$f(01) = 312$$
$$f(010) = 3124$$



Thus, applying Algorithm J to $S_n(132 \wedge 231)$ yields a Hamiltonian path through the $(n-1)-$dimensional hypercube.

Some other important zigzag languages include but are not limited to:

Avoided patterns	**Combinatorial objects and ordering**
$2\underline{31} = 231$	Catalan families • binary trees by rotations → Lucas-van Baronaigien-Ruskey order • triangulations by edge flips • Dyck paths by hill flips
$2\underline{31}$	Bell families • set partitions by element exchanges → Kaye's order
2143 : vexillary permutations	
conjunction of v_k tame patterns with $v_2 = 35, v_3 = 91, v_4 = 2346$: $k-$vexillary permutations $(k \geq 1)$	
$2143 \wedge 3412$: skew-merged permutations	
$2143 \wedge 2413 \wedge 3142$	
$2143 \wedge 2413 \wedge 3142 \wedge 3412$: X-shaped permutations	
$2413 \wedge 3142$: separable permutations	Schröder families • slicing floorplans • topological drawings of $K_{2,n}$
$2\underline{41}3 \wedge 3\underline{14}2$: Baxter, $2\underline{41}3 \wedge 3\underline{41}2$: twisted Baxter, $2\underline{14}3 \wedge 3\underline{14}2$	mosaic floorplans (=diagonal rectangulations= $R-$equivalent rectangulations)
$2\underline{14}3 \wedge 3\underline{41}2$	S-equivalent rectangulations
$2\underline{14}3 \wedge 3\underline{41}2 \wedge 2413 \wedge 3142$	S-equivalent guillotine rectangulations
$35\underline{124} \wedge 35\underline{142} \wedge 24\underline{513} \wedge 42\underline{513}$: $2-$clumped permutations	generic rectangulations (=rectangular drawings)
conjunction of c_k tame patterns with $c_k = 2(k/2)!(k/2+1)!$ for k even and $c_k = 2((k+1)/2)!2$ for k odd: $k-$clumped permutations	
conjunction of 12 tame patterns: permutations with $0-1$ Schubert polynomial	
$2143 \wedge 2413 \wedge 3412 \wedge 314562 \wedge 412563 \wedge 415632 \wedge 431562 \wedge 512364 \wedge 512643 \wedge 516432 \wedge 541263 \wedge 541632 \wedge 543162$: widdershins permutations	

Steinhaus-Johnson-Trotter

Appliying algorithm J to the symmetric group returns a Hamiltonian path through the permutohedron - the Steinhaus-Johnson-Trotter order.

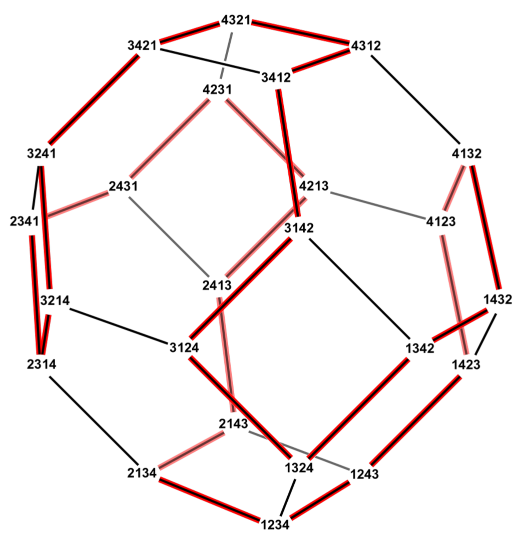


Figure taken from [2]

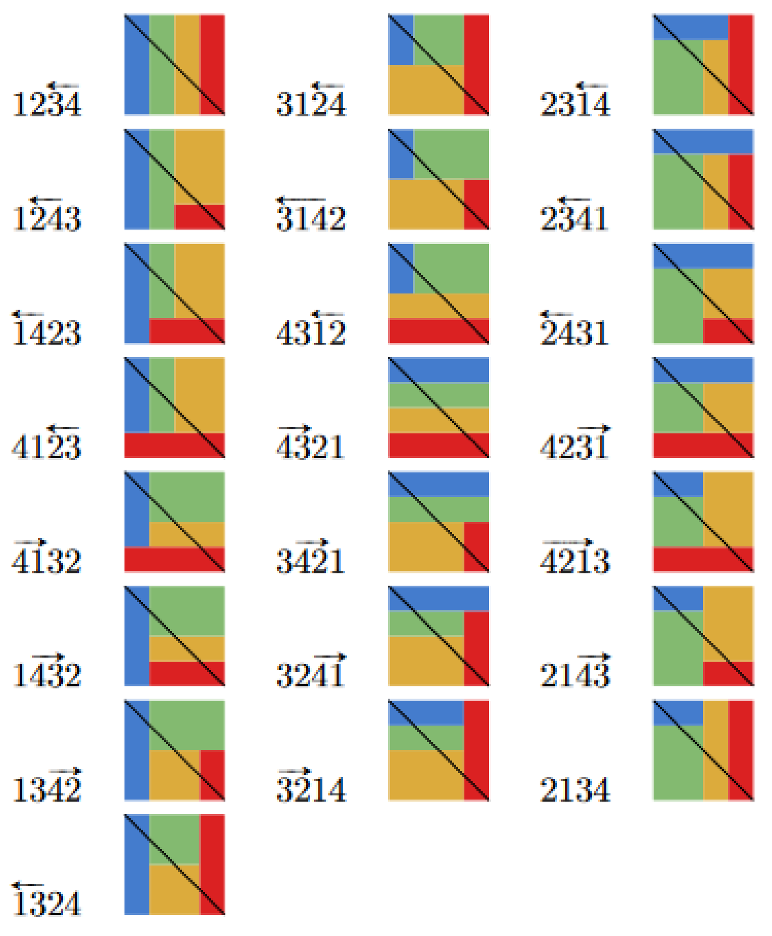
References

[1] E. Hartung, H. P. Hoang, T. Mütze, and A. Williams. *Combinatorial generation via permutation languages*, Symposium on Discrete Algorithms (SODA) 2020

[2] https://en.wikipedia.org/wiki/Steinhaus-Johnson-Trotter_algorithm

Diagonal Rectangulations

The set of permutations avoiding both $2\underline{41}3$ and $3\underline{41}2$, where the underlined entries need to appear next to each other, is a zigzag language.

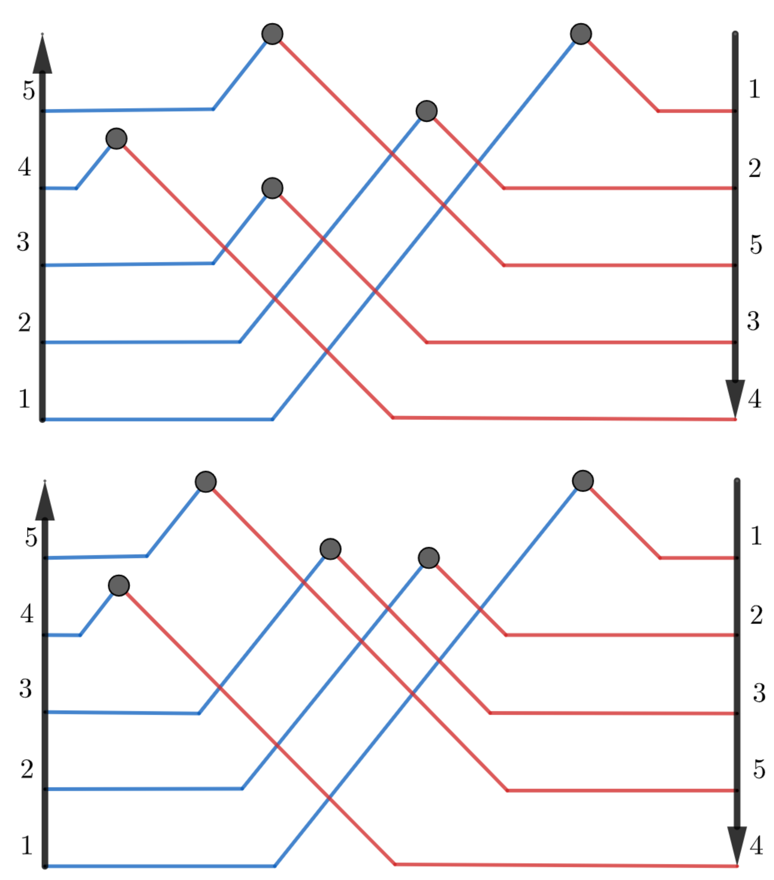


Algorithm J yields an ordering of these permutations and also of diagonal rectangulations, where successors are reached via a flip.

Figure taken from [1]

Drawings of $K_{2,n}$

If all edges are required to meet at most once, including at a common end-point, a permutation avoiding the patterns $2413 \wedge 3142$ corresponds to a drawing of the $K_{2,n}$.



To avoid ambiguity, we have fixed all edges to cross such that the red edge goes from the top left to the bottom right. A jump on a permutation then corresponds to moving one of the grey peaks in the graphic over a single edge.