MULTITRIANGULATIONS AS COMPLEXES OF STAR POLYGONS

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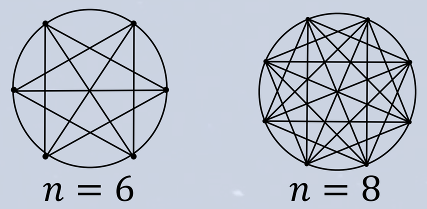
STARTING POINT

Which structures in multitriangulations can play the role of triangles in triangulations?

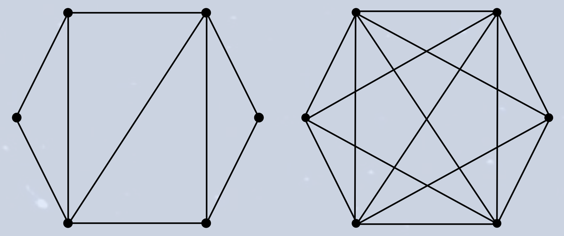
Graph $G = (V_n, E_n)$

Let V_n be the set of vertices of a convex n-gon, i.e. any set of n points on the unit circle. Let E_n be the set of edges of the complete graph on V_n.



Multitriangulation

A multitriangulation T of order k of a convex n-gon is a maximal set of edges such that no $k+1$ of them mutually cross.

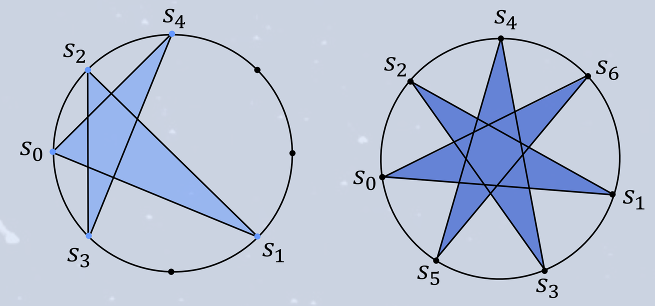


IDEA

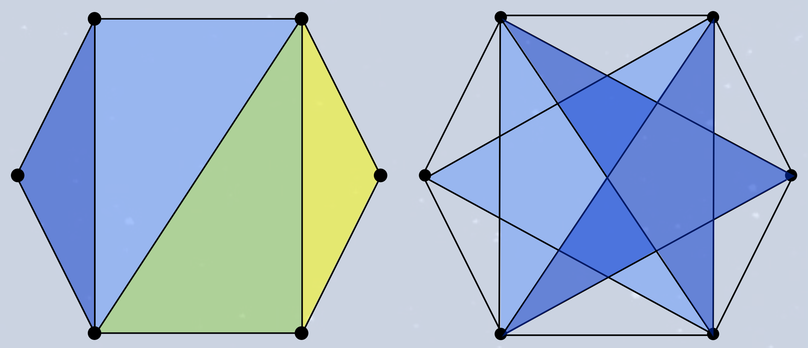
For $k > 1$, consider k-stars in multitriangulations.

***k*-Star**

A k-star is a set of edges of the form $\{[s_j, s_j + k] \mid j \in \mathbb{Z}_{2k+1},\ s_j \in V_n\}$



***k*-Stars vs. Triangles**

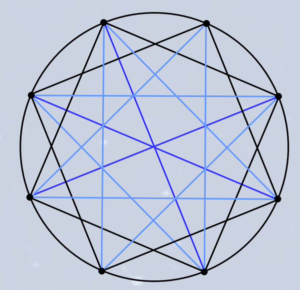


DEFINITIONS

Classification of the Edges

The length of an edge $[u, v]$ is defined as the cyclic distance between its two end nodes $d := \min(|[\![u,v[\![|, |[\![v,u[\![|)$

- ☆ k-relevant edges: $d > k$
- ☆ k-boundary edges: $d = k$
- ☆ k-irrelevant edges: $d < k$



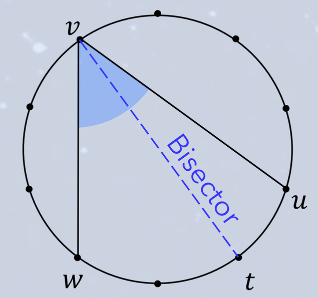
3-triangulation of an octagon

! A multitriangulation always contains all the nk k-irrelevant plus k-boundary edges since they can never be part of a $(k+1)$-crossing.

Angle and Bisector

An angle $\measuredangle(u,v,w)$ is a pair of edges $\{[u,v],[v,w]\}$, such that $\forall\ t \in]\!]w,u[\![$, the edge $[v,t]$ is not in T. We call this edge a bisector of the angle.

- ☆ k-relevant angle: both its edges are either k-relevant or k-boundary edges

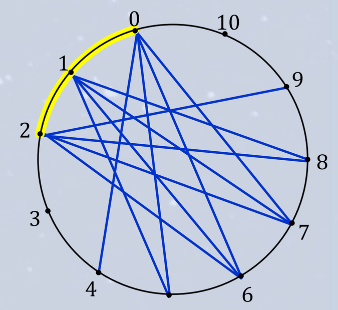


Canonical Form

$T^{min}_{n,k}$ is the canonical k-triangulation of the n-gon if its set of k-relevant edges is

$\{[i,j] \mid i \in [\![0, k-1]\!],$

$j \in]\!]i+k, i-k[\![\ \}$



k-relevant edges of $T^{min}_{11,3}$

FLIPS

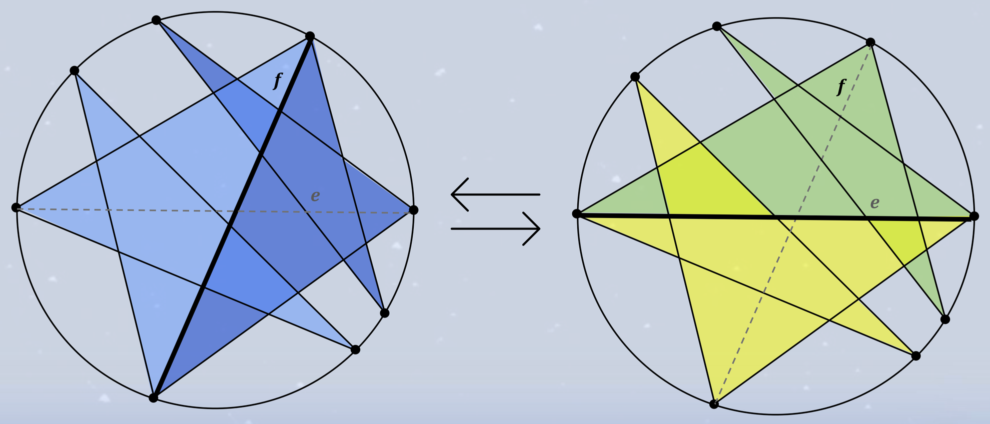
It can be shown that every pair of k-stars in T has a unique common bisector. With this knowledge, we can define flips for multitriangulations.

Flips in Multitriangulations

Let R and S be to k-stars in T with a common edge f and $e = [r_0, s_0]$ being the common bisector of the two stars.

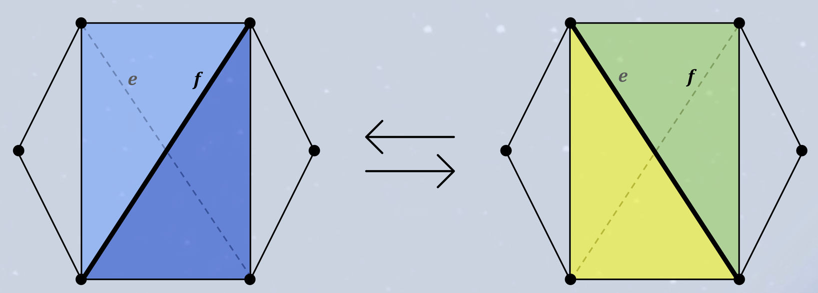
Flip of f consists in building $T \Delta \{e, f\}$

- ☆ The resulting graph is a k-triangulation
- ☆ It contains two new k-stars having e as their common edge and f as bisector



***k*-Stars vs. Triangles**

This idea is analogous to flips in 1 -triangulations, where two triangles also always have a unique common bisector. An edge f can be flipped by replacing it with the common bisector e of the two triangles adjacent to f.



MAIN RESULTS

1

- ☆ Any k-relevant edge of T belongs to exactly 2 k-stars
- ⇒ Every k-relevant edge can be flipped
- ☆ Any k-boundary edge of T belongs to exactly 1 k-star
- ☆ Any k-irrelevant edge of T does not belong to any k-star

2 **Bijections**

Vertices not used in the k-star S of T

⇕

k-stars of T different from S

k-relevant edges not used in T

⇕

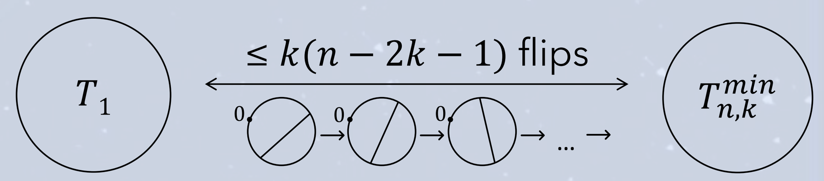
Pairs of k-stars of T

Conclusions from 1 and 2:

- ☆ There are $n - (2k+1)$ k-stars different from S. Hence, any k-triangulation contains $n - 2k$ k-stars.
- ☆ Double counting the number of pairs (S, e) such that e is in S allows us to compute the number of k-relevant edges in T which is $k(n - 2k - 1)$
- ⇒ Every k-triangulation has $k(2n - 2k - 1)$ edges

3

For any k-triangulation of the n-gon $T \neq T^{min}_{n,k}$ there exists a sequence of at most $k(n-2k-1)$ slope-decreasing flips from T to $T^{min}_{n,k}$.



Properties of the Flipgraph

1 2 3 ⟹

The flip graph of k-triangulations of the n-gon is connected, regular of degree $k(n-2k-1)$, and its diameter is at most $2k(n-2k-1)$.